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ON A VARIATION OF THE RAMSEY NUMBER

BY

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ABSTRACT. Let $c(m, n)$ be the least integer p such that, for any graph G of order p, either G has an m-cycle or its complement $\overline{G}$ has an n-cycle. Values of $c(m, n)$ are established for $m, n \leq 6$ and general formulas are proved for $c(3, n)$, $c(4, n)$, and $c(5, n)$.

Introduction. It is a well-known fact that in any gathering of six people, there are three people who are mutual acquaintances or three people who are mutual strangers. This statement has the graph-theoretic formulation that either a given graph of order 6 or its complement contains a triangle. It might further be mentioned that "6" is minimum with respect to this property.

The Ramsey number $r(m, n)$ may be considered a generalization of the above observation. For integers $m, n \geq 2$, the number $r(m, n)$ is defined as the smallest positive integer p such that given any graph G of order p, either G contains the complete subgraph K_m of order m or the complement $\overline{G}$ of G contains K_n. Hence, the aforementioned fact states that $r(3, 3) = 6$. One may easily note that $r(m, n) = r(n, m)$ and that $r(2, n) = n$ for all $n \geq 2$.

It is a result due to Ramsey [3] that the number $r(m, n)$ exists for all $m, n \geq 2$. Despite the fact that a great deal of research has been done on Ramsey numbers, only six values $r(m, n)$ have been determined for $m, n \geq 3$ (see [1]); namely, $r(m, n)$ is known (for $m, n \geq 3$) only when $(m, n) = (3, 3), (3, 4), (3, 5), (3, 6), (3, 7), (4, 4)$.

If we denote an n-cycle (a cycle of length n) by C_n, the original problem may be stated as: Given a graph G of order 6, either G or $\overline{G}$ contains a 3-cycle (triangle). This suggests a generalization different from that which leads to the Ramsey numbers. For $m, n \geq 3$, we define the number $c(m, n)$ to be the least positive integer p, such that for any graph G of order p, either G contains the m-cycle C_m or $\overline{G}$ contains C_n. Of course, we have $c(3, 3) = 6$. The number $c(m, n)$ always exists since $c(m, n) \leq r(m, n)$. It is the object of this paper to determine the value of $c(m, n)$ for several pairs (m, n); in particular, $c(3, n)$, $c(4, n)$, and $c(5, n)$ are determined for all $n \geq 3$. Before proceeding further, we present a few definitions and some additional notation. All terms not defined here may be found in [2].

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The complete bipartite graph $K(m, n)$, $m, n \geq 1$, is that graph G of order $m + n$, whose vertex set may be partitioned as $V_1 \cup V_2$ such that $|V_1| = m$, $|V_2| = n$ and $e = uv$ is an edge of G if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$. For connected graphs G_1 and G_2, we define $G_1 \cup G_2$ to be the disconnected graph having the two components G_1 and G_2. Note that if $G = K(m, n)$, then $\overline{G} = K_m \cup K_n$.

The numbers $c(3, n)$. We have already mentioned that $c(3, 3)$ is the well-known Ramsey number $r(3, 3) = 6$. We consider $c(3, 4)$ next.

Theorem 1. $c(3, 4) = 7$.

Proof. Let $H = K(3, 3)$ so that $\overline{H} = K_3 \cup K_3$. The graph H contains no 3-cycle and its complement $\overline{H}$ fails to contain a 4-cycle; thus, $c(3, 4) \geq 7$. To verify that $c(3, 4) = 7$, we let G be an arbitrary graph of order 7 and assume G contains no 3-cycle. We show that $\overline{G}$ contains a 4-cycle.

Since $c(3, 3) = 6$, either G or $\overline{G}$ has a 3-cycle; hence, $\overline{G}$ contains a 3-cycle, which we represent as C: u_1, u_2, u_3, u_1. (See Figure 1a, where the edges of $\overline{G}$ are represented by dashed lines.) Denote the remaining vertices by v_1, v_2, v_3, and v_4. If some v_i is joined in $\overline{G}$ to more than one vertex of C, then $\overline{G}$ contains a 4-cycle. We may assume, then, that each v_i is adjacent in G to at least two vertices of C. This implies that every two distinct v_i must be joined in G to a common vertex of C. (See Figure 1b, where the edges of G are represented by solid lines.) Because G contains no triangles, every two distinct v_i must be adjacent in $\overline{G}$ (see Figure 1c) which implies that $\overline{G}$ contains K_4 and hence C_4 as a subgraph.

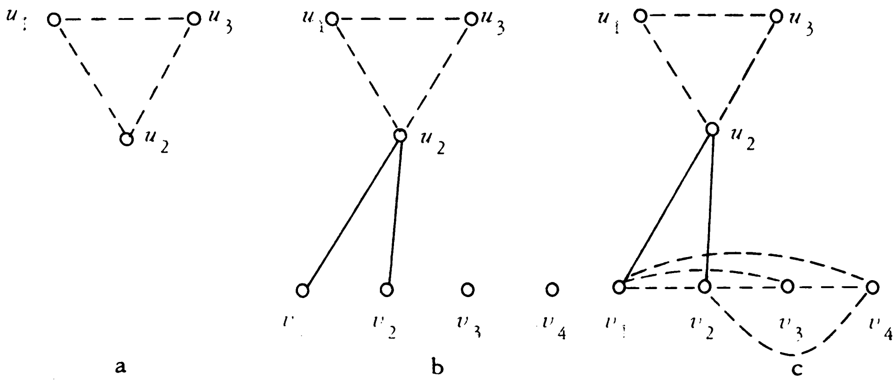

Figure 1

We now proceed to the general situation.

Theorem 2. *For* $n \geq 4$, $c(3, n) = 2n - 1$.

Proof. First, we note that if $H = K(n-1, n-1)$ so that $\overline{H} = K_{n-1} \cup K_{n-1}$, then H contains no 3-cycles and $\overline{H}$ contains no n-cycles. Thus, $c(3, n) \geq 2n - 1$. We prove that $c(3, n) = 2n - 1$, for all $n \geq 4$, by using induction on n.

By Theorem 1, $c(3, 4) = 7$. Assume that $c(3, n) = 2n - 1$ for some $n \geq 4$. It follows, therefore, that if F is any graph of order $2n - 1$, either F contains a 3-cycle or $\overline{F}$ contains an n-cycle. We now show that $c(3, n+1) = 2n + 1$. Let G be a graph of order $2n + 1$, and assume G has no 3-cycles. Because $c(3, n+1) \geq 2n + 1$, it suffices to prove that $\overline{G}$ contains an $(n+1)$-cycle. Since, by the induction hypothesis, $c(3, n) = 2n - 1$, the graph $\overline{G}$ contains an n-cycle, say $C: u_1, u_2, \cdots, u_n, u_1$. Designate the remaining vertices by $v_1, v_2, \cdots, v_n$, and v_{n+1}.

Suppose that some vertex u_i, $1 \leq i \leq n$, is adjacent in G to all vertices v_j, $j = 1, 2, \cdots, n+1$. Since G contains no 3-cycles, every two distinct vertices v_j are adjacent in $\overline{G}$. However, this implies that $\overline{G}$ contains K_{n+1} and therefore C_{n+1} as a subgraph. We henceforth assume that for $i = 1, 2, \cdots, n$, the vertex u_i is adjacent in $\overline{G}$ to some v_j.

We now consider two cases.

Case 1. *Suppose there exist two alternate vertices on C which are respectively joined in $\overline{G}$ to two distinct v_i.* Assume $u_1v_1, u_3v_3 \in E(\overline{G})$. If some vertex v_i is joined in $\overline{G}$ to two consecutive vertices of C, then $\overline{G}$ contains an $(n+1)$-cycle. Otherwise we have $u_2v_1, u_2v_3 \in E(G)$, which implies that $v_1v_3 \in E(\overline{G})$. However, then, $\overline{G}$ contains the $(n+1)$-cycle $u_1, v_1, v_3, u_3, u_4, \cdots, u_n, u_1$.

Case 2. *Suppose no two alternate vertices on C are joined in $\overline{G}$ to distinct vertices v_i.* This implies that u_1 and u_3 are joined in $\overline{G}$ to the same v_i, say v_1. Indeed, every u_i, with i odd, is joined in $\overline{G}$ to v_1. If n is odd, then v_1 is joined in G to both u_1 and u_n which produces an $(n+1)$-cycle in $\overline{G}$. Assume that n is even. It follows here that each u_i, with i even, is adjacent in $\overline{G}$ to the same $v_i \neq v_1$, say v_2.

Each v_j, $3 \leq j \leq n+1$, is necessarily joined in G to every vertex of C; otherwise, we revert back to Case 1. Since G contains no triangles, v_i and v_j, $3 \leq i < j \leq n+1$, are adjacent in $\overline{G}$. For the same reason and because $v_1u_2, v_2u_1 \in E(G)$, all edges v_1v_i and v_2v_i, $3 \leq i \leq n+1$, belong to $\overline{G}$. Now $v_1, v_3, v_2, v_4, v_5, \cdots, v_{n+1}, v_1$ is a desired $(n+1)$-cycle in $\overline{G}$.

The numbers $c(4, n)$. As $c(m, n) = c(n, m)$, it follows that $c(4, 3) = 7$ by Theorem 1. Thus we need only consider $c(4, n)$ for $n \geq 4$. Since the values of $c(4, 4)$ and $c(4, 5)$ do not follow the general formula which we present in this section, we must establish these numbers individually. We begin by doing this.

Theorem 3. $c(4, 4) = 6$.

Proof. Let $H = C_5$ so that $\overline{H} = C_5$. Since neither H nor $\overline{H}$ contains a 4-cycle, $c(4, 4) \geq 6$. Let G be a graph of order 6, and assume neither G nor $\overline{G}$ contains a 4-cycle. Because $c(3, 3) = 6$, either G or $\overline{G}$ contains a triangle. Without loss of generality, we assume G contains the 3-cycle $C: u_1, u_2, u_3, u_1$. Denote the other vertices of G by v_1, v_2, and v_3. No vertex v_i can be joined in G to more than one vertex of C, for otherwise G contains a 4-cycle. Hence each v_i is adjacent in $\overline{G}$ to at least two vertices of C. If there exist two v_i which are adjacent in $\overline{G}$ to the same two vertices of C, then $\overline{G}$ contains a 4-cycle. Hence, we suppose v_1 is adjacent in $\overline{G}$ *to* u_1 and u_2, v_2 is adjacent in $\overline{G}$ to u_2 and u_3, and v_3 is adjacent in $\overline{G}$ to u_1 and u_3; moreover, $v_1u_3, v_2u_1, v_3u_2 \in E(G)$. No two v_i are adjacent in G, for then G has a 4-cycle. This implies that $v_1v_2, v_1v_3, v_2v_3 \in E(\overline{G})$, but then v_1, u_2, v_2, v_3, v_1 is a 4-cycle in $\overline{G}$ which produces a contradiction.

Theorem 4. $c(4, 5) = 7$.

Proof. Let $H = K_3 \cup K_3$ so that $\overline{H} = K(3, 3)$. The graph H has no 4-cycle and $\overline{H}$ has no 5-cycle; thus, $c(4, 5) \geq 7$. Let G be a graph of order 7, and assume G has no 4-cycle. We prove that $\overline{G}$ contains a 5-cycle.

Since $c(4, 4) = 6$ by Theorem 3, the graph $\overline{G}$ contains a 4-cycle, say C: u_1, u_2, u_3, u_4, u_1. Let v_1, v_2, and v_3 denote the other vertices. If any of v_1, v_2, and v_3 is adjacent in $\overline{G}$ to two consecutive vertices of C, then $\overline{G}$ contains a 5-cycle. Suppose, then, that none of v_1, v_2, and v_3 is adjacent in $\overline{G}$ to consecutive vertices of C. Hence each v_i is joined in G to opposite vertices of C. Necessarily, there exist two v_i, say v_1 and v_2, which are adjacent in G to the same opposite vertices of C, say u_1 and u_3. The graph G therefore contains the 4-cycle u_1, v_1, u_3, v_2, u_1, which is contrary to hypothesis.

In order to determine a general formula for $c(4, n)$, we establish the number $c(4, 6)$.

Theorem 5. $c(4, 6) = 7$.

Proof. Let $H = K(1, 5)$; thus, $\overline{H} = K_1 \cup K_5$. Because H has no cycles (hence no 4-cycles) and $\overline{H}$ has no 6-cycles, $c(4, 6) \geq 7$. Let G be a graph of order 7, and assume G has no 4-cycles. We show that $\overline{G}$ contains a 6-cycle. Since $c(4, 5) = 7$ by Theorem 4 and since G has no 4-cycles, it follows that $\overline{G}$ has a 5-cycle C: $u_1, u_2, u_3, u_4, u_5, u_1$. Let the remaining vertices be denoted by v_1 and v_2.

If v_1 or v_2 is adjacent in $\overline{G}$ to two consecutive vertices of C, then $\overline{G}$ has a 6-cycle. Assume neither v_1 nor v_2 is adjacent in $\overline{G}$ to consecutive vertices of C so that each of v_1 and v_2 is joined in G to a set of three vertices of C (not all consecutive). If there exist two vertices of C joined in G to both v_1 and v_2, then G has a 4-cycle which produces a contradiction. However, there must exist one vertex of C joined in G to v_1 and v_2; hence we assume, without loss of generality, that v_1 and v_2 are joined in G to u_1, the edges $v_1u_2, v_1u_4, v_2u_3, v_2u_5 \in E(G)$, while $v_1u_3, v_1u_5, v_2u_2, v_2u_4 \in E(\overline{G})$. Now $\overline{G}$ contains the 6-cycle $v_1, u_3, u_2, v_2, u_4, u_5, v_1$.

We are now prepared to determine the remaining values of $c(4, n)$.

Theorem 6. *For* $n \geq 6$, $c(4, n) = n + 1$.

Proof. Let $n \geq 6$ and let $H = K(1, n-1)$ so that $\overline{H} = K_1 \cup K_{n-1}$. The graph H has no 4-cycles and its complement $\overline{H}$ has no n-cycles; therefore, $c(4, n) \geq n+1$. We proceed by induction on n (≥ 6). That $c(4, 6) = 7$ is the result of Theorem 5. Assume that, for some $n \geq 6$, $c(4, n) = n+1$; hence, for every graph F of order $n+1$, either F contains a 4-cycle or $\overline{F}$ contains an n-cycle. We consider the number $c(4, n+1)$. Since $c(4, n+1) \geq n+2$, it suffices to prove that if G is a graph of order $n+2$, either G has a 4-cycle or $\overline{G}$ has an $(n+1)$-cycle. Suppose G does not contain a 4-cycle. Since $c(4, n) = n+1$ by the induction hypothesis, it follows that $\overline{G}$ contains an n-cycle, say C: $u_1, u_2, \cdots, u_n, u_1$. Designate the other two vertices by v_1 and v_2.

If v_1 or v_2 is adjacent in $\overline{G}$ to consecutive vertices of C, then $\overline{G}$ contains an $(n+1)$-cycle, completing the proof. Assume, therefore, that neither v_1 nor v_2 is adjacent in $\overline{G}$ to consecutive vertices of C, which implies that each of v_1 and v_2 is adjacent in G to some set of $\{n/2\}$ vertices of C such that the set contains at least one of every two consecutive vertices of C. If v_1 and v_2 are adjacent in G to the same two (or more) vertices of C, then G contains a 4-cycle, which is contradictory. Thus we assume that v_1 and v_2 are mutually adjacent in G to one or no vertices of C. We consider these two cases.

Case 1. *Assume that v_1 and v_2 are mutually adjacent in G to no vertices of C.* In this case, it necessarily follows that each of v_1 and v_2 is joined in G to exactly $n/2$ vertices of C such that neither v_1 nor v_2 is adjacent in G to two consecutive vertices of C. Hence, n is even and, without loss of generality, we assume v_1 is joined in G to the vertices of $S_1 = \{u_i \mid i \text{ is odd}\}$ and v_2 is joined in G to the vertices of $S_2 = \{u_i \mid i \text{ is even}\}$. Therefore, v_1 is joined in $\overline{G}$ to the elements of S_2, and v_2 is adjacent in $\overline{G}$ to the elements of S_1. If all edges u_iu_j, with i and j odd, belong to G, then since $n \geq 6$, G contains the 4-cycle u_1, v_1, u_3, u_5, u_1, which is contrary to hypothesis. Therefore, $\overline{G}$ contains an edge u_iu_j with i and j odd such that $1 \leq i < j < n$, say. The graph $\overline{G}$ thus contains the $(n+1)$-cycle $u_i, u_j, u_{j-1}, \cdots, u_{i+1}, v_1, u_{j+1}, u_{j+2}, \cdots, u_{i-1}, u_i$.

Case 2. *Assume that v_1 and v_2 are mutually adjacent in G to exactly one vertex of C, say u_1.* Exactly one of v_1 and v_2 is adjacent in G to u_2, for if this were not the case, then v_1 and v_2 must be joined in G to u_3, which is contrary to our assumption. Without loss of generality, we suppose that $v_1u_2 \in E(G)$. Necessarily, $v_2u_3 \in E(G)$ or else v_2 is joined in $\overline{G}$ to the consecutive vertices u_2 and u_3 of C, which we have previously ruled out. By the same reasoning, $v_1u_4 \in E(G)$, $v_2u_5 \in E(G)$, etc. Hence, if we let S_1 and S_2 be defined as in Case 1, then v_1 is joined in G to the elements of $\{u_1\} \cup S_2$ and joined in $\overline{G}$ to the vertices of $S_1 - \{u_1\}$, while v_2 is joined in G to the vertices of S_1 and joined in $\overline{G}$ to the vertices of S_2.

If G contains all edges u_iu_j with i and j even, then since $n \geq 6$, G contains the 4-cycle v_1, u_2, u_4, u_6, v_1 which produces a contradiction. Therefore, $\overline{G}$ contains some edge u_iu_j, where i and j are even and $1 < i < j \leq n$, say. The graph G then has the $(n+1)$-cycle $u_j, u_i, u_{i+1}, \cdots, u_{j-1}, v_1, u_{i-1}, u_{i-2}, \cdots, u_{j+1}, u_j$, which yields the desired result.

The numbers $c(5, n)$. We have already established the value of $c(5, n)$ for $n = 3$ and $n = 4$. In order to present a formula for $c(5, n)$, $n \geq 5$, we shall first determine $c(5, 5)$.

Theorem 7. $c(5, 5) = 9$.

Proof. Let $H = K(4, 4)$ so that $\overline{H} = K_4 \cup K_4$. Neither H nor $\overline{H}$ contains a 5-cycle; thus, $c(5, 5) \geq 9$. Let G be a graph of order 9, and assume that neither G nor $\overline{G}$ has a 5-cycle.

Since $c(4, 4) = 6$, at least one of G and $\overline{G}$ contains a 4-cycle. Without loss of generality, we assume that G contains the 4-cycle C: u_1, u_2, u_3, u_4, u_1. Denote the other vertices of G by v_1, v_2, v_3, v_4, and v_5. No v_i is adjacent in G to two consecutive vertices of C since G contains no 5-cycle. Hence each v_i is joined in $\overline{G}$ to two opposite vertices of C.

Since G does not contain a 5-cycle, G does not contain K_5 as a subgraph; therefore, not every two distinct v_i are adjacent in G. Assume, say, that $v_1v_2 \in E(\overline{G})$. If there are two vertices u_i and u_j with $v_1u_i, v_2u_j \in E(\overline{G})$ such that there exists a vertex v_k, $3 \leq k \leq 5$, with $v_ku_i, v_ku_j \in E(\overline{G})$, then $\overline{G}$ contains a 5-cycle, which produces a contradiction. Hence, no such vertices u_i, u_j, and v_k exist in $\overline{G}$. We now consider two cases, assuming throughout that $v_1v_2 \in E(\overline{G})$.

Case 1. *Assume* v_1 *and* v_2 *are adjacent in* $\overline{G}$ *to the same pair of opposite vertices of* C, *say* u_1 *and* u_3. None of v_3, v_4, and v_5 is joined in $\overline{G}$ to both u_1 and u_3, for then we have conditions which produce a 5-cycle, as described above. Therefore, each of v_3, v_4, and v_5 is joined in $\overline{G}$ to u_2 and u_4. Now $v_3v_4 \in E(G)$, for otherwise $v_3, v_4, u_2, v_5, u_4, v_3$ is a 5-cycle in $\overline{G}$. Similarly, $v_3v_5, v_4v_5 \in E(G)$.

If v_1 is joined in $\overline{G}$ to one of v_3, v_4, and v_5, and v_2 is joined in $\overline{G}$ to some other vertex of v_3, v_4, and v_5, then $\overline{G}$ contains a 5-cycle. Since $\overline{G}$ has no 5-cycles, it follows that one of v_1 and v_2 is joined in G to all of v_3, v_4, and v_5, say $v_2v_3, v_2v_4, v_2v_5 \in E(G)$. If there are two edges of G from the vertices of C to two distinct vertices of v_3, v_4, and v_5, then G contains a 5-cycle. Because G cannot contain a 5-cycle, there must exist a vertex among v_3, v_4, and v_5, say v_5, which is joined in $\overline{G}$ to all vertices of C. Thus, $\overline{G}$ contains the 5-cycle $v_5, u_3, v_1, v_2, u_1, v_5$, which produces a contradiction.

Case 2. *Assume* v_1 *is adjacent in* $\overline{G}$ *to* u_1 *and* u_3, *and* v_2 *is adjacent in* $\overline{G}$ *to* u_2 *and* u_4. Suppose v_3 is adjacent in $\overline{G}$ to u_1 and u_3. Necessarily, v_3 is adjacent in G to u_2 and u_4, for otherwise we have conditions sufficient to produce a

5-cycle in $\overline{G}$, as mentioned earlier. For the same reason, $v_2u_1, v_2u_3 \in E(G)$. The vertex v_1 is joined in G to u_2 or u_4, for otherwise we return to Case 1. Thus, we assume $v_1u_2 \in E(G)$. Now $v_1v_3 \in E(\overline{G})$, or else $v_1, v_3, u_4, u_3, u_2, v_1$ is a 5-cycle in G. However, this places us in Case 1 again, where v_1 and v_3 are playing the roles of v_1 and v_2, respectively.

This completes the proof.

We conclude this section by presenting a formula for $c(5, n)$ for all $n \geq 5$.

Theorem 8. *For* $n \geq 5$, $c(5, n) = 2n - 1$.

Proof. Let $H = K(n-1, n-1)$ so that $\overline{H} = K_{n-1} \cup K_{n-1}$. The graph H contains no 5-cycle, and $\overline{H}$ has no n-cycle; therefore, $c(5, n) \geq 2n - 1$. We verify that $c(5, n) = 2n - 1$ by induction on n (≥ 5), the result following for $n = 5$ by Theorem 7.

Assume $c(5, n) = 2n - 1$ for some $n \geq 5$, and let G be a graph of order $2n + 1$. Since $c(5, n+1) \geq 2n + 1$, it suffices to show that G contains a 5-cycle or $\overline{G}$ contains an $(n+1)$-cycle. Assume that G has no 5-cycle. Since $c(5, n) = 2n - 1$, the graph $\overline{G}$ contains an n-cycle C: $u_1, u_2, \cdots, u_n, u_1$. Designate the remaining vertices by $v_1, v_2, \cdots, v_n$, and v_{n+1}.

If some v_i $(1 \leq i \leq n+1)$ is adjacent in $\overline{G}$ to two consecutive vertices of C, then $\overline{G}$ contains an $(n+1)$-cycle, completing the proof. Assume, therefore, that no v_i is adjacent in $\overline{G}$ to two consecutive vertices of C. This implies that each v_i is adjacent in G to some set of $\{n/2\}$ vertices of C, where at least one vertex in any pair of consecutive vertices of C belongs to the set. If every two distinct v_i are adjacent in $\overline{G}$, then $\overline{G}$ contains K_{n+1} and hence C_{n+1} as a subgraph. Suppose, then, that there are two distinct v_i, say v_1 and v_2, which are adjacent in G. We now consider three cases, assuming throughout that $v_1v_2 \in E(G)$.

Case 1. *Assume there is a vertex* v_k $(k \neq 1, 2)$ *such that* v_1 *and* v_k *are joined in* G *to a vertex* u_i *on* C, *and* v_2 *and* v_k *are joined in* G *to a vertex* u_j *on* C. If it is possible to select u_i and u_j such that $u_i \neq u_j$, then G contains the 5-cycle $v_k, u_i, v_1, v_2, u_j, v_k$ which is contradictory. Hence we may suppose that v_1 and v_k are joined in G to only one vertex u_i on C, and v_2 and v_k are joined on G to only one vertex on C, namely u_i. Since at least one vertex in every pair of consecutive vertices of C is joined in G to v_1 (respectively v_2), it follows that every vertex of C different from u_i is joined in G to exactly one of v_1 and v_2. The vertex v_k is adjacent in G to at least $\{n/2\}$ vertices of C; therefore, v_k must be adjacent in G to a vertex u_r which is joined in G to v_1, and, furthermore, v_k is adjacent in G to a vertex u_s (different from u_r) which is joined in G to v_2. Hence G contains a 5-cycle, which is contrary to hypothesis.

We note that if n is odd, Case 1 necessarily applies. We may henceforth assume n to be even.

Case 2. *Assume Case* 1 *does not hold and there exists some vertex* v_k $(k \neq 1, 2)$ *which is adjacent in* G *to no vertex of* C *which is joined in* G *to* v_1 *or* v_2.

This implies that whenever $v_1u_i \in E(G)$, $1 \leq i \leq n$, then $v_ku_i \in E(\overline{G})$, and whenever $v_2u_j \in E(G)$, $1 \leq j \leq n$, then $v_ku_j \in E(\overline{G})$. Since v_k is joined in G to at least $\{n/2\}$ vertices of C, and v_k is joined in $\overline{G}$ to at least $\{n/2\}$ vertices of C, it follows that v_k is adjacent in G to exactly $n/2$ vertices of C and is adjacent in $\overline{G}$ to exactly $n/2$ vertices of C. Therefore, we may assume here that v_1 and v_2 are joined in G to the vertices of $S_1 = \{u_i \mid i \text{ is odd}\}$ and joined in $\overline{G}$ to the vertices of $S_2 = \{u_i \mid i \text{ is even}\}$, while v_k is joined in G to the vertices of S_2 and joined in $\overline{G}$ to the elements of S_1.

If i and j are both even, then $u_iu_j \in E(\overline{G})$; for otherwise, we may select an even $t \neq i, j$ (since $n \geq 6$ here) to obtain the 5-cycle $u_i, u_j, v_2, u_t, v_1, u_i$ of G, which is a contradiction. Then $\overline{G}$ contains the $(n+1)$-cycle $v_k, u_{n-1}, u_{n-2}, \cdots, u_4, u_2, v_1, u_n, u_1, v_k$, which gives the desired result.

Case 3. *Assume that Case* 1 *and Case* 2 *do not hold.* Hence each v_k, $k \geq 3$, has the properties that whenever $v_1u_i, v_ku_i \in E(G)$, $1 \leq i \leq n$, then $v_2u_i \in E(\overline{G})$, and whenever $v_2u_j, v_ku_j \in E(G)$, $1 \leq j \leq n$, then $v_1u_j \in E(\overline{G})$. Let S_1 and S_2 be defined as in Case 2. We may assume in this case that v_1 and v_3, say, are joined in G to the elements of S_1 and joined in $\overline{G}$ to the elements of S_2, while v_2 is joined in G to the vertices of S_2 and joined in $\overline{G}$ to the vertices of S_1.

If $v_1v_3 \in E(G)$, then we have the conditions specified in Case 2, where v_1 and v_3 play the roles of v_1 and v_2, respectively. Hence, $v_1v_3 \in E(\overline{G})$, and $\overline{G}$ contains the $(n+1)$-cycle $v_1, v_3, u_2, u_3, \cdots, u_n, v_1$.

The number $c(6, 6)$. We next determine the value of $c(6, 6)$.

Theorem 9. $c(6, 6) = 8$.

Proof. Let $H = K(2, 5)$ so that $\overline{H} = K_2 \cup K_5$. Since neither H nor $\overline{H}$ has a 6-cycle, $c(6, 6) \geq 8$. Let G be a graph of order 8, and suppose neither G nor $\overline{G}$ contains a 6-cycle. We distinguish two cases.

Case 1. *Assume neither G nor $\overline{G}$ has a* 5-*cycle*. Since $c(4, 4) = 6$ by Theorem 3, at least one of G and $\overline{G}$ has a 4-cycle; say G contains the 4-cycle C: u_1, u_2, u_3, u_4, u_1. Denote the remaining vertices of G by v_1, v_2, v_3, and v_4. Since G has no 5-cycle, no v_i is joined in G to two consecutive vertices of C. This implies that every v_i is joined in $\overline{G}$ to some pair of opposite vertices of C. We consider two subcases.

Subcase 1a. *Suppose three or more v_i are joined in $\overline{G}$ to the same pair of opposite vertices of C; say v_1, v_2, and v_3 are joined in $\overline{G}$ to u_1 and u_3.* Every two distinct vertices in $\{v_1, v_2, v_3\}$ are adjacent in G, for otherwise $\overline{G}$ contains a 5-cycle. Also, the vertex v_4 cannot be joined in $\overline{G}$ to two other v_i; otherwise, a 6-cycle exists in $\overline{G}$. Thus, we assume v_2v_4 and v_3v_4 are edges of G. Not both u_2v_2 and u_4v_3 are edges of G, for then G contains a 6-cycle. Without loss of generality, we may assume u_2v_2 is an edge of $\overline{G}$. If u_2v_1 is an edge of $\overline{G}$, then $\overline{G}$

contains a 6-cycle. Furthermore, if $u_2v_3 \in E(\overline{G})$, then $\overline{G}$ contains a 6-cycle. Therefore, $u_2v_1, u_2v_3 \in E(G)$ and G contains the 5-cycle $u_2, v_1, v_2, v_4, v_3, u_2$. This produces a contradiction.

Subcase 1b. *Suppose exactly two* v_i *are joined in* $\overline{G}$ *to the same pair of opposite vertices of* C; *say* v_1 *and* v_2 *are joined in* $\overline{G}$ *to* u_2 *and* u_4 *while* v_3 *and* v_4 *are joined in* $\overline{G}$ *to* u_1 *and* u_3. Assume further that v_1v_3 is an edge of $\overline{G}$. The edge v_3u_2 belongs to G, for otherwise $v_3, u_2, v_2, u_4, v_1, v_3$ is a 5-cycle in $\overline{G}$. Similarly, 5-cycles result in $\overline{G}$ unless v_4u_2 and v_4u_4 are edges of G. Next, $v_4v_3 \in E(\overline{G})$, or else $v_4, v_3, u_2, u_3, u_4, v_4$ is a 5-cycle in G. In a like manner, it follows that $v_2u_1 \in E(G)$ and $v_2u_3 \in E(\overline{G})$. However, $v_2, u_3, v_4, v_3, v_1, u_4, v_2$ is a 6-cycle of $\overline{G}$ which is contradictory. Hence, we must have $v_1v_3 \in E(G)$. By symmetry, we may also conclude that $v_2v_3, v_1v_4, v_2v_4 \in E(G)$.

We observe that not both u_1v_2 and u_3v_1 are edges of G, for otherwise $u_1, v_2, v_4, v_1, u_3, u_4, u_1$ is a 6-cycle of G. However, not both u_1v_2 and u_3v_1 are edges of $\overline{G}$ either, since then $u_1, v_2, u_4, v_1, u_3, v_4, u_1$ is a 6-cycle of $\overline{G}$. Thus, we may assume that $u_1v_2 \in E(G)$ and $u_3v_1 \in E(\overline{G})$. If the edge u_4v_4 is in G, then G contains the contradictory 6-cycle $u_4, v_4, v_1, v_3, v_2, u_1, u_4$. On the other hand, if u_4v_4 is in $\overline{G}$, then $\overline{G}$ contains the 6-cycle $u_4, v_4, u_3, v_1, u_2, v_2, u_4$. We, therefore, have a contradiction in this subcase, also.

Case 2. *Assume that at least one of* G *and* $\overline{G}$ *contains a* 5-*cycle*. Without loss of generality, we assume that G has the 5-cycle $C: u_1, u_2, u_3, u_4, u_5, u_1$ with the remaining vertices denoted by v_1, v_2, and v_3. Since G has no 6-cycles, no v_i $(1 \leq i \leq 3)$ is adjacent in G to two consecutive vertices of C. Thus, each v_i is joined in $\overline{G}$ to three nonconsecutive vertices of C.

We now make use of the following fact: If S_1, S_2, S_3 are 3-element subsets of a 5-element set, then there exist i, j $(i \neq j)$ such that $|S_i \cap S_j| \geq 2$. Hence, if S_i $(i = 1, 2, 3)$ denotes the set of three nonconsecutive vertices of C which are joined in $\overline{G}$ to v_i, then there exist two vertices v_i, say v_1 and v_2, which are mutually adjacent in $\overline{G}$ to at least two vertices of C. This suggests a breakdown into two subcases.

Subcase 2a. *Assume* v_1 *and* v_2 *are joined in* $\overline{G}$ *to the same three vertices of* C; *say* v_1 *and* v_2 *are adjacent in* $\overline{G}$ *to* u_1, u_3, *and* u_4. If v_3 is joined in $\overline{G}$ to any two of the vertices u_1, u_3, and u_4, then it follows directly that $\overline{G}$ contains a 6-cycle, which is contrary to hypothesis. Thus, we may assume that v_3 is joined in $\overline{G}$ to exactly one of u_1, u_3, and u_4. If $v_3u_1 \in E(\overline{G})$, then we must have at least one of the edges v_3u_3 and v_3u_4 in $\overline{G}$ also; therefore, without loss of generality, we assume that $v_3u_4 \in E(\overline{G})$. This further implies that $v_3u_2, v_3u_5 \in E(\overline{G})$ and $v_3u_1, v_3u_3 \in E(G)$. The edge v_2u_5 belongs to G, for otherwise $v_2, u_5, v_3, u_4, v_1, u_3, v_2$ is a 6-cycle of $\overline{G}$. In a like manner, it follows that $v_2u_2, v_1u_5, v_1u_2 \in E(G)$. However, then, $v_2, u_2, u_3, v_3, u_1, u_5, v_2$ is a 6-cycle of G, which is impossible.

Subcase 2b. *No two* v_i *are joined in* $\overline{G}$ *to the same three vertices of* C, *but* v_1 *and* v_2 *are joined in* $\overline{G}$ *to two common vertices of* C. Assume v_1 is adjacent in $\overline{G}$ with each of the vertices u_1, u_3, and u_4; thus, v_2 is adjacent in $\overline{G}$ with exactly two of the three vertices u_1, u_3, and u_4. The vertex v_2 cannot be joined in $\overline{G}$ to u_3 and u_4, for then v_2 must be joined in $\overline{G}$ to u_1 as well. Hence, without loss of generality, we assume v_2 is joined in $\overline{G}$ to u_1 and u_4. Necessarily, then, $v_2u_2 \in E(\overline{G})$. We now consider the location of the edges v_3u_1 and v_3u_4, observing that not both v_3u_1 and v_3u_4 are in $\overline{G}$ (for this is the situation discussed in Subcase 2a).

(i) If v_3 is joined in $\overline{G}$ to neither u_1 nor u_4, then, of course, v_3 is adjacent in $\overline{G}$ to u_2, u_3, and u_5. However, $\overline{G}$ contains the 6-cycle $v_1, u_3, v_3, u_2, v_2, u_4, v_1$, which is impossible.

(ii) Suppose v_3 is joined in $\overline{G}$ to only one of u_1 and u_4. Without loss of generality, we assume that $v_3u_1 \in E(\overline{G})$. Unless $v_3u_3, v_3u_5 \in E(\overline{G})$, we are returned to previously treated cases. However, $v_1, u_3, v_3, u_1, v_2, u_4, v_1$ is now a 6-cycle of $\overline{G}$ which is a contradiction.

We summarize the values established for $c(m, n)$ in the following table.

$m \backslash n$	3	4	5	6	7	8	
3	6	7	9	11	13	15	. . .
4	7	6	7	7	8	9	. . .
5	9	7	9	11	13	15	. . .
6	11	7	11	8			
7	13	8	13				
8	15	9	15				
	.	.	.				
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REFERENCES

1. J. E. Graver and J. Yackel, *Some graph theoretic results associated with Ramsey's theorem*, J. Combinatorial Theory 4 (1968), 125–175. MR 37 #1278.

2. F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass., 1969. MR 41 #1566.

3. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (1930), 264–286.

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