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# ON THE SEMISIMPLICITY OF GROUP RINGS OF SOLVABLE GROUPS

BY

C. R. HAMPTON(1) AND D. S. PASSMAN(2)

ABSTRACT. Let $K[G]$ denote the group ring of $G$ over the field $K$ of characteristic $p > 0$. An interesting unsolved problem is to find necessary and sufficient conditions on $G$ for $K[G]$ to be semisimple. Even the special case in which $G$ is assumed to be a solvable group is still open. In this paper we prove a number of theorems which may be of use in this special case.

We follow the notation of [3]. In addition let $W$ be a subgroup of $G$. We say that $W$ has locally finite index in $G$ and write $[G : W] = \text{l.f.}$ if for all finitely generated subgroups $L$ of $G$ we have $[L : L \cap W] < \infty$. We say that $G$ is a $\Delta$-group if $G = \Delta(G)$. The main result of this paper is

**Theorem.** *Let $K$ be a field of characteristic $p > 0$, let $G$ be a solvable group and let $H$ be a normal $\Delta$-subgroup of $G$. Then $JK[G] \cap K[H] \neq 0$ if and only if $H$ has an element $h$ of order $p$ with $[G : \mathbf{C}_G(h)] = l.f.$*

We remark that this generalizes techniques found in [1], [2] and [5]. Moreover, recent results of A. E. Zalesskiĭ in [5] guarantee that under certain circumstances if $G$ is solvable and if $JK[G] \neq 0$ then such a subgroup $H$ exists with $JK[G] \cap K[H] \neq 0$. In particular this solves the problem in the case of metanilpotent groups.

1. **$\sigma$-nilpotent elements.** Let $\langle \sigma \rangle$ be an infinite cyclic group and let $\Lambda$ denote the ring of rational integers. For convenience set

$$\Lambda^+ = \{1, 2, 3, \cdots, n, \cdots\} \quad \text{and} \quad \Lambda! = \{1!, 2!, 3!, \cdots, n!, \cdots\}.$$

Suppose $\langle \sigma \rangle$ acts on a ring $R$. If $\alpha \in R$ we say that $\alpha$ is $\sigma$-nilpotent if there exists an infinite subset $S$ of $\Lambda!$ such that for each $s \in S$ there exists an integer $r = r(s) \geq 1$ with

$$\alpha\alpha^{\sigma^s}\alpha^{\sigma^{2s}}\cdots\alpha^{\sigma^{rs}} = 0.$$

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Let $I$ be a right ideal of $R$. We say that $I$ is $\sigma$-nil if every element of $I$ is $\sigma$-nilpotent. Observe that if $\langle\sigma\rangle$ acts on a group $H$, then $\langle\sigma\rangle$ acts naturally on the group ring $K[H]$. In this section we study $\sigma$-nil right ideals in $K[H]$. No assumption will be made here on the characteristic of the field.

Suppose $\langle\sigma\rangle$ acts on an abelian group $A$. Then the polynomial ring $\Lambda[\sigma]$ acts in a natural way. If $A$ is torsion (as a $\Lambda$-module) then we set

$$\text{moni } A = \{a \in A \mid a^{f(\sigma)} = 1 \text{ for some monic polynomial } f(\sigma) \in \Lambda[\sigma]\}.$$

If $A$ is torsion free then we set

$$\text{poly } A = \{a \in A \mid a^{f(\sigma)} = 1 \text{ for some } f(\sigma) \in \Lambda[\sigma],\ f(\sigma) \neq 0\},$$

$$\text{nilp } A = \{a \in A \mid a^{(\sigma^n - 1)^m} = 1 \text{ for some } m, n \in \Lambda^+\}.$$

We have the following basic facts.

**Lemma 1.** *Let $A$ be an abelian group acted upon by $\langle\sigma\rangle$.*

(i) *If $A$ is torsion then* moni $A$ *is a $\langle\sigma\rangle$-invariant subgroup of $A$ and* moni$(A/\text{moni } A) = \langle 1\rangle$.

(ii) *If $A$ is torsion free then* poly $A$ *is a $\langle\sigma\rangle$-invariant subgroup of $A$, $A/$poly $A$ is torsion free and* poly$(A/\text{poly } A) = \langle 1\rangle$.

(iii) *If $A$ is torsion free then* nilp $A$ *is a $\langle\sigma\rangle$-invariant subgroup of $A$, $A/$nilp $A$ is torsion free and* nilp$(A/\text{nilp } A) = \langle 1\rangle$.

**Proof.** (i) Let $f(\sigma)$ and $g(\sigma)$ be monic polynomials. Then so is $f(\sigma)g(\sigma)$. Suppose $a, b \in A$. If $a^{f(\sigma)} = 1$, $b^{g(\sigma)} = 1$ then $(ab^{-1})^{f(\sigma)g(\sigma)} = 1$. Thus clearly moni $A$ is a $\langle\sigma\rangle$-invariant subgroup of $A$. If $b^{f(\sigma)} \in$ moni $A$ then $b^{f(\sigma)g(\sigma)} = 1$ for some monic $g(\sigma)$ so moni$(A/\text{moni } A) = \langle 1\rangle$.

(ii) Let $f(\sigma)$ and $g(\sigma)$ be nonzero polynomials. Then so is $f(\sigma)g(\sigma)$. Suppose $a, b \in A$. If $a^{f(\sigma)} = 1$, $b^{g(\sigma)} = 1$ then $(ab^{-1})^{f(\sigma)g(\sigma)} = 1$. Thus clearly poly $A$ is a $\langle\sigma\rangle$-invariant subgroup of $A$. If $b^s \in$ poly $A$, then $(b^{f(\sigma)})^s = (b^s)^{f(\sigma)} = 1$ for some nonzero $f(\sigma)$. Since $A$ is torsion free, $b^{f(\sigma)} = 1$ and $b \in$ poly $A$. Thus $A/$poly $A$ is torsion free. Finally if $b^{f(\sigma)} \in$ poly $A$ then $b^{f(\sigma)g(\sigma)} = 1$ for some $g(\sigma) \neq 0$ so poly$(A/\text{poly } A) = \langle 1\rangle$.

(iii) This follows as in (ii) with the additional observation that if $f(\sigma) = (\sigma^{n_1} - 1)^{m_1}$ and $g(\sigma) = (\sigma^{n_2} - 1)^{m_2}$ then $f(\sigma)g(\sigma)$ divides $(\sigma^{n_1 n_2} - 1)^{m_1 + m_2}$.

The following is a variant of a well-known result.

**Lemma 2.** *Let $\omega_1, \omega_2, \cdots, \omega_t$ be a complete set of nonzero algebraic conjugates over the rational numbers $\mathfrak{Q}$ and suppose that $|\omega_i| \leq 1$ for all $i$. Moreover, suppose there exist nonzero algebraic integers $c_1, c_2, \cdots, c_t$ and an infinite subset $S \subseteq \Lambda^+$ such that $c_i\omega_i^s$ is an algebraic integer for all $i \leq t$ and $s \in S$. Then the $\omega_i$'s are all roots of unity.*

**Proof.** Since the $c_i$'s are all algebraic integers it follows that if $c = c_1 c_2 \cdots c_t$ then $c\omega_i^s$ is an algebraic integer. Moreover, the product of all the algebraic conjugates of $c$ has this same property so we may assume that $c$ is a rational integer.

For each $s \in S$ let $f_s(\zeta) = \prod_{i=1}^t (\zeta - c\omega_i^s)$. Since $c$ is a rational integer and since $\{\omega_i\}$ is a complete set of algebraic conjugates over $\mathfrak{Q}$ it follows that $f_s(\zeta) \in \mathfrak{Q}[\zeta]$. Moreover, $c\omega_i^s$ is an algebraic integer so $f_s(\zeta) \in \Lambda[\zeta]$. Now $|\omega_i| \leq 1$ so $|c\omega_i^s| \leq c$ and it follows that there exists a fixed bound for the coefficients of $f_s(\zeta)$ independent of $s$. Since $S$ is infinite, we conclude that there exists an infinite subset $S_0$ of $S$ such that for all $s \in S_0$ the polynomials $f_s(\zeta)$ are identical.

Fix $s_0 \in S_0$. Then for each $s \in S_0$ there exists a permutation $\tau$ of the subscripts such that $c\omega_i^s = c\omega_{\tau(i)}^{s_0}$. Finally $S_0$ is infinite and the number of such permutations is finite so there exists two distinct elements $s, s' \in S_0$ having the same permutation. Then for all $i$, $c\omega_i^s = c\omega_i^{s'}$ and hence $\omega_i$ is a root of unity since $c \neq 0$ and $\omega_i \neq 0$.

**Lemma 3.** *Let $A$ be an abelian group acted upon by $\langle\sigma\rangle$. Let $a_0 = 1, a_1, a_2, \cdots, a_k$ be a fixed finite set of distinct elements of $A$ and let $S$ be an infinite subset of $\Lambda^+$. Suppose that for each $s \in S$ there exists an integer $r = r(s) \geq 1$ and a set of subscripts $i_0, i_1, \cdots, i_r$ not all zero with*

$$a_{i_0} a_{i_1}^{\sigma^s} a_{i_2}^{\sigma^{2s}} \cdots a_{i_r}^{\sigma^{rs}} = 1.$$

(i) *If $A$ is torsion, then* moni $A \neq \langle 1\rangle$.

(ii) *If $A$ is torsion free, then* poly $A \neq \langle 1\rangle$.

(iii) *If $A$ is torsion free and $A =$ poly $A$, then* nilp $A \neq \langle 1\rangle$.

**Proof.** It is easier to view this result additively. Thus in this proof we will assume that $A$ is an additive abelian group and the action of $\Lambda[\sigma]$ on $A$ will be indicated by right multiplication.

Let us consider the above situation. For each $s \in S$ let $T_s$ denote the set of nonzero subscripts occuring in the $s$-equation. Since $S$ is infinite and the number of subsets of $\{1, 2, \cdots, k\}$ is finite, it follows that there exists an infinite subset $S_0$ of $S$ such that $T_s = T$ is the same for all $s \in S_0$. We can then replace $S$ by $S_0$ and $\{a_0, a_1, \cdots, a_k\}$ by $\{a_i | \ i = 0$ or $i \in T\}$ if necessary to obtain the irredundant situation in which each $a_i$ for $i \geq 1$ occurs at least once in each equation.

Suppose first that $A$ is a vector space over a field $F$. Then $A$ is a right $F[\sigma]$-module and $F[\sigma]$ is a principal ideal domain. Suppose that the above situation is irredundant and set

$$B = a_1 F[\sigma] + a_2 F[\sigma] + \cdots + a_k F[\sigma].$$

We show that there exists $b \in B$, $b \neq 0$ and $f(\sigma) \in F[\sigma]$, $f(\sigma) \neq 0$ with $bf(\sigma) = 0$. If not then $B$ is a finitely generated $F[\sigma]$-module which is torsion free as an $F[\sigma]$-module. Then $B$ is a free $F[\sigma]$-module and hence there is a module homomorphism $\phi$ of $B$ onto $F[\sigma]$. Now $\phi(a_0) = 0$, $\phi(a_1), \cdots, \phi(a_k) \in F[\sigma]$ and these are not all zero since $B$ is generated by $a_1, a_2, \cdots, a_k$. Say $\phi(a_1) \neq 0$. Let $t$ be the maximum degree of the polynomials $\phi(a_i)$. Since $S$ is infinite we can choose $s \in S$ with $s > t$ and deduce that

$$\phi(a_{i_0}) + \phi(a_{i_1})\sigma^s + \phi(a_{i_2})\sigma^{2s} + \cdots + \phi(a_{i_r})\sigma^{rs} = 0.$$

By the irredundancy condition $\phi(a_1) \neq 0$ occurs here and since $s > \deg \phi(a_{i_j})$ we see that the above left-hand term is a nonzero polynomial in $F[\sigma]$, a contradiction. Thus for some $b \in B \subseteq A$, $b \neq 0$ there exists $f(\sigma) \in F[\sigma]$, $f(\sigma) \neq 0$ with $bf(\sigma) = 0$.

(i) Suppose $A$ is torsion and assume that the situation is irredundant. Since each $a_i$ has finite order we can choose an integer $m$ and a prime $q$ such that $a_i mq = 0$ for all $i$ but $a_i m \neq 0$ for some $i$. Say $a_1 m \neq 0$. Let $A_q$ denote the set of elements of $A$ of order 1 or $q$. Then $A_q$ is a $\mathrm{GF}(q)[\sigma]$-module and $a_i m \in A_q$ for all $i$. Now for each $s \in S$ we multiply the $s$-equation by $m$ to obtain a corresponding equation in $A_q$. Since $a_1 m \neq 0$ occurs in each such equation we conclude from the above that there exists $b \in A_q$, $b \neq 0$ and $\bar{f}(\sigma) \in \mathrm{GF}(q)[\sigma]$, $\bar{f}(\sigma) \neq 0$ with $b\bar{f}(\sigma) = 0$. Now $\mathrm{GF}(q)$ is a field so we may suppose that $\bar{f}(\sigma)$ is monic and then we can choose $f(\sigma) \in \Lambda[\sigma]$ monic with $\bar{f}(\sigma) = f(\sigma) \bmod q$. Since $bq = 0$ we have $bf(\sigma) = 0$ so $b \in \operatorname{moni} A$ and $\operatorname{moni} A \neq \langle 0 \rangle$.

(ii) Since $A$ is torsion free, $A$ is naturally embedded in $A \otimes_\Lambda \mathfrak{Q}$ and $(A \otimes_\Lambda \mathfrak{Q})/A$ is torsion. Clearly $\mathfrak{Q}[\sigma]$ acts on $A \otimes_\Lambda \mathfrak{Q}$. Now each of the given equations can be viewed as an equation in $A \otimes_\Lambda \mathfrak{Q}$ so by our earlier observation there exists $\bar{f}(\sigma) \in \mathfrak{Q}[\sigma]$ and $\bar{b} \in A \otimes_\Lambda \mathfrak{Q}$ both not zero with $\bar{b}\bar{f}(\sigma) = 0$. Choose integers $m, n \neq 0$ with $b = \bar{b}m \in A$, $f(\sigma) = n\bar{f}(\sigma) \in \Lambda[\sigma]$. Then $b \neq 0$, $f(\sigma) \neq 0$ and $bf(\sigma) = 0$ so $b \in \operatorname{poly} A$ and $\operatorname{poly} A \neq \langle 0 \rangle$.

(iii) We suppose that the situation is irredundant and we may assume that

$$A = a_1 \Lambda[\sigma] + a_2 \Lambda[\sigma] + \cdots + a_k \Lambda[\sigma].$$

As above $A$ is naturally embedded in $V = A \otimes_\Lambda \mathfrak{Q}$ and $\mathfrak{Q}[\sigma]$ acts on $V$. Since $A = \operatorname{poly} A$ each $a_i$ is annihilated by a nonzero polynomial in $\mathfrak{Q}[\sigma]$ and thus $V$ is a finite dimensional $\mathfrak{Q}$-vector space. Say $\dim_{\mathfrak{Q}} V = t$ and let $\{\omega_1, \omega_2, \cdots, \omega_t\}$ be the set of eigenvalues of $\sigma$ on $V$. Then this is a union of complete sets of algebraically conjugate elements over $\mathfrak{Q}$ and $\omega_i \neq 0$ since $\sigma$ is one-to-one in its action on $A$.

Let $W = V \otimes_{\mathfrak{Q}} \Omega$ where $\Omega = \mathfrak{Q}[\omega_1, \omega_2, \cdots, \omega_t]$. Then $\Omega[\sigma]$ acts on $W$, $\dim_\Omega W = t$ and the eigenvalues of $\sigma$ on $W$ are precisely $\omega_1, \omega_2, \cdots, \omega_t$. Let us study one such eigenvalue $\omega$. From the Jordan canonical form for $\sigma$ we see that there exists a nonzero linear functional $\phi: W \to \Omega$ with $\phi(b\sigma) = \phi(b)\omega$ for all $b \in W$. Multiplying $\phi$ by a rational integer if necessary we may suppose that $\phi(a_1), \phi(a_2), \cdots, \phi(a_k)$ are algebraic integers. Moreover, since the $a_i$'s generate $W$ as an $\Omega[\sigma]$-module we cannot have $\phi(a_i) = 0$ for all $i$. Say $\phi(a_1) \neq 0$. Let $c$ be the product of the nonzero $\phi(a_i)$'s so that $c \neq 0$ is an algebraic integer and let $m = \max\{|\phi(a_i)|/|\phi(a_j)| \mid \phi(a_j) \neq 0\}$.

Let $s \in S$. Then for some $r \geq 1$ we have

$$\phi(a_{i_0}) + \phi(a_{i_1})\omega^s + \phi(a_{i_2})\omega^{2s} + \cdots + \phi(a_{i_r})\omega^{rs} = 0.$$

Since $\phi(a_1) \neq 0$ occurs in here there are at least two nonzero terms. Thus we may suppose that $r \geq 1$ is given above with $\phi(a_{i_r}) \neq 0$. Since the $\phi(a_{i_j})$ are algebraic integers this implies that $\phi(a_{i_r})\omega^s$ is an algebraic integer and hence so is $c\omega^s$. Suppose now that $|\omega| > 1$. Then for all $s \in S$ the equation

$$\omega^{rs} = -\sum_{j=0}^{r-1} \left( \frac{\phi(a_{i_j})}{\phi(a_{i_r})} \right) \omega^{js}$$

yields

$$|\omega^{rs}| \leq m \sum_{j=0}^{r-1} |\omega^s|^j = \frac{m(|\omega|^{rs} - 1)}{(|\omega|^s - 1)} \leq \frac{m|\omega^{rs}|}{(|\omega|^s - 1)}.$$

Thus $|\omega|^s \leq m + 1$ for all $s \in S$, certainly a contradiction.

We have therefore shown that for each $i = 1, 2, \cdots, t$ we have $|\omega_i| \leq 1$ and there exists a nonzero algebraic integer $c_i$ such that $c_i\omega_i^s$ is an algebraic integer for all $s \in S$. By Lemma 2, each $\omega_i$ is a root of unity. Hence for some integer $n \geq 1$, $(\sigma^n - 1)$ has all eigenvalues 0 and thus $(\sigma^n - 1)^t$ annihilates all elements of $A \subseteq W$. Thus nilp $A \neq \langle 0 \rangle$ and the lemma is proved.

If $H$ is a subgroup of $G$ we let $\pi_H^G$ denote the natural projection $\pi_H^G$: $K[G] \to K[H]$. This is of course a module homomorphism if we view $K[G]$ and $K[H]$ as either right or left $K[H]$-modules. Thus in particular if $I$ is a right ideal of $K[G]$, then $\pi_H^G(I)$ is a right ideal of $K[H]$.

**Lemma 4.** *Let $N \lhd H$ be $\langle\sigma\rangle$-invariant groups with $H/N = A$ abelian. Then $\Lambda[\sigma]$ acts on $A$. Suppose that either*

(i) *$A$ is torsion and* moni $A = \langle 1 \rangle$,

(ii) *$A$ is torsion free and* poly $A = \langle 1 \rangle$, *or*

(iii) $A =$ poly $A$ *is torsion free and* nilp $A = \langle 1 \rangle$.

*If* $\alpha \in K[H]$ *is* $\sigma$*-nilpotent, then* $\pi_N^H(\alpha) \in K[N]$ *is* $\sigma$*-nilpotent.*

**Proof.** Write $\alpha = \alpha_0 + \alpha_1 y_1 + \cdots + \alpha_k y_k$ where $\alpha_i \in K[N]$, $\alpha_0 = \pi_N^H(\alpha)$ and $y_0 = 1, y_1, y_2, \cdots, y_k$ are in distinct cosets of $N$ in $H$ and hence represent distinct elements in $H/N = A$. Say $a_0 = 1 \in A$ and $a_i = Ny_i \in A$ for $i = 1, 2, \cdots, k$.

Suppose by way of contradiction that $\alpha_0$ is not $\sigma$-nilpotent. Since $\alpha$ is $\sigma$-nilpotent there clearly exists an infinite subset $S$ of $\Lambda!$ such that, for any $s \in S$ and for some $r = r(s) \geq 1$,

$$\alpha\alpha^{\sigma^s}\alpha^{\sigma^{2s}}\cdots\alpha^{\sigma^{rs}} = 0 \quad \text{and} \quad \alpha_0\alpha_0^{\sigma^s}\alpha_0^{\sigma^{2s}}\cdots\alpha_0^{\sigma^{rs}} \neq 0.$$

Now writing $\alpha$ as above we have the ordered products

$$\prod_{i=0}^{r}(\alpha_0^{\sigma^{is}} + \alpha_1^{\sigma^{is}} y_1^{\sigma^{is}} + \cdots + \alpha_k^{\sigma^{is}} y_k^{\sigma^{is}}) = 0 \quad \text{and} \quad \prod_{i=0}^{r}(\alpha_0^{\sigma^{is}}) \neq 0.$$

Since $\alpha_j^{\sigma^{is}} \in K[N]$ this clearly implies that there exists subscripts $i_j$ not all zero with $y_{i_0}y_{i_1}^{\sigma^s}y_{i_2}^{\sigma^{2s}}\cdots y_{i_r}^{\sigma^{rs}} \in N$. Thus for the same subscripts $a_{i_0}a_{i_1}^{\sigma^s}a_{i_2}^{\sigma^{2s}}\cdots a_{i_r}^{\sigma^{rs}} = 1$. Since our assumptions contradict the results of Lemma 3 we conclude that $\pi_N^H(\alpha) = \alpha_0$ is $\sigma$-nilpotent.

**Lemma 5.** *Let* $A$ *be an abelian group acted upon by* $\langle\sigma\rangle$ *and let* $a_1, a_2, \cdots, a_n$ *be a finite number of elements of* $A$. *We let* $\langle a_1, a_2, \cdots, a_n\rangle^{\langle\sigma\rangle}$ *denote the subgroup of* $A$ *generated by the elements* $a_i$ *and their conjugates under* $\langle\sigma\rangle$.

(i) *If* $A$ *is torsion and* $A = \text{moni } A$, *then* $\langle a_1, a_2, \cdots, a_n\rangle^{\langle\sigma\rangle}$ *is finite.*

(ii) *If* $A$ *is torsion free and* $A = \text{nilp } A$, *then for some integer* $m \geq 1$, $\langle a_1, a_2, \cdots, a_n\rangle^{\langle\sigma\rangle}$ *has an ordering compatible with the action of* $\sigma^m$.

**Proof.** (i) Since $\langle a_1, a_2, \cdots, a_n\rangle^{\langle\sigma\rangle} = \langle a_1\rangle^{\langle\sigma\rangle}\langle a_2\rangle^{\langle\sigma\rangle}\cdots\langle a_n\rangle^{\langle\sigma\rangle}$ it suffices to show that, for any $a \in A$, $\langle a\rangle^{\langle\sigma\rangle}$ is finite. Let $a \in A$. Then $a^m = 1$ for some $m \geq 1$ and $a^{f(\sigma)} = 1$ for some monic polynomial $f(\sigma) \in \Lambda[\sigma]$. It follows that $a^{\Lambda[\sigma]}$ is a homomorphic image of the additive abelian group $\Lambda[\sigma]/(m\Lambda[\sigma] + f(\sigma)\Lambda[\sigma])$.

Let $g(\sigma) \in \Lambda[\sigma]$. Since $f(\sigma)$ is monic we have $g(\sigma) = q(\sigma)f(\sigma) + r(\sigma)$ where $\deg r(\sigma) < \deg f(\sigma)$. Since there are only finitely many possibilities for $r(\sigma)$ modulo $m\Lambda[\sigma]$ it follows that $\Lambda[\sigma]/(m\Lambda[\sigma] + f(\sigma)\Lambda[\sigma])$ is finite. Thus $a^{\Lambda[\sigma]}$ is finite.

Finally $a^{\Lambda[\sigma]}$ is invariant under $\sigma$ and $\sigma$ acts in a one-to-one manner. Since $a^{\Lambda[\sigma]}$ is finite we see that $\sigma$ maps this group onto itself so $a^{\Lambda[\sigma]}$ is also invariant under $\sigma^{-1}$. Thus $\langle a\rangle^{\langle\sigma\rangle} = a^{\Lambda[\sigma]}$ is finite.

(ii) This is essentially the fact that a torsion free nilpotent group can be ordered. Since $A = \text{nilp } A$ for each $i$ there exists $k_i$, $m_i \geq 1$ with $a_i^{(\sigma^{m_i}-1)^{k_i}} = 1$. Since

$$\prod_{i=1}^{n} (\sigma^{m_i} - 1)^{k_i} \Big| (\sigma^{m_1 m_2 \cdots m_n} - 1)^{k_1 + k_2 + \cdots + k_n}$$

we see that $(\sigma^m - 1)^k$ acts trivially on $\langle a_1, a_2, \cdots, a_n\rangle$ where $m = m_1 m_2 \cdots m_n$ and $k = k_1 + k_2 + \cdots + k_n$. Thus clearly $(\sigma^m - 1)^k$ acts trivially on $\langle a_1, a_2, \cdots, a_n\rangle^{\langle\sigma\rangle} = B$.

Let $\tau = \sigma^m$ and for each $j \le k$ set

$$B_j = \{b \in B \mid b^{(\tau-1)^j} = 1\}.$$

Then $\langle 1\rangle = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_k = B$ and clearly $B/B_j$ is torsion free. We now order each of the torsion free abelian quotients $B_j/B_{j-1}$ for $j = 1, 2, \cdots, k$. Observe that $\tau$ centralizes each such quotient so each of these orderings is compatible with the action of $\tau$. Finally we order $B$. Let $b \in B$, $b \ne 1$ and choose $j$ so that $b \in B_j - B_{j-1}$. Then we say $b > 1$ if and only if $bB_{j-1} > B_{j-1}$ under the ordering of $B_j/B_{j-1}$. It is now easy to check that this yields an ordering of $B$ compatible with the action of $\tau = \sigma^m$.

We now come to the main result of this section.

**Proposition 6.** *Let $N \lhd H$ be $\langle\sigma\rangle$-invariant groups with $H/N$ abelian. Let $\alpha \in K[H]$, $\alpha \ne 0$ and suppose that $\alpha K[H]$ is a $\sigma$-nil right ideal. Then there exist $y \in \operatorname{Supp} \alpha$ and a $\langle\sigma\rangle$-invariant subgroup $N^*$ of $H$ with $[N^* : N] < \infty$ such that $\pi^H_{N^*}(\alpha y^{-1})K[N^*]$ is a $\sigma$-nil right ideal of $K[N^*]$.*

**Proof.** Set $A = A_0 = H/N$ and we define a series of $\langle\sigma\rangle$-invariant subgroups of $A$. Let $A_3$ be the torsion subgroup of $A$. Then $A/A_3$ is torsion free and since $\Lambda[\sigma]$ acts on $A/A_3$ we set $A_1/A_3 = \operatorname{poly}(A/A_3)$ and $A_2/A_3 = \operatorname{nilp}(A/A_3)$. Moreover, $A_3$ is torsion and $\Lambda[\sigma]$ acts on $A$ so we set $A_4 = \operatorname{moni} A_3$. We then have $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \langle 1\rangle$. Define $\langle\sigma\rangle$-invariant subgroups $H = N_0 \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq N_4 \supseteq N$ by $N_i/N = A_i$.

Let $\beta \in \alpha K[H]$ so that $\beta$ is $\sigma$-nilpotent. Since $H/N_1 \cong A/A_1$ is torsion free and poly $A/A_1 = \langle 1\rangle$ by Lemma 1(ii) it follows from Lemma 4(ii) that $\pi^H_{N_1}(\beta)$ is $\sigma$-nilpotent. Since $N_1/N_2 \cong A_1/A_2$ is torsion free, $A_1/A_2 =$ poly $A_1/A_2$ and nilp $A_1/A_2 = \langle 1\rangle$ by Lemma 1(iii), it follows from Lemma 4(iii) that $\pi^H_{N_2}(\beta) = \pi^{N_1}_{N_2}\pi^H_{N_1}(\beta)$ is $\sigma$-nilpotent. Thus $\pi^H_{N_2}(\alpha K[H])$ is a $\sigma$-nil right ideal of $K[N_2]$.

Write $\pi^H_{N_2}(\alpha) = \Sigma_1^k \alpha_i y_i$ where $\alpha_i \in K[N_3]$, $1 \in \operatorname{Supp} \alpha_i$ and $y_1, y_2, \cdots y_k$ are in distinct cosets of $N_3$ in $N_2$. Thus $y_1, y_2, \cdots, y_k$ represent distinct elements $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_k \in N_2/N_3$. Now $N_2/N_3 = \operatorname{nilp} N_2/N_3$ so by Lemma 5 (ii) the group $\langle \bar{y}_1, \bar{y}_2, \cdots \bar{y}_k\rangle^{\langle\sigma\rangle}$ has an ordering compatible with the action of $\sigma^m$ for some fixed $m \ge 1$. Fix such an ordering and suppose that the above subscripts are so chosen that $\bar{y}_k > \bar{y}_i$ for $i \ne k$. Then $\bar{1} > \bar{y}_i\bar{y}_k^{-1}$ and $y_k \in \operatorname{Supp} \alpha$ since $1 \in \operatorname{Supp} \alpha_k$.

Now $\pi^H_{N_3}(\alpha y_k^{-1}) = \alpha_k$ and we show that $\alpha_k K[N_3]$ is a $\sigma$-nil right ideal of $K[N_3]$. Let $\gamma \in K[N_3]$. Then $\alpha y_k^{-1}\gamma \in \alpha K[H]$ is $\sigma$-nilpotent and hence so is $\pi^H_{N_2}(\alpha y_k^{-1}\gamma) = \pi^H_{N_2}(\alpha) y_k^{-1}\gamma$ since $y_k^{-1}\gamma \in K[N_2]$. Thus $\sum_{i=1}^k \alpha_i y_i y_k^{-1}\gamma$ is $\sigma$-nilpotent. Let $S$ be the infinite subset of $\Lambda!$ associated with this property. Since there are at most finitely many elements of $\Lambda!$ not divisible by $m$ we may assume that for all $s \in S$ we have $m \mid s$. Thus $\sigma^s$ also preserves the ordering. Finally let $s \in S$. Then for some $r = r(s) \geq 1$ we have the ordered product

$$\prod_{j=0}^{r}\left(\sum_{i=1}^{k} \alpha_i^{\sigma^{js}} y_i^{\sigma^{js}} (y_k^{-1})^{\sigma^{js}} \gamma^{\sigma^{js}}\right) = 0.$$

Since $\alpha_i^{\sigma^{js}} \in K[N_3]$, $\gamma^{\sigma^{js}} \in K[N_3]$ and $\bar{y}_i^{\sigma^{js}}(\bar{y}_k^{-1})^{\sigma^{js}} < \bar{1}$ for $i < k$ we conclude that

$$\prod_{j=0}^{r} \alpha_k^{\sigma^{js}} \gamma^{\sigma^{js}} = 0.$$

Thus $\alpha_k \gamma = \pi^H_{N_3}(\alpha y_k^{-1})\gamma$ is $\sigma$-nilpotent.

We have therefore shown that there exists $y \in \operatorname{Supp} \alpha$ with $\pi^H_{N_3}(\alpha y^{-1})K[N_3] = I$ a $\sigma$-nil right ideal. Let $\beta \in I$. Since $N_3/N_4 \cong A_3/A_4$ is torsion and moni $A_3/A_4 = \langle 1 \rangle$ by Lemma 1(i), we conclude that $\pi^{N_3}_{N_4}(\beta)$ is $\sigma$-nilpotent by Lemma 4(i). Since $\pi^{N_3}_{N_4}(I) \supseteq \pi^H_{N_4}(\alpha y^{-1})K[N_4]$ we see that the latter is a $\sigma$-nil right ideal of $K[N_4]$.

Now $N_4/N \cong A_4$ and $A_4 = \text{moni } A_4$. Since $(\operatorname{Supp} \pi^H_{N_4}(\alpha y^{-1}))N/N$ is finite, there exists by Lemma 5(i) a finite $\langle \sigma \rangle$-invariant subgroup $N^*/N$ of $A_4$ with $\operatorname{Supp} \pi^H_{N_4}(\alpha y^{-1}) \subseteq N^*$. Thus $\pi^H_{N^*}(\alpha y^{-1}) = \pi^H_{N_4}(\alpha y^{-1})$ and $\pi^H_{N^*}(\alpha y^{-1})K[N_4] \supseteq \pi^H_{N^*}(\alpha y^{-1})K[N^*]$ so the latter is a $\sigma$-nil right ideal of $K[N^*]$. This completes the proof.

2. **The main theorems.** In this section we apply the above proposition to the semisimplicity problem. We first generalize it slightly.

**Lemma 7.** *Let $N \lhd H$ be $\langle \sigma \rangle$-invariant groups with $H/N$ a solvable $\Delta$-group. Let $\alpha \in K[H]$, $\alpha \neq 0$ and suppose that $\alpha K[H]$ is a $\sigma$-nil right ideal. Then there exists $y \in \operatorname{Supp} \alpha$ and a $\langle \sigma \rangle$-invariant subgroup $N^*$ of $H$ with $[N^* : N] < \infty$ such that $\pi^H_{N^*}(\alpha y^{-1})K[N^*]$ is a $\sigma$-nil right ideal of $K[N^*]$.*

**Proof.** Let $H = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_k = N$ correspond to the derived series of solvable $H/N$. Then certainly each $N_i$ is $\langle \sigma \rangle$-invariant. We show by induction on $i$ that there exists $\langle \sigma \rangle$-invariant $N_i^*$ with $H \supseteq N_i^* \supseteq N_i$, $[N_i^* : N_i] < \infty$ and $y_i \in \operatorname{Supp} \alpha$ such that $\pi^H_{N_i^*}(\alpha y_i^{-1})K[N_i^*]$ is a $\sigma$-nil right ideal of $K[N_i^*]$. For $i = 0$ we take $N_0^* = H = N_0$, and $y_0$ any element of $\operatorname{Supp} \alpha$.

Let us suppose the result holds for some $i < k$. We consider the quotient $N_i^*/N_{i+1}$ which has an abelian subgroup $N_i/N_{i+1}$ of finite index. Since $H/N = \Delta(H/N)$ this implies easily that $N_i^*/N_{i+1}$ has a center of finite index and hence (see Lemma 2.1 of [3]) $N_i^*/N_{i+1}$ has a finite commutator subgroup $M/N_{i+1}$. Thus $M$ is $\langle\sigma\rangle$-invariant, $[M : N_{i+1}] < \infty$ and $N_i^*/M$ is abelian.

By Proposition 6 applied to $M \lhd N_i^*$ there exists a $\langle\sigma\rangle$-invariant subgroup $N_{i+1}^*$ of $N_i^*$ with $[N_{i+1}^* : M] < \infty$ and an element $z \in \operatorname{Supp} \pi_{N_i^*}^H(\alpha y_i^{-1})$ such that $\pi_{N_{i+1}^*}^{N_i^*}(\pi_{N_i^*}^H(\alpha y_i^{-1}) \cdot z^{-1})K[N_{i+1}^*]$ is a $\sigma$-nil right ideal of $K[N_{i+1}^*]$. Since $[N_{i+1}^* : M] < \infty$ and $[M : N_{i+1}] < \infty$ we have $[N_{i+1}^* : N_{i+1}] < \infty$. Now $z \in N_i^*$ implies that $\pi_{N_i^*}^H(\alpha y_i^{-1}) \cdot z^{-1} = \pi_{N_i^*}^H(\alpha y_i^{-1}z^{-1})$. Also $z \in \operatorname{Supp} \pi_{N_i^*}^H(\alpha y_i^{-1})$ implies that there exists $y_{i+1} \in \operatorname{Supp} \alpha$ with $z = y_{i+1}y_i^{-1}$. Thus $\alpha y_i^{-1}z^{-1} = \alpha y_{i+1}^{-1}$ and by the above $\pi_{N_{i+1}^*}^H(\alpha y_{i+1}^{-1})K[N_{i+1}^*]$ is $\sigma$-nil. The induction step is proved and the $i = k$ case yields the result.

At this point the characteristic of the field comes into play. We assume throughout the remainder of this paper that $K$ has characteristic $p > 0$. If $\alpha \in K[G]$ we define

$$\operatorname{Quot} \alpha = \{xy^{-1} \mid x, y \in \operatorname{Supp} \alpha\},$$

$$p\text{-}\operatorname{Quot} \alpha = \{g \in \operatorname{Quot} \alpha \mid g \neq 1 \text{ has order a power of } p\}.$$

**Proposition 8.** *Let $H$ be a $\langle\sigma\rangle$-invariant solvable $\Delta$-group and let $\alpha \in K[H]$, $\alpha \neq 0$. If $\alpha K[H]$ is a $\sigma$-nil right ideal of $K[H]$ then there exists $g \in p$-Quot $\alpha$ and an integer $n \geq 1$ such that $\sigma^n$ centralizes $g$.*

Proof: We apply Lemma 7 with $N = \langle 1\rangle$. Then there exists a finite $\langle\sigma\rangle$-invariant subgroup $N^*$ of $H$ and $y \in \operatorname{Supp} \alpha$ such that $\pi_{N^*}^H(\alpha y^{-1})K[N^*]$ is a $\sigma$-nil right ideal of $K[N^*]$. Since $N^*$ is finite there exists an integer $n$ such that $\sigma^n$ centralizes $N^*$. Now $\beta = \pi_{N^*}^H(\alpha y^{-1})$ is $\sigma$-nilpotent and since there are only finitely many elements of $\Lambda$! not divisible by $n$ it follows that for some $s$ with $n \mid s$ we have $\beta^{r+1} = \beta\beta^{\sigma^s}\beta^{\sigma^{2s}} \cdots \beta^{\sigma^{rs}} = 0$. Thus $\beta$ is nilpotent. Now $y \in \operatorname{Supp} \alpha$ implies that $1 \in \operatorname{Supp} \alpha y^{-1}$ so $1 \in \operatorname{Supp} \beta$. Then Lemma 3.5 of [3] implies that there exists $g \in \operatorname{Supp} \beta$, $g \neq 1$ such that $g$ has order a power of $p$. Clearly, $g \in p$-Quot $\alpha$ and $\sigma^n$ centralizes $g \in N^*$. The result follows.

We can now obtain our main results. If $W$ is a subgroup of $G$ we let

$$\sqrt{W} = \{x \in G \mid x^n \in W \text{ for some integer } n \geq 1\}.$$

Observe that $\sqrt{W}$ need not be a subgroup of $G$ even if $G$ is assumed to be solvable.

**Theorem A.** *Let $K$ be a field of characteristic $p > 0$ and let $H$ be a normal solvable $\Delta$-subgroup of $G$. Suppose that $\alpha \in JK[G] \cap K[H]$ with $\alpha \neq 0$. Then*

$$G = \bigcup_{g \in p\text{-Quot}\,\alpha} \sqrt{\mathbf{C}_G(g)}.$$

*Moreover, if* $G$ *is solvable then for some* $g \in p$-Quot $\alpha$ *we have* $[G : \mathbf{C}_G(g)] = l.f.$

**Proof:** Let $x \in G$. We distinguish two cases according to whether the image of $x$ in $G/H$ has finite or infinite order.

Suppose first that $x^t \in H$ for some integer $t$. If $G_1 = \langle H, x \rangle$ then clearly $[G_1 : H] < \infty$. Now by Lemma 16.9 of [3], $\alpha \in JK[G] \cap K[H] \subseteq JK[H]$ and thus, by Lemma 19.6 of [3], $\alpha K[H]$ is nilpotent. Choose $y \in$ Supp $\alpha$ so that $1 \in$ Supp $\alpha y^{-1}$ and $\alpha y^{-1}$ is nilpotent. By Lemma 3.5 of [3] there exists an element $g \neq 1$ of order a power of $p$ with $g \in$ Supp $\alpha y^{-1}$. Thus $g \in p$-Quot $\alpha$. Finally $H = \Delta(H)$ so $[H : \mathbf{C}_H(g)] < \infty$ and $[G_1 : \mathbf{C}_{G_1}(g)] < \infty$. This implies that $x^n \in \mathbf{C}_{G_1}(g)$ for some integer $n \geq 1$ and $x \in \sqrt{\mathbf{C}_G(g)}$.

Now suppose that $x$ has infinite order modulo $H$. Then $x$ has infinite order and the map $\sigma^i \to x^i$ affords an isomorphism of the two infinite cyclic groups $\langle \sigma \rangle$ and $\langle x \rangle$. Since $\langle x \rangle$ acts on $H$ we then obtain a natural action of $\langle \sigma \rangle$ on $H$. Observe that for all integers $s \in \Lambda!$, $x^s$ has infinite order modulo $H$ and thus Lemma 21.3 of [3] implies that $JK[G] \cap K[H]$ is a $\sigma$-nil ideal of $K[H]$. By Proposition 8, there exists $g \in p$-Quot $\alpha$ such that $\sigma^n$ centralizes $g$ for some $n \geq 1$. Since $\sigma$ and $x$ act the same way we have $x \in \sqrt{\mathbf{C}_G(g)}$ and the first result follows.

Finally if $G$ is solvable then the main result of [4] implies that for some $g \in p$-Quot $\alpha$ we have $[G : \mathbf{C}_G(g)] =$ l.f.

**Theorem B.** *Let* $K$ *be a field of characteristic* $p > 0$ *and let* $G$ *be a solvable group. Suppose* $H$ *is a normal* $\Delta$*-subgroup of* $G$*. Then* $JK[G] \cap K[H] \neq 0$ *if and only if* $H$ *has an element* $h$ *of order* $p$ *with* $[G : \mathbf{C}_G(h)] = l.f.$

**Proof:** Suppose $JK[G] \cap K[H] \neq 0$ and choose $\alpha \neq 0$ in this intersection. By Theorem A there exists $g \in p$-Quot $\alpha$ with $[G : \mathbf{C}_G(g)] =$ l.f. Let $h \in \langle g \rangle$ have order $p$. Then $\mathbf{C}_G(h) \supseteq \mathbf{C}_G(g)$ so $[G : \mathbf{C}_G(h)] =$ l.f.

Conversely suppose that there exists $h \in H$ of order $p$ with $[G : \mathbf{C}_G(h)] =$ l.f. Let $W = \langle h \rangle^H$. Then $W$ is a finite normal subgroup of $H$ whose order is divisible by $p$. We show that $JK[W] \subseteq JK[G]$. Since $JK[W] \neq 0$ and $JK[W] \subseteq K[H]$, this will yield $JK[G] \cap K[H] \neq 0$.

Since $W$ is finite, it clearly suffices to show that if $L$ is a finitely generated subgroup of $G$ with $L \supseteq W$ then $JK[W] \subseteq JK[L]$. Now by definition $[L : \mathbf{C}_L(h)] < \infty$ so since $\mathbf{C}_L(h)$ clearly normalizes $W$ we have $[L : \mathbf{N}_L(W)] < \infty$. Observe that $\mathbf{N}_L(W) \supseteq W$. Let $N$ denote the core of $\mathbf{N}_L(W)$ in $L$, that is, the intersection of all conjugates of $\mathbf{N}_L(W)$. Then $[L : N] < \infty$ and $N \triangleleft L$. Since $W \triangleleft L \cap$

$H \lhd L$ and $W \subseteq \mathbf{N}_L(W)$ we have $W \subseteq N$. Clearly $W \lhd N$. By Lemma 19.4 of [3], $JK[W] \subseteq JK[N]$ and, by Theorem 16.6 of [3], $JK[N] \subseteq JK[L]$. Thus $JK[W] \subseteq JK[L]$ and the result follows.

It is interesting to consider an example now. Let $A = Z_p$ be the cyclic group of order $p$ and let $B = \Pi Z_p$ be an infinite direct product of copies of $Z_p$. If $G = A \wr B$ and if $K$ has characteristic $p > 0$ then, by Theorem 21.6 of [3], $JK[G]$ is the augmentation ideal of $K[G]$. Now $G$ has a normal abelian subgroup $H \neq \langle 1 \rangle$ so clearly $JK[G] \cap K[H] \neq 0$. By Lemma 21.5(ii) of [3] we see that $H$ has no element $h$ of order $p$ with $[G : \mathbf{C}_G(h)] < \infty$. Thus the condition $[G : \mathbf{C}_G(h)] = \text{l.f.}$ in the above theorem cannot be replaced by the condition $[G : \mathbf{C}_G(h)] < \infty$.

**Theorem C.** *Let $K$ be a field of characteristic $p > 0$ and let $H \lhd G$ with $\Delta(H)$ solvable. Let $I$ be an ideal of $K[G]$ with $I \cap K[H]$ nilpotent. Suppose $\alpha \in I \cap K[H]$ with $\alpha \neq 0$. Then*

$$G = \bigcup_{g \in (p\text{-Quot}\ \alpha) \cap \Delta(H)} \sqrt{\mathbf{C}_G(g)}.$$

*Moreover, if $G$ is solvable then there exists $g \in (p\text{-Quot}\ \alpha) \cap \Delta(H)$ with $[G : \mathbf{C}_G(g)] = l.f.$*

**Proof.** Clearly $I \cap K[H]$ is $G$-invariant and hence so is $\theta(I \cap K[H])$ where $\theta$ is the projection map $\theta\colon K[H] \to K[\Delta(H)]$. By Lemma 20.1(ii) of [3], $\theta(I \cap K[H])$ is a nilpotent ideal of $K[\Delta(H)]$. Observe that $\Delta(H)$ is a normal solvable $\Delta$-subgroup of $G$.

Let $x \in G$ and let $\beta \in \theta(I \cap K[H])$. Then $\beta^{x^{is}} \in \theta(I \cap K[H])$ since $\theta(I \cap K[H])$ is $G$-invariant and hence $\beta\beta^{x^s}\beta^{x^{2s}} \cdots \beta^{x^{rs}} = 0$ for some $r \geq 1$ since the ideal is nilpotent. If $\langle \sigma \rangle$ is an infinite cyclic group we let $\langle \sigma \rangle$ act on $\Delta(H)$ by way of the homomorphism $\sigma^j \to x^j$. By the above $\theta(I \cap K[H])$ is a $\sigma$-nil ideal of $K[\Delta(H)]$.

Let $\alpha \in I \cap K[H]$ with $\alpha \neq 0$ and let $z \in \operatorname{Supp} \alpha$. Then $1 \in \operatorname{Supp} \alpha z^{-1}$ so $\theta(\alpha z^{-1}) \neq 0$. By Proposition 8 there exists $g \in p\text{-Quot}\ \theta(\alpha z^{-1})$ and an integer $n \geq 1$ such that $\sigma^n$ centralizes $g$. Thus $x \in \sqrt{\mathbf{C}_G(g)}$. Since

$$g \in p\text{-Quot}\ \theta(\alpha z^{-1}) \subseteq p\text{-Quot}\ \alpha z^{-1} = p\text{-Quot}\ \alpha$$

the first part follows.

Finally if $G$ is solvable then the main result of [4] implies that for some $g \in p\text{-Quot}\ \alpha$ we have $[G : \mathbf{C}_G(g)] = \text{l.f.}$

At this point it is apparent that a strengthening of some results of A. E. Zalesskiĭ in [5] will solve the semisimplicity problem for solvable groups. As an indication we offer the following

**Proposition 9.** *Let $K$ be a field of characteristic $p > 0$ and let $G$ be metanilpotent. Say $N$ is a normal nilpotent subgroup of $G$ with $G/N$ nilpotent and set $H = \Delta(N\Delta(G))$. Then $JK[G] \neq 0$ if and only if there exists an element $h \in H$ of order $p$ with $[G : C_G(h)] = l.f.$*

**Proof.** If such an element $h \in H$ exists then by Theorem B we have $JK[G] \cap K[H] \neq 0$ so $JK[G] \neq 0$.

Conversely let us assume that $JK[G] \neq 0$. Suppose $K[G]$ has a nonzero nilpotent ideal. Then by Theorem 20.2 of [3], $JK[G] \cap K[\Delta(G)] \neq 0$ so certainly $JK[G] \cap K[H] \neq 0$. On the other hand, if $K[G]$ has no nonzero nilpotent ideal, then by Theorem 2 of [5] we again have $JK[G] \cap K[H] \neq 0$. The result follows from Theorem B.

**Added in proof.** The semisimplicity problem for group rings of solvable groups has been solved. Indeed A. E. Zalesskiĭ has proved the following lovely result.

**Theorem.** *Let $G$ be a solvable group. Then $G$ has a normal $\Delta$-subgroup $H$ with the following property: if $I$ is a nonzero ideal of $K[G]$ then $I \cap K[H] \neq 0$.*

We remark that the subgroup $H$ above is a characteristic subgroup of $G$ given by a specific construction. It seems reasonable to call it the *Zalesskiĭ subgroup* and denote it by $\mathfrak{Z}(G)$. Then the above combines with our Theorem B to prove

**Theorem.** *Let $K$ be a field of characteristic $p > 0$ and let $G$ be a solvable group. Then $JK[G] \neq 0$ if and only if $\mathfrak{Z}(G)$ contains an element $h$ of order $p$ with $[G: C(h)] = l.f.$*

More recently Professor Zalesskiĭ has extended these arguments slightly to show that if $G$ is finitely generated and solvable then $JK[G] = NK[G]$. From this one obtains easily

**Theorem.** *Let $K$ be a field of characteristic $p > 0$ and let $G$ be a solvable group. Then*

$$JK[G] = JK[H] \cdot K[G]$$

*where $H$ is the characteristic locally finite subgroup of $G$ given by*

$$H = \{x \in G \mid [G: C(x)] = l.f. \text{ and } x \text{ has finite order}\}.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

*Current address* (C. R. Hampton): Department of Mathematics, College of Wooster, Wooster, Ohio 44691

*Current address* (D. S. Passman): Department of Mathematics, University of California, Los Angeles, California 90024