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ON THE GENUS OF A GROUP

BY

ARTHUR T. WHITE[1]

ABSTRACT. The *genus of a group* is defined to be the minimum genus for any Cayley color graph of the group. All finite planar groups have been determined, but little is known about the genus of finite nonplanar groups. In this paper two families of toroidal groups are presented; the genus is calculated for certain abelian groups; and upper bounds are given for the genera of the symmetric and alternating groups and for some hamiltonian groups.

Introduction. Corresponding to a presentation P, in terms of generators and relations, for a group G is the *Cayley color graph* $D_P(G)$, defined as follows: The vertices of $D_P(G)$ are the elements of G; with the generators are associated distinct colors, and there is a directed edge in $D_P(G)$ from g_1 to g_2 and colored with the color of generator h, if and only if $g_1 h = g_2$ in G. For each generator of order two, we adopt the standard convention of replacing each pair of directed edges (g_1, g_2) and (g_2, g_1) by the single undirected edge $[g_1, g_2]$. The *genus*, γ, *of a graph* is the minimum genus among the genera of all closed orientable 2-manifolds in which the graph can be imbedded. The *genus of a Cayley color graph*, $\gamma(D_P(G))$, is the genus of the graph that results when all arrows and colors are omitted from $D_P(G)$. We define the *genus of a group* G to be the minimum genus for any Cayley color graph of G; i.e.

$$\gamma(G) = \min_{\text{over } P} \gamma(D_P(G)).$$

If G has no finite genus, we write $\gamma(G) = \infty$. Levinson [10] has shown that an infinite group G has either $\gamma(G) = 0$ or $\gamma(G) = \infty$. Here we consider only finite groups. A *minimal presentation* has no redundant generators. If $\gamma(G) = \gamma(D_P(G))$, we call P a *genus presentation* for G.

Dyck [7] (see also Burnside [4, Chapters 18 and 19]) considered maps on closed orientable 2-manifolds that are transformed into themselves in accordance with a fixed group G, acting transitively on the regions of the map. Any such map gives an upper bound for $\gamma(G)$, as a "dual" formed in terms of Burnside's white regions gives a Cayley color graph for G. Brahana [3] studied groups represented by regular maps on closed orientable 2-manifolds; these maps correspond to presentations on two generators, one of which is of order two. In this context the

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group acts transitively on the edges of the map, and again an upper bound for $\gamma(G)$ is obtained. In this paper we regard G as acting transitively on the vertices of the map induced by imbedding $D_P(G)$ in a closed orientable 2-manifold; indeed, it is well-known that the automorphism group of $D_P(G)$ is isomorphic to G, independent of the presentation P. Our point of view may give a closed orientable 2-manifold of lower genus for a given group; for example the direct product $Z_2 \times Z_4$ (where Z_n denotes the cyclic group of order n) is toroidal for Dyck (or Burnside) and for Brahana, yet $\gamma(Z_2 \times Z_4) = 0$; see Figure 1. Also, our definition has the following intuitive appeal: a Cayley color graph gives a "picture" of its group, from which many group properties (such as commutivity, normality of certain subgroups, the multiplication table) can be discerned; perhaps then it is natural to seek the most efficient surface upon which to "draw" this picture.

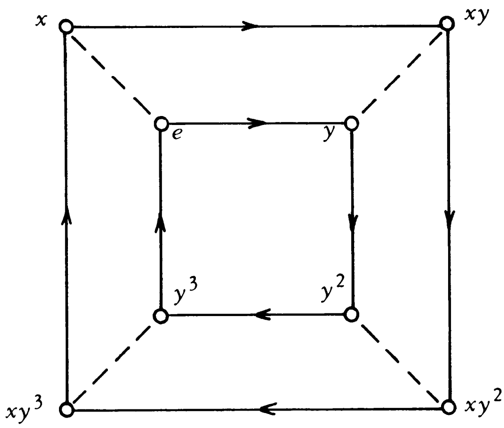

Figure 1. $Z_2 \times Z_4 = \langle x, y | x^2 = y^4 = xyxy^{-1} = e \rangle$

Planar groups. Maschke [11] determined all finite planar Cayley color graphs, and thus all finite groups of genus zero. We summarize his findings (see [5, p. 37] to identify Maschke's extended octahedral and extended icosahedral groups as $Z_2 \times S_4$ and $Z_2 \times S_5$, respectively):

Theorem 1. *The finite group G is planar if and only if $G = G_1 \times G_2$, where $G_1 = Z_1$ or Z_2 and $G_2 = Z_n, D_n, A_4, S_4$ or A_5.*

In this paper we present two infinite families of toroidal groups, determine the genus for certain abelian groups, and derive upper bounds for the genera of several other classes of groups. We will find particularly useful an algebraic description of graphical imbeddings developed by Edmonds [8] (see also Youngs [15]).

Edmonds' permutation technique. Let a connected graph H have vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$. Let $V(i) = \{v_j | v_i v_j \in E(H)\}$, where $E(H)$ denotes the edge set of H. Let p_i: $V(i) \to V(i)$ be a cyclic permutation, of length $|V(i)|$; then

$(p_1, p_2, \cdots, p_n)$ uniquely determines a 2-cell imbedding of H in a closed orientable 2-manifold such that the adjacencies at v_i are given (in the order imposed by the fixed orientation) by p_i. Conversely, corresponding to a 2-cell imbedding of a connected graph H, the p_i are uniquely determined. Furthermore, let $D = \{(v_i, v_j) \mid v_iv_j \in E(H)\}$, and define $F: D \to D$ by $F(v_i, v_j) = (v_j, p_{v_j}(v_i))$; then the orbits under the permutation F correspond to the region boundaries of the imbedding. We will specify each such orbit by listing the constituent vertices in sequence. In the present context, of course, the vertices are group elements.

Two classes of toroidal groups. We are now prepared to compute the genus for two infinite families of groups.

Theorem 2. *The group $Z_m \times Z_n$ is toroidal if $(m, n) \geq 3$ and planar otherwise.*

Proof. If $(m, n) = 1$, $Z_m \times Z_n = Z_{mn}$. If $(m, n) = 2$, $Z_m \times Z_n = Z_2 \times Z_{mn/2}$. For $(m, n) \geq 3$, $\gamma(Z_m \times Z_n) \geq 1$, since $Z_m \times Z_n$ is not one of the groups of Theorem 1. Now consider the following presentation P:

$$Z_m \times Z_n = \langle r, s \mid r^m = s^n = rsr^{-1}s^{-1} = e \rangle;$$

then $\gamma(Z_m \times Z_n) \leq \gamma(D_P(Z_m \times Z_n)) = 1$ (see Figure 2 for the case $m = n = 3$; the generalization to arbitrary $m, n \geq 3$ is entirely obvious).

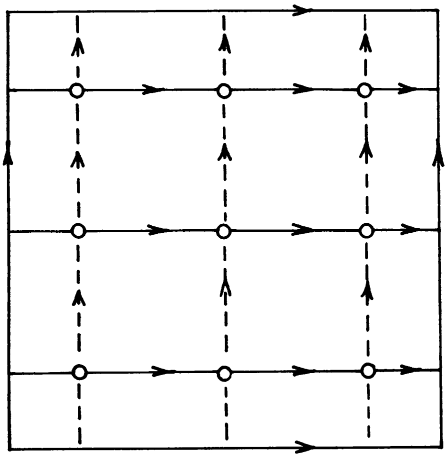

Figure 2. $Z_3 \times Z_3$ on the torus

The next class of groups includes the quaternions, the smallest order nonplanar group.

Theorem 3. *Let G_n be the dicyclic group:*

$$G_n = \langle x, y \mid x^{2n} = x^n y^{-2} = y^{-1}xyx = e \rangle;$$

then $\gamma(G_n) = 1$, for $n > 1$.

Proof. We note that $\gamma(G_n) \geq 1$, since G_n is not planar for $n > 1$ ($G_1 = Z_4$). We show that $\gamma(D_P(G_n)) \leq 1$ for P as in the statement of the theorem. Let $V_1 = \{x^i | 1 \leq i \leq 2n\}$ and $V_2 = \{x^i y | 1 \leq i \leq 2n\}$; then $G_n = V_1 \cup V_2$, a disjoint union. Now in $D_P(G_n)$ every edge colored x joins two vertices in V_1 or two vertices in V_2, since $(x^i)x \in V_1$ and $(x^j y)x \in V_2$, for otherwise $(x^j y)x = x^k$, for some k, and y would be redundant. Also, every edge colored y joins a vertex in V_1 to a vertex in V_2 (or vice-versa), since $(x^i)y \in V_2$ and $(x^j y)y = x^j(y^2) = x^j x^n = x^{j+n} \in V_1$. We select the permutations p_g of Edmonds' algorithm as follows:

$$p_g = \begin{cases} (gx^{-1}, gy^{-1}, gx, gy) & \text{if } g \in V_1, \\ (gy, gx, gy^{-1}, gx^{-1}) & \text{if } g \in V_2. \end{cases}$$

Figure 3 depicts the situation at $g \in V_2$.

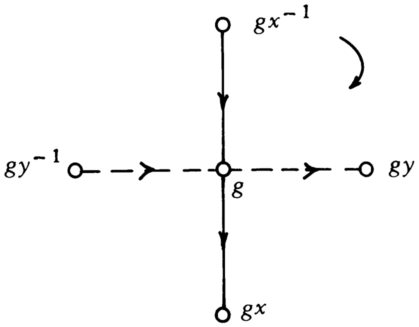

Figure 3. The situation at $g \in V_2$

We now compute orbits under F. Consider the directed edge (a, ay^{-1}), where $a \in V_1$. Then $ay^{-1} \in V_2$, and we compute the next vertex in this orbit to be $p_{ay^{-1}}(a) = ay^{-1}x \in V_2$. Continuing in this fashion, we obtain the orbit: $a - ay^{-1} - ay^{-1}x - ay^{-1}xy - ay^{-1}xyx - ay^{-1}xyxy^{-1} - \cdots$. But $ay^{-1}xyx = a$ and $ay^{-1}xyxy^{-1} = ay^{-1}$, so this orbit has length four. We obtain $2n$ distinct orbits of this form, as a ranges over V_1. Similarly, we obtain $2n$ additional distinct orbits, all of length four, containing directed edges of the form (a, ay^{-1}), where $a \in V_2$. This exhausts all the directed edges of $D_P(G_n)$, so that this 2-cell imbedding has $p = 4n$, $q = 8n$, and $r = 4n$ (where p, q, and r denote the number of vertices, edges, and regions respectively). The euler formula $p - q + r = 2 - 2\gamma$ now gives $\gamma = 1$.

Abelian groups. The following theorem is a standard result in elementary group theory; see for example [2, p. 348].

Theorem 4. *Let G be an abelian group of order n; then G can be expressed (uniquely) as a direct product:*

$$G = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r},$$

where $n = \Pi_{i=1}^{r} m_i$, m_i *is a multiple of* m_{i+1} *for* $1 \leq i \leq r-1$, *and* r *is the rank of* G.

There is a graphical product that corresponds in a natural manner to the direct product for groups. Let H_1 and H_2 be two graphs, with vertex sets $V(H_i)$ and edge sets $E(H_i)$, $i = 1, 2$ respectively. The cartesian product $H_1 \times H_2$ has:

(i) $V(H_1 \times H_2) = V(H_1) \times V(H_2)$,

(ii) $E(H_1 \times H_2) = \{[(u_1, u_2), (v_1, v_2)] | u_1 = v_1$ and $[u_2, v_2] \in E(H_2)$, or $u_2 = v_2$ and $[u_1, v_1] \in E(H_1)\}$.

Similarly, the *cartesian product* $D_{P_1}(G_1) \times D_{P_2}(G_2)$ of two Cayley color graphs has vertex set $V(D_{P_1}(G_1)) \times V(D_{P_2}(G_2))$, and (g_1, g_2) is joined to (g_1', g_2') by an edge colored h if and only if either:

(i) $g_1 = g_1'$ and $g_2 h = g_2'$, for h a generator in P_2, or

(ii) $g_2 = g_2'$ and $g_1 h = g_1'$, for h a generator in P_1.

Now let the presentations P_i for G_i, $i = 1, 2$, be given by:

$$G_1 = \langle k_1, \cdots, k_m | w_1 = \cdots = w_r = e \rangle,$$
$$G_2 = \langle k_{m+1}, \cdots, k_n | w_{r+1} = \cdots = w_{r+s} = e \rangle;$$

then the (external) direct product $G_1 \times G_2$ has *standard* presentation P, given by:

$$G_1 \times G_2 = \langle k_1, \cdots, k_n | w_1 = \cdots = w_{r+s} = k_i k_j k_i^{-1} k_j^{-1} = e,\ 1 \leq i \leq m < j \leq n \rangle,$$

where we represent $g_1 \in G_1$ as (g_1, e_{G_2}) and $g_2 \in G_2$ as (e_{G_1}, g_2), in the product $G_1 \times G_2$.

The next theorem follows directly from these definitions. (Figure 1 illustrates the theorem, for $G_1 = Z_2$, $G_2 = Z_4$.)

Theorem 5. *The cartesian product of the Cayley color graphs for two groups is a Cayley color graph for the direct product of the two groups*; *i.e.*

$$D_P(G_1 \times G_2) = D_{P_1}(G_1) \times D_{P_2}(G_2).$$

Let C_{m_i} denote a directed cycle with all edges colored with color i. Then if $Z_{m_i} = \langle x_i | x_i^{m_i} = e \rangle$ is given by presentation P_i, $D_{P_i}(Z_{m_i}) = C_{m_i}$, and Theorem 4, together with repeated applications of Theorem 5, gives:

Corollary 5a. *If* G *is an abelian group of* rank r, $G = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$, *then*

$$D_P(G) = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}.$$

Thus the genus of an abelian group is no greater than the genus of an appropriate repeated cartesian product of cycles. In [14], the author has investigated the genus of repeated cartesian products of cycles. Here we consider first the case where $m_i = 2$, $1 \leq i \leq r$; C_2 denotes the graph having two vertices and one edge, corresponding to the group Z_2. The following lemma is useful.

Lemma 1. *Suppose that 3 does not divide the order of the group G, and let P be a minimal presentation for G. Then $D_P(G)$ contains no triangles.*

Proof. Suppose $D_P(G)$ contains a triangle; then we find a closed walk $b_1^{a_1}b_2^{a_2}b_3^{a_3} = e$ in $D_P(G)$, where b_i is a generator in P, and $a_i = \pm 1$. If any two of the b_i are distinct, then one of these two is redundant, contradicting the minimality of P. If, on the other hand, $b_1 = b_2 = b_3$, then the a_i all have the same sign; but then $b_1^3 = e$, so that 3 divides the order of G.

Theorem 6. *Let G_r be the group of order 2^r having every nonidentity element of order 2. Then*

$$\gamma(G_r) = 1 + 2^{r-3}(r-4), \qquad r \geq 2.$$

Proof. G_r is necessarily abelian, of rank r. By Lemma 1, an imbedding of a Cayley color graph for G_r giving the genus will have no triangular regions. Hence by a well-known consequence of the euler formula (see, for example [9, p. 118]),

$$\gamma(G_r) \geq 1 - p/2 + q/4,$$

where $p = 2^r$ and $2q \geq r2^r$; thus

$$\gamma(G_r) \geq 1 - 2^{r-1} + r2^{r-3} = 1 + 2^{r-3}(r-4).$$

But G_r is the group of Corollary 5a, with all $m_i = 2$, so that

$$\gamma(G_r) \leq \gamma(D_P(G_r)) = \gamma(Q_r) = 1 + 2^{r-3}(r-4);$$

where the genus of the r-cube Q_r has been computed ([1] and [12]).

We next generalize this result, in accordance with [14], where the following theorem is established:

Theorem 7. *Let the graph H_n be defined by: $H_1 = C_{2m_1}$; $H_n = H_{n-1} \times C_{2m_n}$, $n \geq 2$. If $m_i \geq 2$, $1 \leq i \leq n$, then*

$$\gamma(H_n) = 1 + 2^{n-2}(n-2)\prod_{i=1}^{n} m_i.$$

The constructive method employed in the proof of Theorem 7 is easily modified to allow any or all of the m_i to be 1; suppose $m_i \geqq 2$ for $1 \leqq i \leqq k$, but $m_i = 1$ for $k+1 \leq i \leq n$. Then we obtain:

Theorem 7a. $\gamma(H_n) = 1 + 2^{n-3}(n+k-4)\prod_{i=1}^{n} m_i$.

This leads directly to:

Theorem 8. *Let G be an abelian group of* rank r, *written as*

$$G = Z_{2m_1} \times Z_{2m_2} \times \cdots \times Z_{2m_r},$$

where m_i is a multiple of m_{i+1} for $1 \leq i \leq r-1$.

Suppose that $m_k \neq 1$, *but* $m_{k+1} = 1$, $0 \leq k \leq r$. *Then*

$$\gamma(G) = 1 + 2^{r-3}(r + k - 4)\prod_{i=1}^{r} m_i.$$

Proof. By Corollary 5a and Theorem 7a,

$$\gamma(G) \leq \gamma(D_P(G)) = \gamma(C_{2m_1} \times C_{2m_2} \times \cdots \times C_{2m_r})$$

$$= 1 + 2^{r-3}(r + k - 4)\prod_{i=1}^{r} m_i.$$

The imbedding constructed for Theorem 7a has every region bounded by exactly four sides, and is minimal for G (since G has rank r), unless another imbedding can be found involving triangular regions. But any such imbedding must be of a Cayley color graph for G having larger degree than that corresponding to presentation P, and routine manipulations of the euler formula establish that the genus is then no longer minimal. Hence

$$\gamma(G) \geq 1 + 2^{r-3}(r + k - 4)\prod_{i=1}^{r} m_i.$$

Hamiltonian groups. A nonabelian group G is said to be *hamiltonian* if every subgroup of G is normal in G. For example, the quaternions Q are hamiltonian. In fact, hamiltonian groups have been characterized [6]:

Theorem 9. *A group G is hamiltonian if and only if* $G = Q \times A_1 \times A_2$, *where* A_1 *is abelian of odd order and* A_2 *has every nonidentity element of order* 2.

Suppose that $A_1 = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$, with r the rank of A_1, and that A_2 has order 2^n, with $r \leq n$. Then we have:

Theorem 10. *The hamiltonian group G has genus bounded by*:

$$\gamma(G) \leq 1 + 2^n \prod_{i=1}^{n} m_i (n + r).$$

Proof. Since $K_{4,4}$ (with edges colored and directed appropriately) is a Cayley color graph of minimum genus for Q, we apply Theorem 5 to see that $K_{4,4} \times C_{2m_1} \times \cdots \times C_{2m_r} \times Q_{n-r}$ is a Cayley color graph for G. The constructive techniques of [14] can be further extended to achieve a quadrilateral imbedding of this graph, giving the upper bound.

Formulas for four classes of presentations. In this section we develop genus formulas for four classes of presentations; these formulas collectively describe minimal imbeddings for all planar groups, as well as upper bounds for the genera of the groups S_n and A_n.

Theorem 11. *If G is minimally generated by $\{g_1, \cdots, g_n\}$ and satisfies at least the relations* $g_i^{m_i} = e = (\Pi_{j=1}^n g_j)^k$, $1 \leq i \leq n$, *then*

$$\gamma(G) \leq 1 + \frac{|G|}{2}\left(n - 1 - \frac{1}{k} - \sum_{j=1}^{n} \frac{1}{m_j}\right).$$

Proof. Select $p_g = (gg_1, gg_1^{-1}, gg_2, gg_2^{-1}, \cdots, gg_n, gg_n^{-1})$, for all $g \in G$. Then, using Edmonds' algorithm, we compute orbits as follows:

(i) An orbit containing the directed edge (a, ag_i^{-1}) continues with $p_{ag_i^{-1}}(a) = ag_i^{-2}$; hence this orbit corresponds to the relation $g_i^{m_i} = e$ and has length m_i. (If $m_i = 2$, we draw edges for both $gg_i = g'$ and $g'g_i = g$, obtaining $|G|/2$ 2-sided regions; for each such region, the two sides may be identified and the arrows removed, so that the region is destroyed but the genus is unaffected.)

(ii) An orbit containing the directed edge (a, ag_i) continues with $p_{ag_i}(a) = ag_i g_{i+1}$; hence this orbit corresponds to the relation $(\Pi_{i=1}^n g_i)^k = e$ and has length nk. As there are no other orbits, we find:

$$r = \sum_{i=1}^{n} r_{m_i} + r_{nk} = \sum_{i=1}^{n} \frac{|G|}{m_i} + \frac{|G|}{k},$$

where r_{m_i} denotes the number of m_i-sided regions in the imbedding corresponding to the relation $g_i^{m_i} = e$, and r_{nk} denotes the number of nk-sided regions corresponding to the relation $(\Pi_{j=1}^n g_j)^k = e$. The euler formula now gives the genus.

We note that an equivalent formula was obtained by Burnside [4, p. 398] in a different context. Theorem 11 gives $\gamma(G)$ exactly, for $G = Z_m, A_4, S_4, A_5$, or $Z_3 \times Z_3$. We also obtain the following two corollaries:

Corollary 11a.

$$\gamma(S_n) \leq 1 + \frac{(n-2)!}{4}(n^2 - 5n + 2), \qquad n \geq 2.$$

Proof. Take $S_n = \langle s, t\rangle$, where $s = (1\ 2\ 3 \cdots n)$, $t = (1\ 2)$; then $s^n = t^2 = (st)^{n-1} = e$.

Corollary 11b.

$$\gamma(A_n) \leq \begin{cases} 1 + \dfrac{(n-1)(n-3)!}{8}(n^2 - 6n + 4), & n \text{ odd}, \\[2ex] 1 + \dfrac{n(n-2)!(n-5)}{8}, & n \text{ even}. \end{cases}$$

Proof. Take $A_n = \langle s, r\rangle$, where $s = (1\ 2 \cdots n-2)$, $r = (1\ n-1)(2\ n)$ if n is odd; then $s^{n-2} = r^2 = (sr)^n = e$. If n is even, let $A_n = \langle s, r\rangle$, where $s = (1\ 2 \cdots n-1)$ and $r = (1\ 2)(3\ n)$, so that $s^{n-1} = r^2 = (sr)^{n-1} = e$ (see [3]).

The two formulas given above for S_n and A_n respectively were also found by Brahana [3], using a different method and in a slightly different context.

Theorem 12. *If G is minimally generated by $\{g_1, \cdots, g_n\}$ and satisfies at least the relations $g_i^2 = e = (g_k g_{k+1})^{w_k}$, $1 \le i \le n$, $1 \le k \le n$; and if $D_P(G)$ is bipartite for this presentation P, then*

$$\gamma(G) \le 1 + \frac{|G|}{4}\left(n - 2 - \sum_{k=1}^{n} \frac{1}{w_k}\right).$$

Proof. Select

$$p_g = \begin{cases} (gg_1, gg_2, \cdots, gg_n), & g \in V_1, \\ (gg_n, \cdots, gg_2, gg_1), & g \in V_2, \end{cases}$$

where $V_1 \cup V_2$ is the partition of the vertex set of the bipartite graph $D_P(G)$. Then, computing orbits, we find:

$$r = \sum_{k=1}^{n} r_{2w_k} = \sum_{k=1}^{n} \frac{|G|}{2w_k};$$

the genus now follows from the euler formula.

In applying Theorem 12, it is naturally desirable to order the g_i so that $\Sigma_{k=1}^n 1/w_k$ is a maximum. Theorem 12 gives $\gamma(G)$ exactly for $G = D_m, S_4, Z_2 \times D_m, Z_2 \times S_4, Z_2 \times A_5$, and the group (3, 3, 3; 2) of Coxeter (see [5, p. 134]).

The corollary below gives a bound for $\gamma(S_n)$ that coincides with a result of Jacques; see [13].

Corollary 12a. $\gamma(S_n) \le 1 + (n!/8)(n - 5)$, $n \ge 6$.

Proof. $S_n = \langle (1\ 2), (2\ 3), (3\ 4), \cdots, (n-1\ n)\rangle$; now take $V_1 = A_n$.
Note that this bound is not as sharp as that of Corollary 11a.

Theorem 13. *If $G = \langle g_1, g_2 | g_1^{m_1} = g_2^{m_2} = e = (g_1 g_2 g_1^{-1} g_2^{-1})^k\rangle$, where $m_1 \ge m_2 \ge 3$, then*

$$\gamma(G) \le 1 + \frac{|G|}{2}\left(1 - \frac{1}{k}\right).$$

Proof. We use $p_g = (gg_1, gg_2, gg_1^{-1}, gg_2^{-1})$ for all $g \in G$. Then every orbit has length $4k$, so that $r = |G|/k$, $p = |G|$, $q = 2|G|$, and the euler formula shows that

$$\gamma(G) \le \gamma(D_P(G)) \le 1 + \frac{|G|}{2}\left(1 - \frac{1}{k}\right),$$

where P is the presentation given above for G.

We note that the group G of Theorem 13 is abelian if and only if $k = 1$, and that in this case we have $\gamma(Z_{m_1} \times Z_{m_2}) \le 1$, in agreement with Theorem 2.

We now vary the presentation of Theorem 13 by setting $m_2 = 2$ (we still require $m_1 \ge 3$). Define the multi-graph H^* by omitting all colors and arrows from

$D_p(G)$ and shrinking every cycle representing the relation $g_1^{m_1} = e$ to a single vertex.

Theorem 14. *If H^* is bipartite, then*

$$\gamma(G) \leq 1 + \frac{|G|}{4}\left(1 - \frac{2}{m_1} - \frac{1}{k}\right).$$

Proof. Let V_1 (V_2) be the set of all elements g of G such that every word for g involves g_2 an odd (even) number of times. Then $G = V_1 \cup V_2$, since H^* is bipartite. Select

$$p_g = \begin{cases} (gg_1, gg_2, gg_1^{-1}) & \text{if } g \in V_1, \\ (gg_1^{-1}, gg_2, gg_1) & \text{if } g \in V_2. \end{cases}$$

Computing orbits for this imbedding, we obtain

$$r = r_{m_1} + r_{4k} = \frac{|G|}{m_1} + \frac{|G|}{2k}.$$

We note that $p = |G|$ and $q = 3|G|/2$. Thus

$$\gamma(G) \leq \gamma(D_P(G)) \leq 1 + \frac{|G|}{4}\left(1 - \frac{2}{m_1} - \frac{1}{k}\right).$$

Theorem 14 gives $\gamma(G)$ exactly, for $G = Z_2 \times Z_m$ (m even), S_4, and $Z_2 \times A_4$. The corollary below improves both bounds (given in Corollaries 11a and 12a) for $\gamma(S_n)$, for n odd.

Corollary 14a. *For n odd,*

$$\gamma(S_n) \leq 1 + \frac{(n-1)!}{6}(n-3), \qquad n \geq 3.$$

Proof. Let $S_n = \langle r, s\rangle$, where $r = (1\ 2)$ and $s = (1\ 2\ 3 \cdots n)$, so that $r^2 = s^n = (r^{-1}s^{-1}rs)^3 = e$. We first claim that H^* for $D_P(S_n)$ is bipartite. To see this, consider any cycle C in $D_P(S_n)$; the cycle corresponds to e, which is in A_n. Since n is odd, s is also in A_n. Hence r must appear an even number of times in C; thus all cycles in H^* are even, and H^* is bipartite. Thus Theorem 14 applies, with $m_1 = n$ and $k = 3$.

Asymptotic behavior. Let $\{G_n\}$ be a sequence of groups, with G_i a proper subgroup of G_{i+1}, $i = 1, 2, \cdots$; let j_n and k_n be the degree and average number of sides per region for a genus presentation of G_n respectively. Similarly, consider $\{G_n'\}$ with parameters j_n' and k_n', where G_i' is a proper subgroup of G_{i+1}' and a subgroup of G_i, $i = 1, 2, \cdots$. Let $a(n) \sim b(n)$ indicate that $\lim_{n\to\infty}(a(n)/b(n)) = 1$; i.e. $a(n)$ is asymptotic to $b(n)$. Our final theorem relates the ratio $\gamma(G_n)/\gamma(G_n')$ to the index of G_n' in G_n, under certain conditions.

Theorem 15. *If* (i) $k_n \sim k_n'$,

(ii) *either* $j_n \sim j_n'$, *or* $j_n \to \infty$ *and* $j_n' \to \infty$, *and*

(iii) $\gamma(G_n) \to \infty$ *and* $\gamma(G_n') \to \infty$, *then*

$$\gamma(G_n)/\gamma(G_n') \sim (j_n/j_n')[G_n : G_n'].$$

Proof. The euler formula $\gamma = 1 + \frac{1}{2}(q - p + r)$ applies for both G_n and G_n', with $2q = jp = kr$. Thus

$$\frac{\gamma(G_n)}{\gamma(G_n')} = \frac{1 + \frac{1}{2}\left(\frac{j_n p_n}{2} - p_n - \frac{j_n p_n}{k_n}\right)}{1 + \frac{1}{2}\left(\frac{j_n' p_n'}{2} - p_n' - \frac{j_n' p_n'}{k_n'}\right)} = \frac{1 + \frac{p_n}{2}\left(\frac{j_n}{2} - 1 - \frac{j_n}{k_n}\right)}{1 + \frac{p_n'}{2}\left(\frac{j_n'}{2} - 1 - \frac{j_n'}{k_n'}\right)}$$

$$\sim \frac{j_n}{j_n'}\frac{p_n}{p_n'} = \frac{j_n}{j_n'}[G_n : G_n'].$$

Corollary 15a. *If, in fact,* $j_n \sim j_n'$, *then*

$$\gamma(G_n)/\gamma(G_n') \sim [G_n : G_n'].$$

For example, let G_n and G_n' be two elementary abelian 2-groups, of orders 2^{2n} and 2^n respectively. By Theorem 6,

$$\frac{\gamma(G_n)}{\gamma(G_n')} = \frac{1 + 2^{2n-3}(2n-4)}{1 + 2^{n-3}(n-4)} \sim 2^{n+1} = \frac{j_n}{j_n'}[G_n : G_n'].$$

Also, if equality holds in Corollaries 11a and 11b (for n even), we have an example of Corollary 15a, with $G_n = S_{2n}$, $G_n' = A_{2n}$.

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DEPARTMENT OF MATHEMATICS, WESTERN MICHIGAN UNIVERSITY, KALAMAZOO, MICHIGAN 49001