

SIMPLE GROUPS OF ORDER $2^a 3^b 5^c 7^d p$

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ABSTRACT. Let $\text{PSL}(n, q)$ denote the projective special linear group of degree n over $\text{GF}(q)$, the field with q elements. The following theorem is proved. **Theorem.** Let G be a simple group of order $2^a 3^b 5^c 7^d p$, $a > 0$, p an odd prime. If the index of a Sylow p -subgroup of G in its normalizer is two, then G is isomorphic to one of the groups, $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $\text{PSL}(2, 9)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 16)$, $\text{PSL}(2, 25)$, $\text{PSL}(2, 27)$, $\text{PSL}(2, 81)$, and $\text{PSL}(3, 4)$.

1. Introduction.

1.1. This work was motivated by the work of R. Brauer, D. Mutchler and the author. Brauer [4] classified the simple groups of order $5 \cdot 3^a 2^b$. Mutchler [15] and the author [1] classified simple groups of orders $2^a 3^b 5^c p$ and $2^a 3^b 7^c p$ respectively, in which the index of Sylow p -subgroup in its normalizer is 2. In this paper these results are extended to groups of order $2^a 3^b 5^c 7^d p$.

In §2 we discuss the degrees of the irreducible characters in the principal p -block, $B_0(p)$, of G . In particular, we list the possibilities for the equation relating these character degrees.

In §3 we consider each of the possible character degree equations for $B_0(p)$ listed in §2. Using class algebra coefficients and simple group classification theorems, we show that G is isomorphic to one of the groups, $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $\text{PSL}(2, 9)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 16)$, $\text{PSL}(2, 25)$, $\text{PSL}(2, 27)$, $\text{PSL}(2, 81)$, and $\text{PSL}(3, 4)$.

1.2. *Notation.* In general, upper case letters denote groups, and S_p is used to denote a Sylow p -subgroup. If A is a subgroup of a group G , then $N(A)$, $C(A)$, $[G: A]$, $|A|$ denote the normalizer of A in G , the centralizer of A in G , the index of A in G , and the order of A , respectively.

The notation x_n is used for a group element of order n . Then $C(x_n)$ refers to the centralizer of the element x_n in G .

A character of degree m is denoted by χ_m .

The notation $a(x_n, x_m, x_q)$ denotes the class algebra coefficient which is the

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number of ways each element of the conjugacy class of x_q can be written as a product of an element of the class of x_n by an element of the class of x_m .

2. The degree equation for $B_0(p)$.

2.1. Let G be a simple group satisfying the hypotheses:

I. $|G| = 2^a 3^b 5^c 7^d p$, where $a > 0$ and p is an odd prime.

II. $[N(S_p): S_p] = 2$.

Let x_p be an element of order p , and x_q be a p -regular element. Hypothesis II and Brauer's work [5] yield the following information concerning $B_0(p)$.

$B_0(p)$ contains the identity character, 1, a character χ and $(p-1)/2$ characters $\chi^{(m)}$, $m = 1, 2, \dots, (p-1)/2$. There are signs δ and δ' so that $\chi(x_p) = \delta$, $\sum \chi^{(m)}(x_p) = \delta'$, $m = 1, 2, \dots, (p-1)/2$, $\chi(1) \equiv \delta \pmod{p}$, $\chi^{(m)}(1) \equiv -2\delta' \pmod{p}$, $m = 1, 2, \dots, (p-1)/2$,

$$(2.1) \quad 1 + \delta\chi(x_q) + \delta'\chi^{(m)}(x_q) = 0.$$

If $x_q = 1$ in (2.1) we obtain the following relation between the degrees of the irreducible characters in $B_0(p)$,

$$(2.2) \quad 1 + \delta\chi(1) + \delta'\chi^{(m)}(1) = 0.$$

We call (2.2) the degree equation for $B_0(p)$.

2.2. *Solutions to the degree equation.* Since $\chi(1)$, $\chi^{(m)}(1)$, divide $|G|$ and are relatively prime to p , hypothesis I implies that (2.2) can be written in the form

$$(2.3) \quad 1 + x = y,$$

where x and y are of the form $2^r 3^s 5^t 7^u$. Lehmer [14] and the author [2] have shown independently that the solutions to (2.3) are $(x, y) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (9, 10), (14, 15), (15, 16), (20, 21), (24, 25), (27, 28), (35, 36), (48, 49), (49, 50), (63, 64), (80, 81), (125, 126), (224, 225), (2400, 2401),$ and $(4374, 4375)$.

3. Proof of main theorem.

3.1. *Preliminary results.* Before considering the possible degree equations we list some lemmas which are extremely useful in using the degree equation for $B_0(p)$ to determine the order of the group and the structures of various subgroups of G . The first lemma follows from the work [9] of Brauer and Tuan.

Lemma 3.1. *Let G be a simple group of order $p \cdot q^b \cdot r$, where p and q are primes, $(pq, r) = 1$. Suppose the degree equation for $B_0(p)$ is $\sum \delta_i \chi_i(1) = 0$, and G has no elements of order pq . Then for any q -block, $B(q)$,*

$$\sum \delta_i \chi_i(1) \equiv 0 \pmod{q^b},$$

where the summation is taken over all characters in $B_0(p) \cap B(q)$.

The following lemma follows from the work [9] of Brauer and Tuan and Lemma 3.1.

Lemma 3.2. *Let G be a group satisfying hypotheses I and II with degree equation $q^r = w \pm 1$, where q is 2, 3, 5 or 7. Then q^r is the full power of q dividing $|G|$ and a character of degree q^r vanishes on all q -singular elements.*

The following lemmas follow immediately from Lemmas 3.4 and 3.5 of [1].

Lemma 3.3. *Let G be a simple group satisfying hypotheses I and II. Let x_r, x_s be p -regular elements of G , and x_p be an element of order p . If (2.3) is the degree equation for $B_0(p)$, then the class algebra coefficient $a(x_r, x_s, x_p)$ satisfies*

$$(3.1) \quad a(x_r, x_s, x_p) = \frac{|G|[x - \chi_x(x_r)][x - \chi_x(x_s)]}{|C(x_r)||C(x_s)|x(x+1)},$$

where χ_x is a character in $B_0(p)$ of degree x .

Lemma 3.4. *If G is a simple group satisfying hypotheses I and II then G has one class of involutions and, if x_2 is an involution in G and x_p is an element of order p in G , then*

$$(3.2) \quad a(x_2, x_2, x_p) = p.$$

Finally we list the main results of [1] and [15].

Lemma 3.5. *Let G be a simple group of order $2^a 3^b 7^c p$, $a, b > 0$, p a prime distinct from 2, 3 and 7. If G satisfies hypothesis II then G is isomorphic to one of the groups, $\text{PSL}(2, 5)$, $\text{PSL}(2, 9)$, $\text{PSL}(2, 27)$ and $\text{PSL}(3, 4)$.*

Lemma 3.6. *Let G be a simple group of order $2^a 3^b 5^c p$, $a, b > 0$, p a prime distinct from 2, 3 and 5. If G satisfies hypothesis II then G is isomorphic to one of the groups, $\text{PSL}(2, 8)$, $\text{PSL}(2, 16)$, $\text{PSL}(2, 25)$, and $\text{PSL}(2, 81)$.*

3.2. Proof of main theorem. Next the possible degree equations are considered and the simple groups, G , which satisfy hypotheses I and II are determined.

First of all, the equation $1 + 1 = 2$ is impossible since G is a simple group, and the equation $1 + 2 = 3$ is inconsistent with the relations above (2.1).

Lemma 3.7. *If G is a simple group satisfying I and II, then*

(1) *If G has degree equation $1 + 3 = 4$ or $1 + 4 = 5$, then G is isomorphic to $\text{PSL}(2, 5)$.*

(2) *G cannot have degree equation $1 + 5 = 6$ or $1 + 6 = 7$.*

(3) If G has degree equation $1 + 7 = 8$, then G is isomorphic to $\text{PSL}(2, 7)$.

(4) If G has degree equation $1 + 8 = 9$, then G is isomorphic to $\text{PSL}(2, 8)$ or $\text{PSL}(2, 9)$.

(5) G cannot have degree equation $1 + 9 = 10$ or $1 + 14 = 15$.

(6) If G has degree equation $1 + 15 = 16$, then G is isomorphic to $\text{PSL}(2, 16)$.

Proof. Statement (1) follows from the works of Blichfeldt [3] and Brauer [6]. Statement (2) is an immediate consequence of Brauer [6] and Wales [17]. Statement (3) follows from the work [17] of Wales.

When the degree equation is $1 + 8 = 9$, Lemma 3.2 and the relations above (2.1) imply that $|G| = 2^3 3^2 5^c 7^d p$, where $p = 5$ or 7 . Then since a Sylow 2-subgroup of G is of order 8, statement (4) follows from the works of Brauer-Suzuki [8], Gorenstein [12], and Gorenstein-Walter [13].

When the degree equation is $1 + 9 = 10$, the relations above (2.1) imply that $p = 11$, and then the work [11] of Feit implies that G is isomorphic to $\text{PSL}(2, 11)$. But $\text{PSL}(2, 11)$ has no character of degree 9, and so we reach a contradiction.

When the degree equation is $1 + 14 = 15$, the relations above (2.1) and the class algebra coefficient $a(x_7, x_7, x_{13})$, $x_7 \in C(S_7)$, yield $|G| = 2^a 3^b 5^c 7 \cdot 13$. Then Lemma 3.1 applied to $B_0(7) \cap B_0(13)$ implies that the identity character and the 6 characters of degree 15 are in $B_0(7)$. This is inconsistent with the relations above (2.1) with $p = 7$. This proves statement (5).

When the degree equation is $1 + 15 = 16$, Lemma 3.2 and the relations above (2.1) yield $|G| = 2^4 3^b 5^c 7^d p$, where $p = 7$ or 17 . When $p = 7$, $d = 0$ and then Lemma 3.6 leads to a contradiction. When $p = 17$, the work [11] of Feit implies that G is isomorphic to $\text{PSL}(2, 16)$. This proves statement (6).

It is an easy matter to verify that the groups $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 9)$ and $\text{PSL}(2, 16)$ satisfy hypotheses I and II.

We next consider the degree equation $1 + 20 = 21$.

Lemma 3.8. *There is no simple group G satisfying hypotheses I and II with degree equation $1 + 20 = 21$.*

Proof. Let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, $x_5 \in C(S_5)$, and $x_7 \in C(S_7)$. Then the relations above (2.1) and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5, 7$, yield $|G| = 2^a 3^b 5^c 7 \cdot p$ where $a \leq 8$ and even, $b \leq 5$ and odd, $c \leq 3$ and odd, and $p = 11$ or 19 . When $a = 8$, the coefficient $a(x_2, x_2, x_p)$ yields $\chi_{20}(x_2) = -12$. Thus G has no elementary Abelian subgroup of order 8 and then [10, Theorem 4.2.7] leads to a contradiction.

When $p = 11$, a count of Sylow 11-subgroups yields $|G| = 2^2 3 \cdot 5 \cdot 7 \cdot 11$, $2^2 3^5 5^3 7 \cdot 11$, $2^4 3 \cdot 5^3 7 \cdot 11$, or $2^6 3^3 5 \cdot 7 \cdot 11$. In the first two cases [13] leads to a contradiction. When $|G| = 2^4 3 \cdot 5^3 7 \cdot 11$, Lemma 3.1 applied to

$B_0(11) \cap B_0(3)$ yields a contradiction. If $|G| = 2^6 3^3 5 \cdot 7 \cdot 11$, Lemma 3.1 applied to $B_0(11) \cap B_0(7)$, the relations above (2.1) with $p = 7$ and the coefficient $a(x_2, x_2, x_{11})$ yield

$$[N(S_7): C(S_7)] = 6 \text{ and } |C(x_2)| = 2^6 3.$$

But then a count of Sylow 7-subgroups leads to a contradiction.

When $p = 19$, a count of Sylow 19-subgroups yields $|G| = 2^2 3 \cdot 5 \cdot 7 \cdot 19$, $2^4 3^5 5 \cdot 7 \cdot 19$ or $2^6 3 \cdot 5^3 7 \cdot 19$. In the first two cases a contradiction is obtained by the application of Lemma 3.1 to $B_0(5) \cap B_0(19)$. When $|G| = 2^6 3 \cdot 5^3 7 \cdot 19$, the coefficients $a(x_2, x_2, x_{19})$ and $a(x_5, x_5, x_{19})$ yield

$$(3.3) \quad |C(x_2)| = 2^6 5, |C(x_5)|/5^3 2^2 \text{ and } \chi_{20}(x_5) = -5.$$

Clearly $\exists y_5 \in S_5$ so that $\chi_{20}(y_5) \geq 0$. Since y_5 cannot be in $C(S_5)$, the class algebra coefficient $a(y_5, y_5, x_{19})$ yields

$$(3.4) \quad |C(y_5)|/2^4 5^2 \text{ and } \chi_{20}(y_5) = 0.$$

Furthermore, every noncentral element y_5 in S_5 satisfies (3.4). Now since $\chi_{20}(x_5) = -5$, equation (2.1) implies that $\chi_{21}(x_5) = -4$. Then $\sum \chi^2(x_5) \geq 170$ where the summation is taken over all characters, χ , of G . Thus $|C(x_5)| \geq 170$ and then (3.3) implies that $2/|C(x_5)|$. Then Lemma 3.4 and equations (3.3) and (3.4) imply that $|C(y_5)| = 5^2$. Now it is clear that the Sylow 5-subgroups of $C(x_5)$ and $C(y_5)$ are normal which in turn implies that the set, S , of Sylow 5-subgroups of G is a trivial intersection, T.I., set. Next we represent a single S_7 subgroup, A , as a permutation group on the set S . Since the S_5 's are a T.I. set, the subgroup fixing any other S_5 , B , is the identity. The following congruence is immediate, $r \equiv 1 \pmod{125}$, where $r = |S|$. This congruence is incompatible with $|G|$. Thus $|G| \neq 2^6 3 \cdot 5^3 7 \cdot 19$ and Lemma 3.8 is proved.

Lemma 3.9. *Let G be a simple group satisfying hypotheses I and II, then*

- (1) *If G has degree equation $1 + 24 = 25$, then G is isomorphic to $\text{PSL}(2, 25)$.*
- (2) *If G has degree equation $1 + 27 = 28$, then G is isomorphic to $\text{PSL}(2, 27)$.*
- (3) *G cannot have degree equation $1 + 35 = 36$.*
- (4) *G cannot have degree equation $1 + 48 = 49$.*

Proof. Let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, and $x_7 \in C(S_7)$. If the degree equation is $1 + 24 = 25$, Lemma 3.1, the relations above (2.1), and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 7$, yield $|G| = 2^a 3^b 5^2 7^c p$ where $a \leq 7$ and odd, $b \leq 5$ and odd, $d \leq 2$ and even and $p = 13$ or 23 . When $a = 7$, $\chi_{24}(x_2) = -8$ and then [10, Theorem 4.2.7] leads to a contradiction. When $c = 0$, Lemma 3.5 implies that G is isomorphic to $\text{PSL}(2, 25)$. Otherwise, a count of Sylow p -subgroups

yields $|G| = 2^5 3 \cdot 5^2 7^2 \cdot 13$. Then the coefficient $a(x_7, x_7, x_{13})$ yields

$$(3.5) \quad |C(x_7)|/2 \cdot 7^2.$$

Thus the S_7 subgroup of $C(x_7)$ is normal and then the set of S_7 's is a T.I. set. Then as in the proof of Lemma 3.8 we see that the number of S_7 's is congruent to 1 modulo 49. This is incompatible with (3.5) and $|G|$. This proves (1).

When the degree equation is $1 + 27 = 28$, Lemma 3.1, the relations above (2.1) and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 5, 7$, yield $|G| = 2^a 3^3 5^c 7^d p$ where $a \leq 8$ and even, $c \leq 4$ and even, $d \leq 3$ and odd and $p = 13$ or 29. If $d = 3$, $\chi_{27}(x_7) = -22$ whence $(\chi_{27} | \langle x_7 \rangle, 1 \langle x_7 \rangle) < 0$, a contradiction. When $c = 0$, Lemma 3.6 implies that G is isomorphic to $\text{PSL}(2, 27)$. Otherwise, a count of Sylow p -subgroups yields $|G| = 2^2 3^3 5^4 7 \cdot 13$. Then the work [13] leads to a contradiction. This proves (2).

Next, if the degree equation is $1 + 35 = 36$, the relations above (2.1) and the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5, 7$, yield $|G| = 2^a 3^b 5^c 7^d p$ where $a \leq 10$ and even, $b \leq 4$ and even, $c \leq 3$ and odd, $d \leq 3$ and odd and $p = 37$ or 17. Now if $d = 3$, $\chi_{35}(x_7) = -14$ so that

$$(\chi_{35} | \langle x_7 \rangle, 1 \langle x_7 \rangle) < 0,$$

a contradiction. Thus $d = 1$. When $p = 37$, $B_0(37)$ contains a character of degree less than 37 and then the work [11] of Feit leads to a contradiction. Next, when $p = 17$, Lemma 3.1 applied to $B_0(7) \cap B_0(17)$ yields a contradiction. Thus (3) is proved.

Finally, if the degree equation is $1 + 48 = 49$, the relations above (2.1), Lemma 3.1, and the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5$, yield $|G| = 2^a 3^b 5^c 7^2 p$, where $a \leq 8$ and even, $b \leq 7$ and odd, $c \leq 4$ and even, and $p = 5$ or 47. If $p = 5$ or $c = 0$, Lemma 3.5 yields a contradiction. Otherwise, a count of Sylow 47-subgroups yields a contradiction. This proves (4).

It is an easy matter to verify that $\text{PSL}(2, 25)$ and $\text{PSL}(2, 27)$ satisfy hypotheses I and II.

Lemma 3.10. *There is no simple group, G , satisfying hypotheses I and II with degree equation $1 + 49 = 50$.*

Proof. As usual, let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, $x_5 \in C(S_5)$ and $x_7 \in C(S_7)$. The relations above (2.1), Lemma 3.2 and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5$, yield $|G| = 2^a 3^b 5^2 7^2 p$, where $a \leq 11$ and odd, $b \leq 8$ and even, and $p = 3$ or 17. When $a = 11$, $\chi_{49}(x_2) = -15$, and so G has no elementary Abelian subgroup of order 8. Then, [10, Theorem 4.2.7] leads to a contradiction. When $b = 8$, $\chi_{49}(x_3) = -32$ so that $(\chi_{49} | \langle x_3 \rangle, 1 \langle x_3 \rangle) < 0$, a contradiction.

If $p = 17$, a count of S_{17} subgroups yields $|G| = 2^3 3^4 5^2 7^2 17$ or $2^9 5^2 7^2 17$. In

the former case, the coefficient $a(x_2, x_2, x_{17})$ implies that

$$(3.6) \quad \sum \chi^2(x_2) > |C(x_2)|,$$

where the summation is taken over all irreducible characters, χ , of G . This is a contradiction. In the latter case, the coefficient $a(x_2, x_2, x_{17})$ yields $|C(x_2)| = 2^9$, and then the work [15] of Suzuki leads to a contradiction.

If $p = 3$, a count of S_3 subgroups yields

$$|G| = 2^9 3 \cdot 5^2 7^2, \quad 2^7 3 \cdot 5^2 7^2, \quad 2^5 3 \cdot 5^2 7^2, \quad \text{or} \quad 2^3 3 \cdot 5^2 7^2.$$

In the first two cases, the coefficient $a(x_2, x_2, x_3)$ and [10, Theorem 4.2.7] imply that (3.6) is satisfied, a contradiction. When $|G| = 2^5 3 \cdot 5^2 7^2$, the coefficient $a(x_2, x_2, x_3)$ implies that (3.6) is satisfied unless $\chi_{49}(x_2) = -7$. Then equation (2.1), Lemma 3.4 and the fact that χ_{49}, χ_{50} are of defect zero for the primes 7 and 5 respectively yields $a(x_2, x_3, x_5) = 129$. This is absurd since this coefficient must be a multiple of 5. Finally when $|G| = 2^3 3 5^2 7^2$, it follows from the coefficient $a(x_2, x_2, x_3)$ that $C(x_2)$ is solvable and then the works [8], [12], and [13] lead to a contradiction. This proves Lemma 3.10.

Lemma 3.11. *If G is a simple group satisfying hypotheses I and II with degree equation $1 + 63 = 64$, then G is isomorphic to $\text{PSL}(3, 4)$.*

Proof. Here if x_3, x_5 , and x_7 are as usual, Lemma 3.2, the relations above (2.1) and the coefficients $a(x_i, x_i, x_p)$, $i = 3, 5, 7$, yield $|G| = 2^6 3^b 5^c 7^d p$, where $b \leq 6$ and even, $c \leq 6$ and even, $d \leq 3$ and odd, and $p = 5, 31$, or 13 . If $p = 5$, $c = 0$ and then Lemma 3.5 implies that G is isomorphic to $\text{PSL}(3, 4)$. When $c = 6$, $\chi_{63}(x_5) = -62$, and then $(\chi_{63}|_{\langle x_5 \rangle}, 1|_{\langle x_5 \rangle}) < 0$, a contradiction.

When $p = 31$, a count of S_{31} subgroups yields

$$|G| = 2^6 3^6 7^3 31, \quad 2^6 3^2 7^3 31, \quad 2^6 3^2 5^4 7^3 31, \quad \text{or} \quad 2^6 3^6 5^4 7^3 31.$$

In the first two cases, Lemma 3.5 leads to a contradiction. In the third case, Lemma 3.1 applied to $B_0(7) \cap B_0(31)$ implies that the 15 characters of degree 64 are in $B_0(7)$ which is impossible. Finally, if $|G| = 2^6 3^6 5^4 7^3 31$, the coefficient $a(x_7, x_7, x_{31})$ yields $|C(x_7)|/3^2 5^2 7^3$, and $\chi_{63}(x_7) = 14$. Furthermore,

$$|C(x_7)| = \sum \chi^2(x_7) \geq 3572.$$

Thus $|C(x_7)| = 5^2 7^3, 3 \cdot 5^2 7^3, 3^2 5^2 7^3, 3 \cdot 5^2 7^3$, or $3^2 5^2 7^3$. In each of these cases there is an element, y_7 , of order 5 in $C(x_7)$ which normalizes and, hence, centralizes an S_7 subgroup of $C(x_7)$. Then the coefficient $a(y_5, y_5, x_{31})$ is not integral, a contradiction.

When $p = 13$, a count of S_{13} subgroups yields $|G| = 2^6 3^2 7^3 13, 2^6 3^2 5^4 7^3 13$, or $2^6 3^4 5^2 7^3 13$. In the first case Lemma 3.5 leads to a contradiction.

When $|G| = 2^6 3^2 5^4 7 13$, the coefficients $a(x_i, x_i, x_{13})$, $i = 3, 5$, yield $|C(x_7)|/3^2 5^2 7$ and $\chi_{63}(x_5) \equiv 13 \pmod{25}$. Then $\sum \chi^2(x_5) > |C(x_5)|$, unless

$$\chi_{63}(x_5) = 13 \quad \text{and} \quad |C(x_5)| = 2 \cdot 5^4,$$

or

$$\chi_{63}(x_5) = -12 \quad \text{and} \quad |C(x_5)| = 3 \cdot 5^4.$$

Let y_5 be an element of order 5 in $C(x_7)$. Then $|C(y_5)| = 5^{2k}$, $k \geq 1$. Consideration of the coefficient $a(y_5, y_5, x_{13})$ yields $\chi_{63}(y_5) \equiv 28 \pmod{35}$. Then it is clear that $y_5 \notin Z(S_5)$, and $\sum \chi^2(y_5) > |C(y_5)|$ unless

$$(3.7) \quad \chi_{63}(y_5) = -7 \quad \text{and} \quad |C(y_5)| = 2 \cdot 5^2 7.$$

Now let A be an S_5 subgroup of $C(y_5)$. Clearly A is a normal subgroup of $C(y_5)$, and thus A centralizes the elements of order 7 in $C(y_5)$. But this implies that all nonidentity elements of A satisfy (3.7). This is a contradiction since there must be nonidentity elements of A conjugate to x_5 . Thus $5 \nmid |C(x_7)|$. Now a count of S_7 subgroups yields $|N(S_7)| = 21$. Then Burnside's Theorem and the relations above (2.1) with $p = 7$ imply that $B_0(7)$ contains 3 characters whose degrees are congruent to ± 1 modulo 7 and 2 characters whose common degree is congruent to ± 3 modulo 7. Lemma 3.1 applied to $B_0(7) \cap B_0(13)$ implies that the identity character and χ_{64} are in $B_0(7)$. Then consideration of the tree for $B_0(7)$ (cf. [9]) yields that the other characters in $B_0(7)$ must be of degrees 13, 52, 52. Then consideration of the coefficient $a(x_2, x_2, x_7)$ yields $\chi_{13}(x_2) = -3$. Thus G has no elementary Abelian subgroup of order 8. Now [10, Theorem 4.2.7] leads to a contradiction.

Finally if $|G| = 2^6 3^4 5^2 7^3 13$, the coefficient $a(x_7, x_7, x_{13})$ yields $|C(x_7)|/3 \cdot 5 \cdot 7^3$ and $\chi_{63}(x_7) = 14$. Furthermore $5 \nmid |C(x_7)|$ else $\sum \chi^2(x_7) > |C(x_7)|$. But then if y_5 is an element of order 5 in $C(x_7)$, $7^3 \nmid |C(y_5)|$, and hence the coefficient $a(y_5, y_5, x_{13})$ leads to a contradiction.

It is shown in [1] that $\text{PSL}(3, 4)$ satisfies hypotheses I and II. This proves Lemma 3.11.

Lemma 3.12. *If G is a simple group satisfying hypotheses I and II, then*

- (1) *If G has degree equation $1 + 80 = 81$, then G is isomorphic to $\text{PSL}(2, 81)$, and*
- (2) *G cannot have degree equation $1 + 125 = 126$.*

Proof. As usual, let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, $x_5 \in C(S_5)$, and $x_7 \in C(S_7)$. When the degree equation is $1 + 80 = 81$, the coefficients $a(x_i, x_i, x_p)$, $i = 2, 5, 7$, the relations above (2.1) and Lemma 3.2 yield $|G| = 2^a 3^4 5^c 7^d p$, where $a \leq 10$ and even, $c \leq 5$ and odd, $d \leq 4$ and even and $p = 79$ or 41 . If $c = 5$, the

coefficient $a(x_5, x_5, x_p)$ implies that $\chi_{80}(x_5) = -45$, and then $(\chi_{80} | \langle x_5 \rangle, 1 \langle x_5 \rangle) < 0$, a contradiction. Now a count of S_p subgroups yields

$$|G| = 2^4 3^4 5^7 9, \quad 2^{10} 3^4 5^3 41, \quad 2^4 3^4 5^4 41, \quad \text{or} \quad 2^4 3^4 5^3 7^4 41.$$

In the first two cases, Lemma 3.6 leads to a contradiction, and in the third case Lemma 3.6 implies that G is isomorphic to $\text{PSL}(2, 81)$. In the last case, the coefficient $a(x_7, x_7, x_{41})$ yields $\chi_{80}(x_7) = 31, -18$ or -67 . Now $\sum \chi^2(x_7) > |C(x_7)|$, a contradiction, when $\chi_{80}(x_7) = 31$. When $\chi_{80}(x_7) = -18$ or -67 , $(\chi_{80} | \langle x_7 \rangle, 1 \langle x_7 \rangle) < 0$, also a contradiction.

It is an easy matter to verify that $\text{PSL}(2, 81)$ satisfies hypotheses I and II. This proves (1).

When the degree equation is $1 + 125 = 126$, the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 7$, the relations above (2.1), and Lemma 3.2 yield $|G| = 2^a 3^b 5^3 7^d p$, where $a \leq 13$ and odd, $b \leq 8$ and even, $d \leq 3$ and odd, and $p = 127$ or 31 . If $p = 127$, the work [11] of Feit leads to a contradiction. If $p = 31$, a count of S_{31} subgroups yields $|G| = 2^{11} 3^{25} 37 31, 2^7 3^{85} 37 31$, or $2^{11} 3^{65} 37^3 31$. In the first case, consideration of $a(x_2, x_2, x_{31})$ yields $\sum \chi^2(x_2) > |C(x_2)|$, a contradiction. In the second case, the coefficient $a(x_3, x_3, x_{31})$ yields $\chi_{125}(x_3) = -118$. But then $(\chi_{125} | \langle x_3 \rangle, 1 \langle x_3 \rangle) < 0$, a contradiction. When $|G| = 2^{11} 3^{65} 37^3 31$, Lemma 3.4, the coefficients $a(x_2, x_2, x_{31})$ and $a(x_5, x_5, x_{31})$, and the fact that χ_{125} is of defect 0 for the prime 5 yield

$$(3.8) \quad |C(x_5)|/3^2 7 5^3.$$

Furthermore, (3.8) holds for any 5-element, x_5 . Let y_7 be an element of order 7 in $C(x_5)$. Consider $\langle y_7 x_5 \rangle = H$ of order 35. Since $(\chi_{125} |_H, 1_H)$ is a nonnegative integer, $\chi_{125}(y_7) \equiv 20 \pmod{35}$. Now the coefficient $a(y_7, y_7, x_{31})$ yields $\sum \chi^2(y_7) > |C(y_7)|$, a contradiction. Thus $7 \nmid |C(x_5)|$, for any 5-element, x_5 . Thus an S_5 of $C(x_5)$ is normal in $C(x_5)$. Thus the set of S_5 subgroups of G is a T.I. set. Thus the number of S_5 's in G is congruent to 1 modulo 125. Thus $|N(S_5)| = 2^7 3^5 5^3$. But then there is an element, z_7 , of order 7 in $N(S_5)$ so that $5^3 \mid |C(z_7)|$. The coefficient $a(z_7, z_7, x_{31})$ now leads to a contradiction. This proves (2).

Lemma 3.13. *If G is a simple group satisfying hypotheses I and II, then*

- (1) G cannot have degree equation $1 + 224 = 225$;
- (2) G cannot have degree equation $1 + 2400 = 2401$ and
- (3) G cannot have degree equation $1 + 4374 = 4375$.

Proof. As usual, let $x_i \in C(S_i)$, $i = 2, 3, 5$ and 7 . When the degree equation is $1 + 224 = 225$, the coefficients $a(x_j, x_j, x_p)$, $j = 2, 3, 5, 7$, and the relations above (2.1) yield $|G| = 2^a 3^b 5^c 7^d p$, where $a \leq 11$ and odd, $b \leq 8$ and even, $c \leq 4$

and even, $d \leq 5$ and odd, and $p = 223$ or 113 . A count of S_p subgroups yields $|G| = 2^5 3^2 5^2 7^{223}$ or $2^7 3^4 5^4 7^{113}$. In the former case, Lemma 3.1 applied to $B_0(7) \cap B_0(223)$ implies that $B_0(7)$ contains the 111 characters of degree 225, which is absurd. In the latter case, the coefficient $a(x_7, x_7, x_{113})$ yields $\chi_{225}(x_7) = -118$ so that $(\chi_{225} | \langle x_7 \rangle, 1 | \langle x_7 \rangle) < 0$, a contradiction. This proves (1).

When the degree equation is $1 + 2400 = 2401$, the coefficients $a(x_i, x_j, x_p)$, $j = 2, 3, 5$, the relations above (2.1) and Lemma 3.2 yield $|G| = 2^a 3^b 5^c 7^d p$, where $a \leq 19$ and odd, $b \leq 13$ and odd, $c \leq 8$ and even, and $p = 2399$ or 1201 . Next a count of S_p subgroups yields $|G| = 2^{15} 3^3 5^6 7^4 2399$, $2^{13} 3^3 5^6 7^4 1201$, $2^5 3^5 5^2 7^4 2399$, or $2^5 3^5 5^2 7^4 1201$. In the first two cases the coefficient $a(x_2, x_2, x_p)$ implies that $\sum \chi^2(x_2) > |C(x_2)|$, a contradiction. In the third case, Lemma 3.1 applied to $B_0(3) \cap B_0(2399)$ implies that the 1199 characters of degree 2401 are in $B_0(3)$, which is absurd. In the final case, application of Lemma 3.1 to $B_0(3) \cap B_0(1201)$ implies that $B_0(3)$ contains a character of degree 2402. But then $\sum \chi^2(1) > |G|$, a contradiction. This proves (2).

Finally if the degree equation is $1 + 4374 = 4375$, the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5, 7$, and the relations above (2.1) yield $|G| = 2^a 3^b 5^c 7^d p$, where $a \leq 25$ and odd, $b \leq 9$ and odd, $c \leq 6$ and even, $d \leq 3$ and odd and $p = 4373$ or 547 . When $p = 4373$, if $b = 9$ then the coefficient $a(x_3, x_3, x_{4373})$ implies that $\sum \chi^2(x_3) > |C(x_3)|$, a contradiction. Then a count of S_{4373} subgroups of G yields no choices for $|G|$. When $p = 547$, if $d = 1$, then Lemma 3.1 applied to $B_0(7) \cap B_0(547)$ implies that the 273 characters of degree 4374 are in $B_0(7)$, a contradiction. Then a count of S_{547} subgroups of G yields no choices for $|G|$. This proves (3).

We have now considered all of the possible degree equations which result from the solutions listed in §2, and we have used them to show that $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $\text{PSL}(2, 9)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 16)$, $\text{PSL}(2, 25)$, $\text{PSL}(2, 27)$, $\text{PSL}(2, 81)$ and $\text{PSL}(3, 4)$ are the only simple groups which satisfy hypotheses I and II.

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