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CANONICAL NEIGHBORHOODS FOR TOPOLOGICALLY EMBEDDED POLYHEDRA

BY

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ABSTRACT. D. R. McMillan has shown that in any neighborhood of a compact two sided surface in a 3-manifold there is a closed neighborhood of the surface which is the sum of a solid homeomorphic to the cartesian product of the surface with the unit interval and some small disjoint cubes-with-handles each of which intersects the cartesian product in a disk on its boundary. In the present paper the author generalizes this notion of canonical neighborhood so that it applies to topological embeddings of arbitrary polyhedra in 3-manifolds. This is done by replacing the cartesian products by small regular neighborhoods of polyhedral approximations to the topological embeddings.

In [20] McMillan established the following result on neighborhoods of surfaces in 3-manifolds:

Theorem. *Suppose* S *is a compact, boundaryless, polyhedral surface,* M *is a pwl 3-manifold, and* f *is a homeomorphism of* S *onto a two sided surface in* Int M.

Then for each $\epsilon > 0$ *there is a pwl embedding* g *of* $S \times [0, 1]$ *into* Int M *and there is a finite collection* $\{H_i\}$ *of mutually exclusive, polyhedral cubes-with-handles in* Int M *such that*

(1) *for each* $y \in S$, $\rho(f(y), g(y, e)) < \epsilon$ $(e = 0, 1)$,

(2) *each* dia $H_i < \epsilon$ *and each* $H_i \cap g(S \times [0, 1])$ *is a disk in* Bd H_i, *and*

(3) $g(S \times [0, 1]) \cup \bigcup H_i$ *contains a neighborhood of* $f(S)$ *in* M.

In this paper we generalize McMillan's notion of a canonical neighborhood so that it applies to embeddings of arbitrary polyhedra in 3-manifolds. In our system regular neighborhoods of approximating polyhedra take the place of embeddings of cartesian products. In §8 we obtain existence and uniqueness theorems for canonical neighborhoods. In §9 we sharpen McMillan's theorem for surfaces by removing the compactness and no boundary assumptions and by showing that the diameters of the fibers of $g(S \times [0, 1])$ can be kept small.

Notation. Some of our conventions are taken from [8], [9], [10]; however,

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we have changed our use of several terms in this paper.

Manifold and *manifold-with-boundary* are taken as synonymous. A surface is a 2-manifold. If $Y \subseteq X$ then Fr Y denotes the set $(\mathrm{Cl}\ Y) \cap \mathrm{Cl}(X - Y)$. If we write that a function, say f, takes a pair (X, Y) into a pair (U, V) then we mean that $f^{-1}(U) = X$ and $f^{-1}(V) = Y$. We use ρ to denote a metric and $d(Y, Z)$ to denote the distance between sets Y and Z.

We use $C(X)$ to denote the set of continuous, nonnegative functions μ on X such that, for each $t > 0$, $\mu^{-1}([t, \infty))$ is compact. We use $C(X, Y)$ and $C(X, Y, Z)$ to denote respectively the subset of $C(X)$ consisting of the functions which are positive on Y and the subset of $C(X)$ consisting of the functions which are positive on Y and 0 on Z. We write $\mu_1 \le \mu_2$ if $\mu_1(x) \le \mu_2(x)$ for each x in the common domain of μ_1 and μ_2. Let $\mu \in C(X)$. A map f of a subset Y of X into X is a *μ-map* if, for each $y \in Y$, $\rho(y, f(y)) \le \mu(y)$. If f and g are maps of Y into X, then g is a μ-approximation to f provided gf^{-1} is a μ-map of $f(Y)$. A subset $Z \subseteq X$ is a μ-set if Z is contained in an open set $O(Z)$ with dia $O(Z) \le \inf \mu$ on $O(Z)$.

For μ_1 and $\mu_2 \in C(X)$ and a natural number k, we say that (X, μ_1, μ_2, k) has *Property* S if, for every sequence of points $x_1, \cdots, x_k$ such that each $\rho(x_i, x_{i+1}) \le \mu_2(x_i)$, $\mu_2(x_k) \le \mu_1(x_1)$ and $\mu_1(x_1) \le 2\mu_1(x_k)$. The composition of k μ-maps is not necessarily a $(k\mu)$-map, and the inverse of a μ-map is not necessarily a μ-map. The two conditions in the definition of Property S are used to get around these two facts.

We follow unpublished notes of M. H. A. Newman in using $\theta\binom{a_1 \cdots a_n}{b_1 \cdots b_n}$ to denote the map of $[a_1, a_n]$ onto $[b_1, b_n]$ which takes each a_i onto b_i and is affine on each subinterval $[a_i, a_{i+1}]$. If $Y \subseteq X$ then a *collar* on Y in X is a closed embedding ϕ: $Y \times [0, r] \to X$ for some $r \ne 0$ such that, for each $y \in Y$, $\phi(y, 0) = y$ and $\phi(Y \times [0, r])$ contains a neighborhood of Y. If X and Y are polyhedra then ϕ is a pwl *collar* provided the map ϕ is pwl. We say that ϕ is a *proper collar* provided that any homeomorphism h of $Y \times [0, r]$ onto itself which takes each fiber $y \times [0, r]$ onto itself and is the identity on $Y \times \{0, r\}$ induces a homeomorphism of X which is an extension of $\phi h \phi^{-1}$. If $\mu \in C(X)$ then ϕ is a *μ-collar* provided each $\phi(y \times [0, r])$ is a μ-set.

We emphasize that here polyhedra are not necessarily compact. We write $<$ to indicate a subcomplex or subpolyhedron, and we use $\lhd$ to indicate a full subcomplex. For a complex K, $|K|$ denotes the carrier of K, and if X is a set in $|K|$ then $[X]$ denotes the intersection of all subcomplexes of K whose carriers contain X, and $N(X, K)$ denotes the polyhedron underlying the intersection of all subcomplexes of K whose carriers contain neighborhoods of X. For a simplex s in a complex K, $\mathrm{lk}(s, K) = \{t\colon s * t \in K\}$, $\mathrm{st}(s, K) = s * \mathrm{lk}(s, K)$, and $\mathrm{st}^\circ(s, K) = |\mathrm{st}(s, K)| - |\dot{s} * \mathrm{lk}(s, K)|$. For a complex K, K_i denotes the i-skeleton of K. For a simplex s

of K, $b(s)$ denotes the barycenter of s, or if we are dealing with a derived subdivision of K, $d(s)$ denotes the vertex in $\operatorname{Int} s$. Here K^n denotes an nth derived subdivision of K, and for $L < K$, $(K, L)^n$ denotes an nth derived subdivision of K modulo L. In $(K, L)^1$ there is one vertex in $\operatorname{Int} s$ for each simplex s of $K - L$.

The following definitions on collapsing, although somewhat restrictive, are sufficient for our needs. First $\searrow^s$, $\searrow^e$, and $\searrow^c$ denote respectively simplicial collapse, elementary collapse, and collection of disjoint collapses with compact supports. We explain this last term more fully: $X \searrow^c Y$ means that X and Y are polyhedra with $Y \subset X$, $\mathrm{Cl}(X - Y)$ is a polyhedron, and if C is a component of $\mathrm{Cl}(X - Y)$ then C is compact and $C \searrow C \cap Y$. Finally, for polyhedra X and Y which are not necessarily compact, $X \searrow Y$ means that there is a finite sequence of collapses $X = X_0 \searrow^c X_1 \searrow^c \cdots \searrow^c X_k = Y$. For polyhedra X, Y, and Z, an elementary collapse $X \searrow^e Y$ is *admissible* with respect to Z if $(X - Y) \cap Z \neq \emptyset$ implies that $X - Y \subseteq Z$. Similarly, a collapse $X \searrow^c Y$ is *admissible* with respect to Z if for each component C of $\mathrm{Cl}(X - Y)$, the collapse $C \searrow C \cap Y$ can be made by way of a finite sequence of elementary collapses which are admissible with respect to Z. Admissibility of arbitrary collapses is defined in the obvious fashion. For $X \subseteq M$ and $\mu \in C(M)$, we say that a collapse $X \searrow Y$ is a *μ-collapse* if each component of $\mathrm{Cl}(X - Y)$ is a compact μ-set. We write $X \searrow 0$ if X collapses to a point.

A *triangulation* of a polyhedron P is a pair (J, ϕ) consisting of a rectilinear simplicial complex J in some Euclidean space together with a pwl homeomorphism ϕ of $|J|$ onto P. If $P(1), \cdots, P(k)$ are subpolyhedra of P then $(J, J(1), \cdots, J(k), \phi)$ *triangulates* $(P, P(1), \cdots, P(k))$ if each $J(i) < J$ and $\phi(|J(i)|) = P(i)$.

We follow Cohen [7] in defining *regular neighborhood* as follows: If $P < Q < R$ are polyhedra (not necessarily compact) then Q is a regular neighborhood of P in R if there is a triangulation (J, L, ϕ) of (R, P) in which $L \lhd J$ and there is a first derived subdivision J^1 of J such that $Q = \phi(N(L^1, J^1))$. If $\mu \in C(R)$ then Q is a *μ-regular neighborhood* of P in R if, for some triangulation (J, L, ϕ) as above, $\phi(N(t, J))$ is a μ-set for each $t \in L$.

Suppose P is a polyhedron, $P_a < P$, and f is a closed embedding of (P, P_a) into $(M, \mathrm{Bd}\, M)$. Suppose $\mu \in C(M, f(P))$. A *canonical neighborhood system* for (M, P, f, μ) is a triple $(g, N, \{H_i\})$ consisting of a pwl homeomorphism $g\colon (P, P_a) \to (M, \mathrm{Bd}\, M)$ which μ-approximates f, a μ-regular neighborhood N of $g(P)$ in M, and a discrete collection of polyhedral cubes-with-handles $\{H_i\}$ in M such that each H_i is a μ-set in $\operatorname{Int} M$ which intersects N in a disk and $N \cup \bigcup H_i$ contains a neighborhood of $f(P)$ in M. The polyhedron $N \cup \bigcup H_i$ will be called a *canonical neighborhood* for (M, P, f, μ).

Next we describe a notion of *reduction* of a complex. Reduction is used to alter a complex so that it becomes more like a combinatorial 3-manifold. We begin with complexes $K_a \lhd K$ and $L_a \lhd L$ and a pwl map $r\colon L \to K$ such that $r(L_a) = K_a$.

We say that (L, L_a, r) *reduces* (K, K_a) *at a set of* 2-*simplexes* $\{\Delta_i\}$ of K provided (1) $K < L$ and $K_a = L_a$, (2) each $\mathrm{lk}(\Delta_i, K) = \emptyset$ and each $\mathrm{lk}(\Delta_i, L)$ is a vertex w_i in $L_0 - K_0$ where $w_i \neq w_j$ for $i \neq j$, (3) $L = K \cup \bigcup w_i * \Delta_i$, and (4) r is a retraction which is affine on each simplex of L and sends each w_i to $b(\Delta_i)$.

We say that (L, L_a, r) is an *elementary reduction* of (K, K_a) *at a pair* (σ, S) where σ is a 1-simplex of K if (1) $\mathrm{lk}(\sigma, K)$ is a 1-manifold and S is a 0-sphere in $\dot{\mathrm{lk}}(\sigma, K) - K_a$ which does not bound in $\mathrm{lk}(\sigma, K)$, (2) $L = K \cup w * \sigma * S$ where w is a vertex in $L_0 - K_0$ and $L_a = K_a$, and (3) r is a retraction which is affine on each simplex of L and sends w to $b(\sigma)$.

We say that (L, L_a, r) is an *elementary reduction* of (K, K_a) *at a pair* (v, J) where v is a vertex of K and J is a 1-sphere in $\mathrm{lk}(v, K)$ provided (1) each component of $\mathrm{lk}(v, K)$ is either a punctured 2-sphere or a point, and $\mathrm{lk}(v, K) \cap K_a = \emptyset$ unless $v \in K_a$ in which case $\mathrm{lk}(v, K_a)$ is a 1-sphere in the boundary of some component of $\mathrm{lk}(v, K)$, (2) $\mathrm{lk}(v, K)$ is the sum of disjoint complexes E and F where J is in the boundary of some component of E and $J \cap K_a = \emptyset$, (4) $L = \{t \in K\colon v \notin t\} \cup v * E \cup u * F \cup v * u * J$ where $u \in L_0 - K_0$ and $L_a = K_a$, and (3) r is a simplicial map from L to K which sends u to v and leaves all other vertices fixed. Figure 1.1 illustrates an elementary reduction for a vertex.

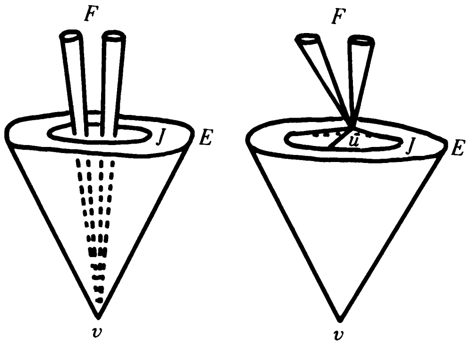

Figure 1.1

We say that (L, L_a, r) *reduces* (K, K_a) *at a set of simplexes* $\{t_k\}$ of K where the t_k's are either all 1-simplexes or all vertices if (1) the complexes $\mathrm{lk}(t_k, K)$ are mutually exclusive, (2) r is the identity on $|L| - \bigcup r^{-1}(\mathrm{st}^\circ(t_k, K))$, (3) for each t_k there is a sequence of complexes $(\mathrm{st}(t_k, K), \mathrm{st}(t_k, K) \cap K_a) = (L(k, 0), L_a(k, 0)), \cdots, (L(k, m), L_a(k, m)) = (r^{-1}(\mathrm{st}(t_k, K)), r^{-1}(\mathrm{st}(t_k, K) \cap K_a))$, there is a sequence $r_{k1}, \cdots, r_{km}$ of pwl maps $r_{ki}\colon L(k, i) \to L(k, i-1)$ such that $r|L(k, m) = r_{k1} \cdots r_{km}$ and each $(L(k, i), L_a(k, i), r_{ki})$ is an elementary reduction of $(L(k, i-1), L_a(k, i-1))$ at a pair $(t(k, i), S(k, i))$ where $S(k, i) < \mathrm{lk}(t_k, K)$

and $r_{k1} \cdots r_{ki}(t(k, i)) = t_k$, and (4) if the t_k's are 1-simplexes then each $\mathrm{lk}(t_k, L)$ is a 1-ball and if the t_k's are vertices then for each vertex w in $r^{-1}(\bigcup t_k)$, every nondegenerate component of $\mathrm{lk}(w, L)$ is a disk.

Finally let P and Q be polyhedra, r a pwl map from Q to P, M a pwl 3-manifold, f and g closed embeddings of P and Q respectively into M, and $\mu \in C(M)$. A *special map* for (M, f, g, r, μ) is a continuous μ-map π of M onto itself such that

(1) $\pi g = fr$,

(2) π takes $(M - g(Q))$ homeomorphically onto $M - f(P)$, and

(3) for each $\mu_1 \in C(M, \mu^{-1}(0, \infty))$ there is a homeomorphism H of M onto itself which μ_1-approximates π.

2. Some lemmas on Property S.

Lemma 2.1. *Suppose X is a locally compact metric space, $\mu \in C(X)$, and k is a natural number.*

Then there is a function $\mu_1 \in C(X, \mu^{-1}(0, \infty))$ such that (X, μ, μ_1, k) has Property S.

Proof. By taking infimums over functions defined for a locally finite collection of compact sets we can reduce the problem to the case where X is compact. For this case define μ_2 by

$$\mu_2(x) = \inf(\{1\} \cup \{\rho(x, y): y \notin \mu^{-1}(2^{-1/k}\mu(x), 2^{1/k}\mu(x))\}).$$

The compactness of X can be used to show that μ_2 is lower semicontinuous. By the Baire insertion theorem (see [14, p. 171]) there is a function $\mu_3 \in C(X, \mu^{-1}(0, \infty))$ with $\mu_3 \leq \mu_2$. Define μ_1 by $\mu_1(x) = \mathrm{minimum}(\mu_3(x), \mu(x)/2)$. It is easily verified that (X, μ, μ_1, k) has Property S.

Calculations involving Property S will almost always implicitly involve one of the next two lemmas which follow immediately from the definition of Property S.

Lemma 2.2. *Suppose $(X, \mu, \mu_1, k+1)$ has Property* S *and suppose $Y \subseteq X$.*

Then if g is a μ_1-homeomorphism of Y into X, g^{-1} is a 2μ-homeomorphism, and if $h_1, \cdots, h_k$ are μ_1-maps of sets $Y = Y_1, \cdots, Y_k$ where each $h_{i-1}(Y_{i-1}) \subseteq Y_i$, then $h = h_k \cdots h_1$ is a $(k\mu)$-map of Y into X.

Lemma 2.3. *Suppose S is a metric space and μ and $\mu_1 \in C(X)$ are such that $(X, \mu/6, \mu_1, 2)$ has Property* S.

Then if Y is a μ_1-set in X and f is a μ_1-map of Y into X, $f(Y)$ is a μ-set.

Lemma 2.4. *Suppose M is a (pwl) n-manifold, S is a locally flat (pwl) $(n-1)$-manifold in* Bd M *which is a closed subset of* Bd M, *and $\mu \in C(M, S)$.*

Then for any $r \neq 0$ there is a proper (pwl) μ-collar $\phi: S \times [0, r] \to M$ on S in M.

Proof. The topological version is obtained from Brown's collaring theorem [6] by cutting down the lengths of fibers. It follows from invariance of domain that the fibers can be cut back enough so that the collar is proper. The pwl case follows similarly from [7, 4.2].

3. **On μ-regular neighborhoods.** Here we use Property S to help identify μ-regular neighborhoods.

Lemma 3.1. *Suppose that* M *is a pwl manifold,* P *is a polyhedron in* M, *and* μ, μ_1, *and* $\mu_2 \in C(M)$ *are such that* $(M, \mu/2, 2\mu_1, 3)$ *and* $(M, \mu_1, \mu_2, 2)$ *have Property* S.

Then if N_1 *and* N_2 *are* μ_2*-regular neighborhoods of* P *in* M, *there is a pwl* μ*-homeomorphism* h *of* M *onto itself which is the identity on a polyhedral neighborhood of* P *so that* $h(N_1) = N_2$.

Proof. Without the μ's this lemma is a standard result in regular neighborhood theory. (See [7, 3.1 and 3.2] for example.) In the proof a third regular neighborhood N_3 of P is constructed and pwl homeomorphisms h_1 and h_2 of M onto itself are defined which are the identity on polyhedral neighborhoods of P and take N_1 and N_2 onto N_3. If (J_1, L_1, ϕ_1) and (J_2, L_2, ϕ_2) are the triangulations of (M, P) providing the μ_2-structures for N_1 and N_2 then we can suppose that for each i and each $t \in J_i$, h_i takes $\phi_i(t)$ onto itself and h_i is the identity on $\phi_i(t)$ if $t \cap L_i = \emptyset$; thus we can suppose that each h_i is a μ_2-map. But then by Lemma 2.2 each of h_1 and h_2^{-1} is a $2\mu_1$-map and so $h = h_2^{-1}h_1$ is a pwl μ-homeomorphism of M onto itself which takes N_1 onto N_2 and is the identity on a polyhedral neighborhood of P.

Lemma 3.2. *Suppose* M *is a pwl manifold,* P *is a polyhedron in* M, *and* μ *and* $\mu_1 \in C(M)$ *are such that* $(M, \mu/6, \mu_1, 2)$ *has Property* S.

If N *is a* μ_1*-regular neighborhood of* P *in* M *and* h *is a pwl* μ_1*-homeomorphism of* M *onto itself which is the identity on* P, *then* $h(N)$ *is a* μ*-regular neighborhood of* P *in* M.

Proof. This follows directly from Lemma 2.3.

Lemma 3.3. *Suppose that* M *is a pwl manifold,* $Y \subseteq M$, $\mu \in C(M, Y)$, *and* k *is a natural number.*

There is a function $\mu_1 \in C(M, Y)$ *such that if* Q *and* P *are polyhedra in* Y *with* $Q = Q_0 \searrow Q_1 \searrow \cdots \searrow Q_k = P$ *where each* $Q_{i-1} \searrow Q_i$ *is a* μ_1*-collapse admissible with respect to* Bd M *and if* N *is a* μ_1*-regular neighborhood of* Q *in* M, *then* N *is a* μ*-regular neighborhood of* P *in* M.

Proof. Use Lemma 2.1 to find $\mu_2, \cdots, \mu_5, \mu_1 \in C(M, Y)$ *so that* $(M, \mu/6, \mu_2, 2)$, $(M, \mu_2/2k, \mu_3, 2k+1)$, $(M, \mu_3/2, \mu_4, 2)$, $(M, \mu_4/2, 2\mu_5, 3)$, and $(M, \mu_5, \mu_1, 2)$ all have Property S. Then choose a triangulation $(J, \{J_i\}, \phi)$ of $(M, \{Q_i\})$ so that, for each simplex t of J, $\phi(t) \cap Q \neq \emptyset$ implies that $\phi(N(t, J))$ is a μ_1-set. From the proofs of Theorems 8.1 and 9.1 of [7] and Theorem 2 of [23] (each time a triangulation is mentioned make sure that the triangulation induces a subdivision of J) there correspond to each collapse $Q_{i-1} \searrow Q_i$ a pair of μ_1-regular neighborhoods N_{i-1} of Q_{i-1} and N_i' of Q_i and a pwl μ_1-homeomorphism h_{2i} of M onto itself which is the identity on Q_i and takes N_{i-1} onto N_i'. From Lemma 3.1 there is a pwl μ_4-homeomorphism h_1 of M onto itself which is the identity on Q and takes N onto N_0 and for each $i < k$ there is a pwl μ_4-homeomorphism h_{2i+1} of M onto itself which is the identity on Q_i and takes N_i' onto N_i. Set $h = h_{2k}h_{2k-1} \cdots h_1$. Each h_i^{-1} is a μ_3-homeomorphism so h^{-1} is a μ_2-homeomorphism. It follows from Lemma 3.2 that $N = h^{-1}(N_k')$ is a μ-regular neighborhood of P.

Lemma 3.4. *Suppose M is a pwl n-manifold, Q is a pwl sub-n-manifold which is closed in M and has nonempty frontier, $Q \cap \operatorname{Bd} M$ is either a pwl $(n-1)$-manifold in $\operatorname{Bd} Q$ or the empty set, and $\mu \in C(M)$.*

If λ: $(\operatorname{Fr} Q) \times [0, r] \to \operatorname{Cl}(M - Q)$ is a proper, pwl μ-collar on $\operatorname{Fr} Q$ in $\operatorname{Cl}(M - Q)$, then $Q \cup \operatorname{image} \lambda$ is a μ-regular neighborhood of Q.

Proof. It follows from the Newman theorem (see [27, Theorem 3] for example) that $\operatorname{Cl}(M - (Q \cup \operatorname{image} \lambda))$ is a pwl n-manifold. Use Lemma 2.4 to extend λ to λ' so that λ' takes $(\operatorname{Fr} Q) \times [0, r+1]$ into $\operatorname{Cl}(M - Q)$ as a proper, pwl μ-collar. Let (τ, ϕ) be a triangulation of $\operatorname{Fr} Q$ such that $\lambda'(\phi(N(t, \tau)) \times [0, r+1])$ is a μ-set for each $t \in \tau$. Let T be a subdivision of the cell complex $\tau \times [0, r+1]$ which adds no vertices (see [27, Lemma 1]), and let ψ: $|T| \to M$ be given by $\psi(x, t) = \lambda'(x, \theta(\begin{smallmatrix} 0 & \frac{1}{2} & 1 \\ 0 & r & r+1 \end{smallmatrix})(t))$. Extend (T, ψ) to a triangulation (L, η) of M in which $\eta(N(t, L))$ is a μ-set for each $t \in [\eta^{-1}(Q)]$. Note that $[\eta^{-1}(Q)]$ is automatically a full subcomplex K of L. Then $\eta(N(K^1, L^1)) = Q \cup \operatorname{image} \lambda$ where L^1 is the first barycentric subdivision of L. Thus $Q \cup \operatorname{image} \lambda$ is a μ-regular neighborhood of Q.

Corollary 3.5. *Suppose M is a pwl n-manifold, $Y \subseteq M$, $\mu \in C(M, Y)$, and k is a natural number.*

There is a function $\mu_1 \in C(M, Y)$ such that if P and N are polyhedra in Y where N is a pwl n-manifold containing a neighborhood of P, $N \cap \operatorname{Bd} M$ is either a pwl $(n-1)$-manifold or the empty set, and $N = N_0 \searrow \cdots \searrow N_k = P$ is a sequence of k μ_1-collapses admissible with respect to $\operatorname{Bd} M$, then N is a μ-regular neighborhood of P.

Proof. Choose μ_1 from Lemma 3.3 corresponding to M, Y, μ, and k. Let $N = N_0 \searrow \cdots \searrow N_k = P$ be given as in the hypothesis. Choose $\mu_2 \in C(M, Y)$ so small that $\mu_2 \leq \mu_1$ and so that if h is a μ_2-homeomorphism of M onto itself then each of the collapses $h(N_{i-1}) \searrow h(N_i)$ is still a μ_1-collapse. The Newman theorem cited in the preceding proof shows that $\mathrm{Cl}(M - N)$ is a pwl n-manifold and that $\mathrm{Fr}\, N$ is a pwl $(n-1)$-manifold. Use Lemma 2.4 to find a pwl embedding $\lambda: (\mathrm{Fr}\, N) \times [-1, 2] \to M - P$ so that each $\lambda(y \times [-1, 2])$ is a μ_2-set, $\lambda \mid (\mathrm{Fr}\, N) \times [-1, 0]$ is a proper collar on $\mathrm{Fr}\, N$ in $\mathrm{Cl}(M - N)$, and $\lambda \mid (\mathrm{Fr}\, N) \times [0, 2]$ is a proper collar on $\mathrm{Fr}\, N$ in N.

Define a pwl μ_2-homeomorphism h of M onto itself by $h = I$ on $M -$ image λ and $h\lambda(y, t) = \lambda(y, \theta(\begin{smallmatrix}-1 & 0 & 2\\ -1 & 1 & 2\end{smallmatrix})(t))$. Set $Q = h(N)$. By Lemma 3.4, N is a μ_2-regular neighborhood of Q, and by Lemma 3.3 and the conditions on μ_2, N is a μ-regular neighborhood of P.

Lemma 3.6. *Suppose M is a pwl n-manifold, Q is a polyhedron in M, and $\mu \in C(M, Q)$.*

Then there are μ-regular neighborhoods of Q, and if N is any μ-regular neighborhood of Q, then $N = N_0 \searrow \cdots \searrow N_n = Q$ by a sequence of n μ-collapses each of which is admissible with respect to Bd M.

Proof. The existence of μ-regular neighborhoods follows from [7] or [23] and the fact there are triangulations of neighborhoods of Q in M which partition the neighborhoods into appropriately small sets.

Let N be a μ-regular neighborhood of Q in M with $N = \phi(N(K^1, J^1))$ where (J, K, ϕ) is a triangulation of (M, Q) with $K \lhd J$, J^1 is a first derived subdivision of J, and for each $t \in K$, $\phi(N(t, J))$ is a μ-set. Define another first derived subdivision J' of J as in [27, Chapter 3] by first defining a simplicial map $\lambda: J \to [0, 1]$ with $\lambda(v) = 0$ if $v \in K_0$ and $\lambda(v) = 1$ if $v \in J_0 - K_0$. Choose J' so that if σ is a simplex of J with $\lambda(\sigma) = [0, 1]$ then $d(\sigma) \in \lambda^{-1}(1/2)$.

Let J'' be the first barycentric subdivision of J' and K'' the induced subdivision of K. Let $\{t_i^j\}$ denote the j-simplexes of K. Set $D_j = \bigcup_i N(d(t_i^j), K'')$, and for each simplex t of K let $t(j) = D_j \cap t$. Set $N_0' = N(K', J')$ and $N_k' = \phi^{-1}(Q) \cup (N_0' \cap (\bigcup\{t(j) * s\colon t \in K,\ \lambda(s) = 1,\ t * s \in J, \text{ and } j < n - k\}))$. We claim that each N_{k-1}' collapses to N_k' by a collapse that is admissible with respect to $\phi^{-1}(\mathrm{Bd}\, M)$. To see this look at a simplex $s * t$ of J with $\lambda(s) = 1$ and $\lambda(t) = 0$. Then for $j = n - k$, $C = (t(j) * s) \cap N_0' \subseteq N_{k-1}'$ is a pwl cell. Furthermore, $C \cap N_k'$ is a collapsible polyhedron in Bd C, so $C \searrow C \cap N_k'$. An induction argument now shows that $N_{k-1}' \searrow N_k'$ and that this collapse is admissible with respect to $\phi^{-1}(\mathrm{Bd}\, M)$.

Let $\eta: J' \to J^1$ be a simplicial isomorphism which is the identity on J_0. Set $N_k = \phi\eta(N_k')$ $(k = 0, \cdots, n)$. Each $N_{k-1} \searrow N_k$ is a μ-collapse which is admissible with respect to Bd M, and $N_n = Q$.

4. **Extending the Lininger-Hosay theorem.** Here we use a result of Lister's [18] to generalize the theorem of Lininger and Hosay's [15], [17] on re-embedding crumpled cubes in E^3. The trick of using [18] to simplify the proof of the

re-embedding theorem was observed by Daverman in [12].

Lemma 4.1. *Suppose S is a 2-sphere in E^3, D is a disk on S such that* $\mathrm{Cl}(S - D)$ *is tame, and U is a connected open set in E^3 such that* $U \cap S = \mathrm{Int}\, D$ *and $U - S$ has two components—U_0 and U_1.*

There is a homeomorphism g of S onto a tame 2-sphere in E^3 and there is a homeomorphism h of $E^3 - U_1$ into E^3 such that

(1) $h \mid E^3 - U = I$,

(2) $g \mid S - \mathrm{Int}\, D = I$,

(3) $g(\mathrm{Int}\, D) \subseteq U$, *and*

(4) $h(E^3 - U_1) \cap g(\mathrm{Int}\, D) = \emptyset$.

Proof. There is no loss in supposing that $U_0 \subseteq \mathrm{Ext}\, S$ and $\mathrm{Fr}\, U$ is connected. Since E^3 is unicoherent the second condition can be achieved by cutting down U to a smaller open set if necessary. Define $\mu \in C(S)$ by $\mu(x) = d(x, E^3 - U)/2$. From [18, Theorem 5] there is a homeomorphism f of S into E^3 and there is a null sequence $E_1, \cdots, E_i, \cdots$ of mutually exclusive disks in Int D so that (1) $\bigcup E_i$ is closed in Int D, (2) f μ-approximates the identity, (3) $f(\mathrm{Int}\, D)$ is locally polyhedral, (4) $f((\mathrm{Int}\, D) - \bigcup E_i) \subseteq U_1$, and (5) $f(\mathrm{Int}\, D) \cap S = \bigcup (f(\mathrm{Int}\, E_i) \cap \mathrm{Int}\, E_i)$. By [3, Theorem 8.5], $f(S)$ is tame, so $B = \mathrm{Cl}\,\mathrm{Int}\, f(S)$ is a 3-cell.

For each i, set $F_i = f(E_i)$. We can suppose that each F_i is a polyhedron. Let G_i denote the sum of all components of $B \cap \mathrm{Cl}\,\mathrm{Ext}\, S$ which intersect F_i. Because $\mathrm{Fr}\, G_i \subseteq U$ and $\mathrm{Fr}\, U$ is connected, $G_i \subseteq \mathrm{Cl}\, U_0$. It follows as in [12] that the G_i's are mutually exclusive.

For each F_i let B_i be a 3-cell in $F_i \cup \mathrm{Int}\, B$ such that $B_i \cap f(S) = F_i$, $G_i \subseteq B_i$, and $G_i \cap \mathrm{Bd}\, B_i \subseteq F_i$. Let $C_1, \cdots, C_i, \cdots$ be a null sequence of mutually exclusive polyhedral cubes in $B \cap U$ so that each $C_i \cap f(S) = F_i$. For each B_i let ϕ_i be a homeomorphism of B_i onto C_i which is the identity on F_i.

Define g to be f on $S - \bigcup E_i$ and let g take each E_i onto $(\mathrm{Bd}\, C_i) - \mathrm{Int}\, F_i$. Set $h(x) = x$ for $x \in \mathrm{Cl}\,\mathrm{Ext}\, f(S)$ and $h(x) = \phi_i(x)$ for $x \in G_i$. Just as in [12], g and h satisfy conditions (1)–(4).

Lemma 4.2. *Suppose M is a connected 3-manifold without boundary, S is a connected boundaryless surface in M which is a closed subset of M, and S separates M into two components—V_0 and V_1.*

Suppose $\mu \in C(M, S)$.

Then there is a μ-homeomorphism g of S onto a tame surface in M and there is a μ-homeomorphism h of $\mathrm{Cl}\, V_0$ *into M so that* $g(S) \cap h(\mathrm{Cl}\, V_0) = \emptyset$.

Proof. First cover S with a collection of open 3-cells in M. Then use [5, Theorem 6.3] and [4, Theorem 5] to find a curvilinear triangulation T of S with tame 1-skeleton T_1 and 2-simplexes $\{\Delta_i\}$ so that each $\Delta_i \subseteq S_i \subseteq N_i$ where S_i is a 2-sphere and N_i is an open 3-cell which is a $\mu/2$-set. From [5] we can assume that each S_i is locally tame modulo Δ_i. Use [8, Lemma 2.1] to find for each Δ_i a connected open set U_i in N_i such that Int $\Delta_i \subseteq U_i$, $S_i \cap \mathrm{Cl}\, U_i = \Delta_i$, and $U_i - \Delta_i$ has exactly two components. Choose the U_i's so that they are mutually exclusive. For each U_i set $U_{i0} = U_i \cap V_0$ and $U_{i1} = U_i \cap V_1$.

Use Lemma 4.1 to find for each Δ_i a homeomorphism f_i of S_i into N_i and a homeomorphism h_i of $N_i - U_{i1}$ into N_i such that $f_i = I$ on S − Int Δ_i, $f_i(\mathrm{Int}\, \Delta_i)$ is locally tame, $h_i = I$ on $N_i - U_i$, and $h_i(N_i - U_{i1})$ fails to intersect $f_i(\mathrm{Int}\, \Delta_i)$. Set $h = I$ on $M - \bigcup U_i$ and $h = h_i$ on each Cl U_{i0}. Set $f = f_i$ on each Δ_i. From [5, Theorem 3.1] and [1], [22], $f(S)$ is tame. The condition on the diameters of the N_i's shows that h is a μ-homeomorphism and that f is a $\mu/2$-approximation to the identity.

Since each $f(\Delta_i)$ separates U_i it follows that $f(S)$ separates M into components W_0 and W_1 where W_0 is the component which contains the connected set $h(M - (V_1 \cup T_1))$. By Lemma 2.4 we can push $f(S)$ off into W_1 to get a μ-homeomorphism g of S onto a tame surface in W_1. This completes the proof of the theorem.

We can relax somewhat the requirement that S have no boundary:

Corollary 4.3. *Suppose M is a connected 3-manifold, S is a connected surface in M with $S \cap \mathrm{Bd}\, M = \mathrm{Bd}\, S$ and S is closed in M, and suppose S separates M into two components—V_0 and V_1.*

Suppose $\mu \in C(M, S)$.

Then there is a μ-homeomorphism g of $(S, \mathrm{Bd}\, S)$ onto a tame pair in $(M, \mathrm{Bd}\, M)$ and there is a μ-homeomorphism h of $(\mathrm{Cl}\, V_0, (\mathrm{Bd}\, M) \cap \mathrm{Cl}\, V_0)$ into $(M, \mathrm{Bd}\, M)$ so that $g(S) \cap h(\mathrm{Cl}\, V_0) = \emptyset$.

Proof. Let $S' \subseteq M'$ be the doubles of S and M along Bd S and Bd M. Here M' is obtained by attaching two copies of M together with the identity homeomorphism on Bd M. Extend μ to $\mu' \in C(M', S')$. Carry out the constructions in the proof of Lemma 4.2 for S' in M' to get f', g', and h' in place of f, g, and h with the following modifications: (1) Require that Bd S be contained in T_1, the tame 1-skeleton of the curvilinear triangulation of S'; (2) choose the U_i's so that each $U_i \cap \mathrm{Bd}\, M = \emptyset$; and (3) in defining g' from f' by pushing off T_1 require that $(g')^{-1}(\mathrm{Bd}\, M) = \mathrm{Bd}\, S$. Then for h and g just take $h' \mid M$ and $g' \mid S$.

The theorem which follows, the promised extension of the Lininger-Hosay theorem, is obtained from Lemma 4.3 in much the same way as Theorem 2 is established in [17]. We give only a sketch of the proof.

Theorem 4.4. *Suppose M is a connected 3-manifold, S is a connected surface in M with $S \cap \mathrm{Bd}\, M = \mathrm{Bd}\, S$ and S is a closed subset of M, and suppose S separates M into two components—V_0 and V_1.*

Suppose $\mu \in C(M, S)$.

Then there is a μ-*homeomorphism* h *of* $(M - V_1, (\mathrm{Bd}\, M) - V_1)$ *into* $(M, \mathrm{Bd}\, M)$ *such that* $h(S)$ *is collared in* $M - h(V_0)$.

Proof. By combining Corollary 4.3 with [9, Theorem 9.2] it is possible to define inductively a sequence of re-embeddings h_i $(i = 1, 2, \cdots)$ of $(\mathrm{Cl}\, V_0, (\mathrm{Bd}\, M) \cap \mathrm{Cl}\, V_0)$ into $(M, \mathrm{Bd}\, M)$, a sequence of homeomorphisms g_i $(i = 1, 2, \cdots)$ of $(S, S \cap \mathrm{Bd}\, M)$ onto tame pairs in $(M, \mathrm{Bd}\, M)$, and an embedding λ of $(S \times [0, 1], (S \cap \mathrm{Bd}\, M) \times [0, 1])$ into $(M, \mathrm{Bd}\, M)$ so that (1) $h_i \to h$ a μ-homeomorphism of $\mathrm{Cl}\, V_0$ into M, (2) $g_i \to h \mid S$ and $g_i(S) \cap h(\mathrm{Cl}\, V_0) = \emptyset$ for each i, and (3) $\lambda(y, 1/i) = g_i(y)$ $(y \in S,\ i = 1, 2, \cdots)$. But this says that $h(S)$ is collared in $M - h(V_0)$.

5. **Some combinatorial preliminaries.** We omit a proof of the first lemma here. It is easily obtained by building up a homeomorphism over blocks of convex linear cells.

Lemma 5.1. *Suppose* $u * v * S = u * B$ *is a combinatorial* n-*ball where* u *and* v *are vertices and* S *is a combinatorial* $(n-2)$-*sphere. Let* r *denote the simplicial retraction* $u * B \to B$ *which sends* u *to* v.

Then there is a pwl homeomorphism ϕ *of* $|u * B| - |S|$ *onto* $|\mathrm{Int}\, B| \times [0, 1]$ *such that, for each* $y \in \mathrm{Int}\, B$, $\phi r^{-1}(y) = y \times [0, 1]$ *and* $\phi(y) = (y, 0)$.

Lemma 5.2. *Suppose* $K_a \triangleleft K$ *and* $L_a \triangleleft L$ *are complexes,* r *is a simplicial map from* L *to* K, v *is a vertex of* K, *and* S *is a* 1-*sphere in* $\mathrm{lk}(v, K)$ *such that* (L, L_a, r) *is an elementary reduction of* (K, K_a) *at* (v, S).

If H *and* J *are finite subcomplexes of* K *containing* v *such that* $H \searrow^s J$ *by a collapse that is admissible with respect to* K_a, *then* $r^{-1}(H) \searrow^s r^{-1}(J)$ *by a collapse that is admissible with respect to* L_a. *Furthermore,* $r^{-1}(v) \searrow^s 0$ *by a collapse that is admissible with respect to* L_a.

Proof. First collapse $r^{-1}(H) \searrow^s H(0) \cup r^{-1}(J)$ where $H(0) = [|r^{-1}(H)| - (|v * u * S| - |v * S|)]$ and u is the new vertex added to obtain L. Do this by collapsing simplexes $v * u * r^{-1}(t)$ across free faces $u * r^{-1}(t)$ in order of decreasing dimension for simplexes t of $S \cap (H - J)$. Then for each simplex $t \in H$ with $t \neq v$, there is a unique simplex $\tilde{t}$ in $H(0)$ which is taken isomorphically onto t so $H \searrow^s J$ induces a collapse $H(0) \cup r^{-1}(J) \searrow^s r^{-1}(J)$ via r^{-1}. It is clear that this collapse is admissible with respect to L_a and that $r^{-1}(v) \searrow^s 0$ by a collapse that is admissible with respect to L_a.

6. **Splitting a surface.** Theorem 6.1 is the key to reducing the canonical neighborhood problem to one that can be solved using McMillan's techniques. It says roughly that a two sided surface S in a 3-manifold can be split open and

blown up into $S \times [0, 1]$. McMillan uses this idea in [21] in the proof of Theorem 2 when he constructs the auxiliary manifold M^*. Daverman and Eaton [13] have sketched out a weaker version of this theorem for interiors of disks.

Theorem 6.1. *Suppose* $P_a < P$ *and* $Q_a < Q$ *are polyhedra,* r *is a pwl map from* (Q, Q_a) *to* (P, P_a), S *is a connected polyhedral surface in* P *with* $S \cap P_a =$ Bd S, *and* ϕ *is a pwl homeomorphism from* $(S \times [0, 1], (\mathrm{Bd}\, S) \times [0, 1])$ *to a polyhedral pair* $(C, C_a) < (Q, Q_a)$ *so that, on* C, $r\phi$ *is the projection of* $S \times [0, 1]$ *onto* S. *Suppose that* r *takes* $Q - C$ *homeomorphically onto* $P - S$.

Suppose M *is a 3-manifold and* $f\colon (P, P_a) \to (M, \mathrm{Bd}\, M)$ *is a closed embedding, and suppose there is a connected open set* W *in* M *containing* $f(S)$ *such that* $W - S$ *has two components—*W_0 *and* W_1 *with* $\mathrm{Cl}(r^{-1}f^{-1}(W_i)) \cap C \subseteq S_i = \phi(S \times i)$ $(i = 0, 1)$.

Suppose $\mu \in C(M, f(S))$.

Then there is an embedding $g\colon (Q, Q_a) \to (M, \mathrm{Bd}\, M)$ *and there is a special map* π *for* (M, f, g, r, μ).

Proof. The idea of the proof is to use Theorem 4.4 to obtain from f embeddings of $f^{-1}(\mathrm{Cl}\, W_i)$ $(i = 0, 1)$ whose boundaries are collared on one side and then to use [10, Theorem 8.2] to separate the two copies of $f(S)$ so that the collars attach along corresponding endpoints. See Figure 6.1.

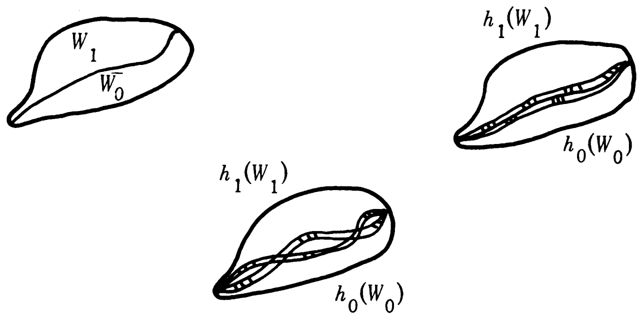

Figure 6.1

Let U be a connected open set in M such that $S \subseteq U \subseteq W$ and $X_i = (\mathrm{Fr}\, U) \cap W_i \neq \emptyset$ $(i = 0, 1)$. Set $U_i = U \cap W_i$ $(i = 0, 1)$. Let μ_1 and $\mu_2 \in C(M, f(S), M - U)$ be such that $(M, \mu_1, 2\mu_2, 2)$ and $(M, \mu_1, \mu_2, 2)$ have Property S. Substitute $(U \to M, S \to K, \mathrm{Bd}\, S \to K_a, f \to f, \mu_2 \to \mu)$ in Theorem 8.2 of [10] to find an associated $\nu \in C(S, S)$. Use Theorem 4.4 to find μ_2-homeomorphisms h_0 of $M - U_1$ and h_1 of $M - U_0$ so that $h_i f(S)$ is locally collared in $M - h_i(U_i)$ and $\rho(f(x), h_i f(x)) < \nu(x)$ $(x \in S,\ i = 0, 1)$. Set $S_{1/2} = \phi(S \times \frac{1}{2})$, and set $C_i = \phi(S \times [i, \frac{1}{2}])$ $(i = 0, 1)$. (Here $[1, \frac{1}{2}] = [\frac{1}{2}, 1]$.) From Lemma 2.4 there are homeomorphisms g_i of C_i into $U - h_i(U_i)$ $(i = 0, 1)$ so that (1) $g_i(C_i - S_i)$ is locally tame, (2) for each $x \in S_i$, $g_i(x) = h_i fr(x)$, (3) for each $x \in S$, $g_i(r^{-1}(x) \cap C_i)$ is a $\mu_2 f(x)$-set, and (4) for each $x \in C_i$, $\rho(g_i(x), fr(x)) < \nu r(x) \leq \mu_2(fr(x))$.

Define homeomorphisms f_i of S into U $(i = 0, 1)$ by $f_i(x) = g_i(r^{-1}(x) \cap S_{1/2})$. For each $x \in S$, $\rho(f(x), f_i(x)) < \nu(x)$ by (4) in the preceding paragraph. Now $f_i(S)$ is tame in U by [1], [22] so by our application of [10] there is a μ_2-homeomorphism h of M such that $hf_0 = f_1$.

Note that $f_1(S)$ separates X_0 from X_1 in Cl U because $h_1(\text{Cl } U_1) \cup g_1(C_1 - S_{1/2})$ is a sum of components of (Cl U) $- f_1(S)$ which contains X_1 and misses X_0. Furthermore, $h_1(\text{Cl } U_1) \cap f_1(S) = hh_0(\text{Cl } U_0) \cap f_1(S) = \emptyset$. Thus $h_1(\text{Cl } U_1) \cap hh_0(\text{Cl } U_0) = \emptyset$, and g is a homeomorphism where g is defined by $g = fr$ on $Q - r^{-1}f^{-1}(U)$, $g = hh_0 f$ on $r^{-1}f^{-1}(U_0)$, $g = h_1 f$ on $r^{-1}f^{-1}(U_1)$, $g = hg_0$ on C_0, and $g = g_1$ on C_1. Notice that $U - g(C) = hh_0(U_0) \cup h_1(U_1)$.

Define π by $\pi = I$ on $M - U$, $\pi = h_0^{-1}h^{-1}$ on $hh_0(\text{Cl } U_0)$, $\pi = h_1^{-1}$ on $h_1(\text{Cl } U_1)$, and $\pi = frg^{-1}$ on $g(C)$.

If $x \in h_1(\text{Cl } U_1)$ with $x = h_1(y)$ then $\rho(x, \pi(x)) = \rho(x, y) \leq \mu_2(y) \leq \mu(x)$. If $x \in hh_0(\text{Cl } U_0)$ then $x = hh_0(y)$ and $\pi(x) = y$. Now $\rho(y, h_0(y)) \leq \mu_2(y) \leq \mu_1(h_0(y)) \leq \mu(x)/2$, and $\rho(h_0(y), x) \leq \mu_2(h_0(y)) \leq \mu_1(x) \leq \mu(x)/2$. Thus $\rho(x, \pi(x)) \leq \mu(x)$. If $x \in g(C)$, then $x = g(y)$ and $\pi(x) = fr(y)$. By an analysis similar to the preceding one we find that $\rho(x, \pi(x)) \leq \mu(x)$ for $x \in g(C)$.

It remains to show that π can be approximated arbitrarily closely by a homeomorphism. Since point inverses $\pi^{-1}(x)$ are cellular [19, Theorem 6] this actually follows from work of Voxman's [25, Theorem 1] (Bd M presents no real problem); however it is instructive to look at a direct proof.

Let $\lambda \in C(M, \mu^{-1}(0, \infty))$. Choose λ_1 and $\lambda_2 \in C(M, \mu^{-1}(0, \infty))$ so that, for $x \in M$, $\lambda_1(x) \leq \inf \lambda \mid \pi^{-1}(x)$ and so that $(M, \lambda_1, 2\lambda_2, 2)$ has Property S. Repeat the construction of g and π with λ_2 in place of μ to get λ_2-homeomorphisms h_0' of $M - U_1$, h_1' of $M - U_0$, h' of M, a homeomorphism g' of Q and a λ_2-map π' such that $\pi' g' = fr$. Define an approximation H to π by $H = I$ on $M - U$, $H = h_1'h_1^{-1}$ on $h_1(\text{Cl } U_1)$, $H = h'h_0'(hh_0)^{-1}$ on $hh_0(\text{Cl } U_0)$, and $H = g'g^{-1}$ on $g(C)$. Clearly, H is a homeomorphism. If $x \in h_1(\text{Cl } U_1)$, then $x = h_1(y)$ and $H(x) = h_1'(y) = h_1'\pi(x)$. Thus $\rho(H(x), \pi(x)) \leq \lambda_2(y) \leq \lambda_1(y) \leq \lambda(x)$ by the definition of λ_1. If $x \in hh_0(\text{Cl } U_1)$, then $x = hh_0(y)$, and $H(x) = h'h_0'(y) = h'h_0'\pi(x)$. Now $\rho(h'h_0'\pi(x), \pi(x)) \leq \lambda_1(x)$ since $h'h_0'$ is a λ_1-homeomorphism. Thus $\rho(h'h_0'h_0^{-1}h^{-1}(x), \pi(x)) \leq \lambda(x)$. For $x \in g(C)$ we have $x = g(y)$ and $\pi(x) = gr(y)$. Now $\rho(H(x), \pi(x)) = \rho(g'(y), \pi'g'(y)) \leq \lambda_2(g'(y)) \leq \lambda_1(\pi'g'(y)) = \lambda_1(\pi(x)) \leq \lambda(x)$.

For our work here we need to allow $f^{-1}(\text{Bd } M)$ to contain points in Int S. The corollary which follows enables us to do this:

Corollary 6.2. *Suppose $P_a < P$ and $Q_a < Q$ are polyhedra, r is a pwl map from Q to P which takes Q_a onto P_a, S is a connected polyhedral surface in P*

with Bd $S = \emptyset$, *and* $\phi: S \times [0, 1] \to C \subseteq Q$ *is a pwl homeomorphism such that* $r\phi$ *is the projection of* $S \times [0, 1]$ *on* S. *Suppose* r *takes* $Q - C$ *homeomorphically onto* $P - S$. *Set* $S_i = \phi(S \times i)$ $(i = 0, 1)$. *Suppose* $Q_a \cap C = S_1 \cap r^{-1}(P_a)$.

Suppose M *is a pwl* 3-*manifold*, $f: (P, P_a) \to (M, \mathrm{Bd}\, M)$ *is a closed embedding, and suppose* $\mu \in C(M, f(S))$.

Let M' *denote the sum of* M *and* $(\mathrm{Bd}\, M) \times [0, 1]$ *with the identification* $y = (y, 0)$ *on* Bd M. *Suppose that there is a connected open set* W *in* M' *containing* $f(S)$ *which is separated by* $f(S)$ *into two components—*W_0 *and* W_1 *with* $W_0 \subseteq \mathrm{Int}\, M$ *and* $(\mathrm{Cl}\, r^{-1}f^{-1}(W_i)) \cap C \subseteq S_i$ $(i = 0, 1)$. *Suppose that for each* $x \in P_a \cap \mathrm{Cl}(S - P_a)$ *there is a subpolyhedron* $P(x)$ *in* $f^{-1}(\mathrm{Cl}\, W_1)$ *such that* $P(x)$ *contains a neighborhood of* x *in* S, $P(x)$ *has no local cut points*, $P_a(x) = P(x) \cap P_a$ *has no point components, and* $P(x) - P_a(x)$ *is a* 3-*manifold.*

Then there is a closed embedding $g: (Q, Q_a) \to (M, \mathrm{Bd}\, M)$ *and there is a special map* π *for* (M, f, g, r, μ) *which is the identity on* Bd M.

Proof. Extend the metric on M to one on M', and extend μ to a function in $C(M')$. Replace M by M' in the proof of Theorem 6.1 and carry out the constructions there. Since $\pi \mid h_1(M' - \mathrm{Int}\, M) = h_1^{-1} \cdots$, it is sufficient to show that h_1 can be chosen so that it is the identity on $M' - \mathrm{Int}\, M$.

Following the notation in the proof of Theorem 6.1, define a μ_2-homeomorphism h_3 of $M' - U_0$ such that $h_3 \mid M' - M = I$, $h_3 f(S - P_a)$ is locally collared in $M' - h_3(U_1)$, and $\rho(f(x), h_3 f(x)) < \nu(x)$ $(x \in S)$. We claim that $h_3 f(S)$ is locally collared in $M - h_3(U_1)$. The only points where this could fail to be so are points in $f(P_a \cap \mathrm{Cl}(S - P_a))$. Let $x \in f(P_a \cap \mathrm{Cl}(S - P_a))$. Then from [10, Lemma 10.1], $h_3(P(x))$ is locally tame at x and so $h_3 f(S)$ is locally collared at x in $M - h_3(U_1)$.

This shows that we could have used h_3 in place of h_1 and the proof of the corollary is complete.

Theorem 6.3. *Suppose* K *is a complex in* E^7, $K_a \lhd K$, f *is a closed embedding of* $(|K|, |K_a|)$ *into* $(M, \mathrm{Bd}\, M)$ *where* M *is a* 3-*manifold, and* s *is a* 0-, 1-, *or* 2-*simplex of* K. *If* s *is a* 1-*simplex suppose that* $\mathrm{lk}(s, K)$ *has more than one component, each component of* $\mathrm{lk}(s, K)$ *is an arc, and* $\mathrm{lk}(s, K) \cap K_a = \emptyset$ *unless* $s \in K_a$ *in which case* $\mathrm{lk}(s, K_a)$ *is a* 0-*sphere in* $\dot{\mathrm{l}}\mathrm{k}(s, K)$. *If* s *is a vertex suppose that* $\mathrm{lk}(s, K)$ *has some nonsimply connected component, each nondegenerate component of* $\mathrm{lk}(s, K)$ *is a punctured* 2-*sphere, and* $\mathrm{lk}(s, K) \cap K_a = \emptyset$ *unless* $s \in K_a$ *in which case* $\mathrm{lk}(s, K_a)$ *is a* 1-*sphere in the boundary of one of the components of* $\mathrm{lk}(s, K)$.

Suppose $\mu \in C(M, f(\mathrm{st}^\circ(s, K)))$ *and* O *is an open convex set in* E^7 *containing* $\mathrm{st}(s, K)$.

Then there are complexes $L_a < L$ *in* E^7 *and there is a pwl map* $r: L \to K$ *such that* (L, L_a, r) *reduces* (K, K_a) *at* s *and* $r^{-1}(\mathrm{st}(s, K)) \subseteq O$, *there is a closed* *embedding* g *of* $(|L|, |L_a|)$ *into* $(M, \text{Bd } M)$, *and there is a special map* π *for* (M, f, g, r, μ) *which is the identity on* Bd M.

Proof. If s is a 2-simplex the conclusion follows easily from Corollary 6.2. For the remaining two cases let k be equal to the difference between the total number of boundary components of the nondegenerate components of $\text{lk}(s, K)$ and the number of nondegenerate components of $\text{lk}(s, K)$. Use Lemma 2.1 to find $\mu_1 \in C(M, \mu^{-1}(0, \infty))$ so that $(M, \mu/k, \mu_1, k+1)$ has Property S. Set $(L(0), L_a(0)) = (K, K_a)$, $g_0 = f$, and $r_0 = I: L(0) \to L(0)$. Let M' denote the sum of M and $(\text{Bd } M) \times [0, 1]$ with the identification $x = (x, 0)$.

We consider two cases:

Case I. *s is a vertex.* We define inductively a sequence of complex pairs $(L(i), L_a(i))$, a sequence of pwl maps r_i, a sequence of embeddings $g_i: (|L(i)|, |L_a(i)|) \to (M, \text{Bd } M)$, and a sequence of special maps π_i for $(M, g_{i-1}, g_i, r_i, \mu_1)$ which are the identity on Bd M so that each $(L(i), L_a(i), r_i)$ is an elementary reduction of $(L(i-1), L_a(i-1))$ at a pair (s_i, S_i) where $S_i < \text{lk}(s, K)$, $r_1 \cdots r_{i-1}(s_i) = s$, and $(r_1 \cdots r_{i-1})^{-1}(\text{st}(s, K)) \subseteq O$.

Suppose that for some $1 \leq j < k$ the appropriate items have been defined for all $i < j$. Then for some vertex s_j in $(r_0 \cdots r_{j-1})^{-1}(s)$ and some component R_j of $\text{lk}(s_j, L(j-1))$, R_j has two boundary components. Choose R_j so that if $s_j \in L_a(j-1)$ then $\text{lk}(s_j, L_a(j-1)) < \dot{R}_j$. An invariance of domain argument can be used to show that this is always possible. Let S_j be a 1-sphere in $\dot{R}_j$ which misses $L_a(j-1)$. Notice that $S_j < \text{lk}(s, K)$. Use [8, Lemma 2.1] to find a connected open set W in M' containing $f(\text{Int } s_j * S_j)$ so that $W - f(s_j * S_j)$ has two components—W_0 and W_1, $W_0 \subseteq \text{Int } M$, $\text{Cl } W_1$ contains a neighborhood of $g_{j-1}(\text{Int } s_j * S_j)$ in $g_{j-1}(s_j * R_j)$, and for each component R of $\text{lk}(s_j, L(j-1))$ one of $W_0 \cap g_{j-1}(s_j * R)$ and $W_1 \cap g_{j-1}(s_j * R)$ is empty. Let $R_{j1}, \cdots, R_{jq}, \cdots, R_{jm}$ be an indexing of the components of $\text{lk}(s_j, L(j-1))$ such that $g_{j-1}(s_j * R_{ji}) \cap W_1 \neq \emptyset \iff i \leq q$.

Let u_j be a point of $(E^7 - L(j-1)) \cap O$ which is joinable to every finite subcomplex of $L(j-1)$. Let $L(j)$ denote the complex $[|L(j-1)| - \text{st}^\circ(s_j, L(j-1))] \cup \bigcup_{i \leq q} s_j * R_{ji} \cup \bigcup_{i > q} u_j * R_{ji} \cup s_j * u_j * S_j$, and set $L_a(j) = L_a(j-1)$. Let r_j denote the simplicial map from $L(j)$ to $L(j-1)$ which sends u_j to s_j and leaves every other vertex fixed. Then $(L(j), L_a(j), r_j)$ is an elementary reduction of $(L(j-1), L_a(j-1))$ at (s_j, S_j). Lemma 5.1 shows that r_j is equivalent to a projection on $|s_j * u_j * S_j| - |S_j|$, so from Corollary 6.2 there is a closed embedding $g_j: (|L(j)|, |L_a(j)|) \to (M, \text{Bd } M)$ and there is a special map π_j for $(M, g_j, g_{j-1}, r_j, \mu_1)$.

After k steps, the process terminates since for every vertex v in $(r_k \cdots r_1)^{-1}(s)$ and for every nondegenerate component R of $\text{lk}(v, L(k))$, R is a disk. Set

$(L, L_a) = (L(k), L_a(k))$, $g = g_k$, $r = r_k \cdots r_1$, and $\pi = \pi_k \cdots \pi_1$. Now (L, L_a, r) reduces (K, K_a) at s, and from Lemmas 2.1 and 2.2, π is a special map for (M, f, g, r, μ).

Case II. *s is a* 1-*simplex.* The items L, L_a, r, g, and π can be obtained by an iterative construction analogous to the one in the preceding case. We show how to define an open set W and a triple $(L(1), L_a(1), r_1)$ to be used in Corollary 6.2 to get the process started, and we leave all other details to the reader.

From an invariance of domain argument, if $s \in K_a$, then the two components of $\operatorname{lk}(s, K_a)$ belong to different components of $\operatorname{lk}(s, K)$. It follows then from [8, Lemma 2.1] that there are among the components of $\operatorname{lk}(s, K)$ two—$\dot{R}_1$ and $\dot{R}_2$; there are vertices $u \in \dot{R}_1 - K_a$ and $v \in \dot{R}_2 - K_a$, and there is a connected open set W in M' containing $f(\operatorname{Int}(u * s \cup v * s))$ so that $W - f(u * s \cup v * s)$ has two components—W_0 and W_1 with $W_0 \subseteq \operatorname{Int} M - f(|K|)$. For the complex $L(1)$ choose a point $w \in O$ which is joinable to each finite subcomplex of K and set $L(1) = K \cup s * w * (u \cup v)$ and $L_a(1) = K_a$. Define r_1 to be a retraction which is affine on each simplex of $L(1)$ and sends w to $b(s)$.

After $(k-1)$ steps the link of s is a 1-cell and a reduction of (K, K_a) at s is achieved.

7. **Neighborhoods for pinched 3-manifolds.** Call a polyhedron P a *pinched* 3-*manifold* if it fails to be a three manifold at no more than a discrete set of points and at each of these points P is locally equivalent to a cone over a finite collection of disjoint disks. In this section we rework McMillan's proofs from [21] to show how to find canonical neighborhoods for embeddings of polyhedra which are sums of pinched manifolds and graphs.

Lemma 7.1. *Suppose P is a polyhedral* 3-*manifold, P_a is either a polyhedral surface in* Bd P *or the empty set, and f is a closed embedding of (P, P_a) into* $(M, \operatorname{Bd} M)$ *such that $f(P)$ is locally tame at $f(P_a)$. Set* $P_b = (\operatorname{Bd} P) - \operatorname{Int} P_a$.

Suppose $\mu \in C(M, f(P))$.

Then there is a pwl homeomorphism g of (P, P_a) into $(M, \operatorname{Bd} M)$ *and there is a discrete collection $\{C_i\}$ of polyhedral 3-cells in P with each $C_i \cap \operatorname{Bd} P$ a disk D_i in* $\operatorname{Int} P_b$ *so that*

(1) *$g(P)$ is tame and g μ-approximates f,*

(2) *each $g(D_i)$ and each $f(C_i)$ is a μ-set,*

(3) *$f(P - \bigcup (C_i - \operatorname{Fr} C_i)) \subseteq g(P - P_b)$, and*

(4) *for each i, $f(P) \cap g(D_i) \subseteq f(C_i)$.*

Proof. Choose $\mu_1 \in C(M, f(P))$ so that $(M, \mu/2, \mu_1, 2)$ has Property S. Substitute $(M \to M, P_b \to S, \operatorname{Bd} P_b \to R, f \to f, \mu_1 f \to \mu)$ in [9, Theorem 9.1] to get an associated $\nu \in C(P_b, \dot{P}_b)$. Extend νf^{-1} to $\mu_2 \in C(M)$ so that $\mu_2 \leq \mu_1$.

From Lemma 2.4 there is a proper collar $\phi: P_b \times [0, 1] \to P$ so that $f\phi(y \times [0, 1])$ is a $\mu_2/6$-set $(y \in P_b)$. From [1], [2], [22] we can suppose that $f\phi$ is locally pwl on $P_b \times (0, 1]$.

From [18, Theorem 4] and [1], [2], [22] there is a homeomorphism $f_0: (P_b, \text{Bd } P_b) \to (M, \text{Bd } M)$ which $\mu_2/2$-approximates $f \mid P_b$ and there is a discrete collection of polyhedral disks $\{D_i\}$ in Int P_b (discrete in Int P_b) so that (1) $f_0(x) = f(x)$ $(x \in \text{Bd } P_b)$, (2) f_0 is locally pwl on Int P_b, (3) each $f(D_i) \cup f_0(D_i)$ is a $\mu_2/2$-set, (4) $f(P_b) \cap f_0(\text{Int } P_b) \subseteq \bigcup (f(\text{Int } D_i) \cap f_0(\text{Int } D_i))$, and (5) $f(P) \cap f_0(P_b) \subseteq f\phi(P_b \times [0, \frac{1}{2}))$. For each D_i set $C_i = \phi(D_i \times [0, \frac{1}{2}])$. We suppose that (6) each $f_0(D_i)$ is in general position with respect to the open polyhedron $f((\text{Bd } C_i) - D_i)$. Since $f(P)$ is locally tame on $f(P_a)$ the proof of Theorem 4 of [18] shows that f_0 can be chosen so that $\{D_i\}$ is discrete not only in Int P_b but also in P_b. Thus by pushing off $f(P)$ near $f(\text{Bd } P_b)$ it is easy to define a pwl homeomorphism f_1 of $(P_b, \text{Bd } P_b)$ into $(M, \text{Bd } M)$ which has properties (2)–(6) when f_0 is replaced by f_1 and which has the additional property (7) that $f_1(\text{Bd } P_b) \cap f(P) = \emptyset$.

For each D_i, $f(\text{Fr } C_i) \cap f_1(D_i)$ separates $f_1(D_i) \cap f(P)$ from $f_1(D_i) \cap f(P - C_i)$ in $f_1(D_i)$. Thus by cutting along components of $f_1(D_i) \cap f(\text{Fr } C_i)$, sewing in new disks from $f(\text{Fr } C_i)$, and then pushing the adjusted portions of $f_1(D_i)$ into $f(\text{Int } C_i)$ for each D_i, we get a pwl homeomorphism f_2 of P_b into M which agrees with f_1 on the complement of $f_2^{-1}(\bigcup \text{Int } C_i)$ so that each $f_2(D_i) \cap f(P) \subseteq f(\text{Int } C_i \cup \text{Int } D_i)$. Since each $f(C_i)$ is a $\mu_2/2$-set it follows that f_2 is a $\mu_2/2$-approximation to f.

Define $f_3: P_b \to M$ by $f_3(x) = f\phi(x, 1)$. Since f_3 is also a $\mu_2/2$-approximation to f our application of [9] provides a pwl homeomorphism h of $(P_b \times [0, 1], (\text{Bd } P_b) \times [0, 1])$ into $(M, \text{Bd } M)$ such that $h(y, 0) = f_2(y)$, $h(y, 1) = f_3(y)$, and $h(y \times [0, 1])$ is a $\mu_1 f(y)$-set for each $y \in P_b$.

Define a homeomorphism g_0 of P into M by $g_0 \mid P - \text{image } \phi = f \mid P - \text{image } \phi$ and $g_0 = h\phi^{-1}$ on $\phi(P_b \times [0, 1])$. If $x = \phi(y, t)$ then $\rho(f(x), g_0(x)) \leq \rho(f(x), f(y)) + \rho(f(y), g_0(x)) \leq \mu_2(f(x))/6 + \mu_1(f(y)) \leq \mu_2(f(x))/6 + \mu(f(x))/2 \leq (2/3)\mu(f(x))$. Thus by using [1], [2], [22] to take a close pwl approximation to g_0 which agrees with g_0 on P_b and takes P onto $g_0(P)$ we get a g for which conditions (1)–(4) in the conclusion of the lemma are satisfied.

Theorem 7.2. *Suppose P is a polyhedral 3-manifold, P_a is either a polyhedral surface in* Bd P *or the empty set, M is a pwl 3-manifold, and f is a closed embedding of (P, P_a) into $(M, \text{Bd } M)$. Suppose $\mu \in C(M, f(P))$.*

Then there is a pwl homeomorphism g of (P, P_a) into $(M, \text{Bd } M)$ and there is a discrete collection of polyhedral cubes-with-handles $\{H_i\}$ in M such that

(1) *g is a μ-approximation to f,*

(2) *each H_i is a μ-set in* Int M,

(3) *each* $H_i \cap g(P)$ *is a disk in* $g(\text{Bd } P)$, *and*

(4) $g(P) \cup \bigcup H_i$ *is a closed neighborhood of* $f(P)$.

Proof. The proof is divided into two cases:

Case I. $f(P)$ *is locally tame on* $f(P_a)$. This case follows from Lemma 7.1 exactly as Theorem 1 of [21] follows from Lemmas 1 and 2 there.

Case II. *General case.* Only Case I of the theorem is used in the remainder of this paper; thus we can use Theorem 9.1 in obtaining the general case. We omit the μ's and just sketch the proof.

Define a proper collar $\lambda\colon P_b \times [0, 1] \to P$ on P_b in P so that $f\lambda$ is locally pwl on $P_b \times (0, 1]$, and apply Theorem 9.1 to get a pwl embedding $h\colon (P_b \times [0, \frac{1}{2}], (\text{Bd } P_b) \times [0, \frac{1}{2}]) \to (M, \text{Bd } M)$ and a discrete collection of polyhedral cubes-with-handles $\{G_i\}$ so that $h(P_b \times [0, \frac{1}{2}]) \cup \bigcup G_i$ contains a neighborhood of $f(P_b)$ and $h(P_b \times \frac{1}{2})$ lies between $h(P_b \times 0)$ and $f\lambda(P_b \times 1)$. Then apply [9] as in the proof of the preceding lemma to extend h to a pwl embedding of $(P_b \times [0, 1], (\text{Bd } P_b) \times [0, 1])$ into $(M, \text{Bd } M)$ so that $h(y, 1) = f\lambda(y, 1)$ for $y \in P_b$. Define $g'\colon P \to M$ by $g' = f$ on $P -$ image λ and $g'\lambda(y, t) = h(y, t)$. Take a pwl approximation to g' which agrees with g' on P_b to get g. The H_i's are obtained from the G_i's by throwing out those which are contained in $g(\text{Int } P)$.

Theorem 7.3. *Suppose* $Q_a = P_a < P < Q$ *are polyhedra such that* P *is a pinched* 3-*manifold,* P_a *is either a surface which has a* 3-*manifold neighborhood in* P *or is the empty set,* $\text{Cl}(Q - P) \cap P_a = \emptyset$, *and dimension* $(Q - P) \leq 1$.

Suppose f *is a closed embedding of* (Q, Q_a) *into* $(M, \text{Bd } M)$ *such that* $f(P)$ *is locally tame on* $f(P_a)$ *and* $\mu \in C(M, f(Q))$.

Then there is a canonical neighborhood system for (M, Q, f, μ).

Proof. Let $\mu_1 \in C(M, f(Q))$ correspond to M, $f(Q)$, μ, and $k = 5$ in Corollary 3.5. Let (L, K, K_a, ϕ) be a triangulation of (Q, P, P_a) with $K, K_a \lhd L$ such that, for each $t \in L$, $f\phi(N(t, L))$ is a μ_1-set. Let $\{v_i\}$ denote the vertices of K whose links are not connected, and let $\{u_k\}$ denote the vertices of $L_0 - K_0$. Further, let $\{\sigma_j\}$ denote the 1-simplexes of $L - K$. Let L^2 denote the second barycentric subdivision of L.

Let $\{O(t)\colon t \in L\}$ be a collection of orientable, open μ_1-sets such that (1) for each t, $f\phi(N(t, L^2)) \subseteq O(t)$, (2) $O(s) \cap O(t) = \emptyset$ if $s \cap t = \emptyset$, (3) $O(t) \subseteq O(s)$ if $t < s$, and (4) $O(t) \cap \text{Bd } M = \emptyset$ if $t \cap K_a = \emptyset$. For each u_k let $B(u_k)$ be a pwl 3-cell in $O(u_k)$ such that (5) $f\phi(u_k) \in \text{Int } B(u_k)$ and (6) $f(Q) \cap \text{Bd } B(u_k)$ is totally disconnected.

Set $P(0) = \phi(\bigcup v_i)$, $P(1) = P - P(0)$, and $P(2) = \text{Cl}(P - \bigcup \phi(N(v_i, K^2)))$. Set $M(1) = M - (f(P(0) \cup (Q - P)) \cup \bigcup B(u_k))$. Choose $\mu_2 \leq \mu_1$ in $C(M, f(P(1)), M - M(1))$ so that (7) for each $t \in L$, $\sup\{\mu_2(x)\colon x \in O(t)\} < d(f\phi(N(v_i, L^2)), M - O(t))$.

Apply Theorem 7.2 for the substitution $(M(1) \to M,\ P(1) \to P,\ f \to f,\ \mu_2 \to \mu)$ to get a pwl embedding g_1 of $(P(1), P_a)$ into $(M(1), \text{Bd } M)$ and a discrete collection of polyhedral cubes-with-handles $\{G_k\}$ in $M(1)$ so that the four conditions are satisfied in the conclusion of the theorem. We suppose that any G_k is thrown out if it does not intersect $f(P)$. From (7) if $G_k \cap f\phi(N(v_i, L^2)) \neq \emptyset$ then $G_k \subseteq O(v_i)$. We suppose that g_1 has been adjusted so that (8) each $g_1\phi(|\text{lk}(v_i, K^2)|)$ misses $\bigcup G_k$ and (9) each $g_1\phi(N(v_i, K^2))$ fails to intersect $f(P(2))$. Extend g_1 to all of Q by setting $g_1 = f$ on $Q - P(1)$.

In each $O(v_i)$ let $C(v_i)$ be a compact, polyhedral 3-manifold such that (10) $C(v_i) \cap g_1(P) = g_1\phi(N(v_i, K^2))$, (11) $(\text{Bd } C(v_i)) \cap g_1(P) = g_1\phi(|\text{lk}(v_i, K^2)|)$, (12) for each G_k, $G_k \subseteq C(v_i)$ or $G_k \cap C(v_i) = \emptyset$ accordingly as $G_k \cap g_1\phi(N(v_i, K^2)) \neq \emptyset$ or $G_k \cap g_1\phi(N(v_i, K^2)) = \emptyset$, (13) $N(1) = g_1(P) \cup \bigcup C(v_i) \cup \bigcup G_k$ contains a neighborhood of $f(P)$, and (14) $(\text{Bd } C(v_i)) \cap f(Q - P)$ is totally disconnected.

Each $D(v_i) = g_1(P) \cap \text{Bd } C(v_i)$ is a collection of mutually exclusive disks; thus from [20, Lemma 1] there is a finite collection $\{A_{ij}\}$ of mutually exclusive polyhedral arcs in $C(v_i) - D(v_i)$ which span $\text{Bd } C(v_i)$ so that the closure of $C(v_i)$ minus a regular neighborhood of $\bigcup_j A_{ij}$ in $C(v_i) - D(v_i)$ is a cube-with-handles. We can suppose each $\text{Bd } A_{ij}$ misses $f(Q)$. Each $\text{st}(v_i, L^2) \searrow^s v_i * V_i \cup \text{lk}(v_i, L^2)$ where V_i is a complex having one vertex in each component of $\text{lk}(v_i, L^2)$. From [20, Corollary 1.1] there is a pwl homeomorphism h_1 of M onto itself which is the identity on $(M - N(1)) \cup (M - \bigcup O(v_i))$ so that (15) each $h_1(A_{ij}) \cap f(Q) = \emptyset$ and (16) $h_1 \mid G_k \neq I \Rightarrow G_k \subseteq O(v_i)$ for some i. Condition (16) can be obtained by requiring that h_1 be the identity on all G_k's which fail to intersect any $f\phi(N(v_i, L^2))$. Set $g_2 = h_1 g_1$.

In each $h_1(C(v_i))$ subtract a regular neighborhood of $h_1(\bigcup_j A_{ij})$ which misses both $g_2\phi(|\text{lk}(v_i, K^2)|)$ and $f(Q)$ and take the closure of what remains to get a polyhedral cube-with-handles $E(v_i)$. By (8), (11), and (14), $E(v_i) = B(v_i) \cup F(v_i)$ where (17) $B(v_i)$ is a pwl 3-cell, (18) $F(v_i)$ is a polyhedral cube-with-handles missing $g_2(P(2))$, (19) $B(v_i) \cap F(v_i)$ is a disk, and (20) $f(Q - P) \cap \text{Bd } E(v_i) \subseteq (\text{Bd } B(v_i)) - F(v_i)$. Set $N(2) = g_2(P(2)) \cup \bigcup B(v_i)$. Note that $N(3) = N(2) \cup \bigcup \{G_k : G_k \cap C(v_i) = \emptyset \text{ for every } v_i\} \cup \bigcup F(v_i)$ contains a neighborhood of $f(P)$ in M.

Using the same constructions as in the preceding two paragraphs we find a collection $\{E(\sigma_j)\}$ of mutually exclusive polyhedral cubes-with-handles in $\text{Int } M$ such that (21) $N(3) \cup \bigcup B(u_k) \cup \bigcup E(\sigma_j)$ contains a neighborhood of $f(Q)$ in M, and for each σ_j, (22) $E(\sigma_j) \subseteq O(\sigma_j)$, (23) $N(3) \cap E(\sigma_j)$ is a pair of disks in $\text{Bd}(N(3) \cup \bigcup B(u_k))$, one in each $B(t)$ for which $t < \sigma_j$, (24) $E(\sigma_j) \cap F(v_i) = \emptyset$ for every v_i, and (25) $E(\sigma_j) = B(\sigma_j) \cup F(\sigma_j)$ where $B(\sigma_j)$ is a pwl 3-cell, $F(\sigma_j)$ is a polyhedral cube-with-handles missing $N(3) \cup \bigcup B(u_k)$, and $B(\sigma_j) \cap F(\sigma_j)$ is a disk in $\text{Bd } B(\sigma_j)$.

Let S_1 denote the polyhedral surface $(\text{Bd } P(2)) - (\phi(\bigcup |\text{lk}(v_i, K^2)|) \cup$

Int P_a), and set $S = \mathrm{Cl}\, S_1$. From Lemma 2.4 there is a proper pwl collar $\lambda: S \times [0, 2] \to P(2)$ such that (26) for each simplex t of K with $\phi(t) \cap S = \emptyset$, $g_2^{-1}(O(t))$ contains $\lambda((S \cap \phi(t)) \times [0, 2])$.

Define a pwl homeomorphism g of (Q, Q_a) into $(M, \mathrm{Bd}\, M)$ as follows: On $P(2)$ − image λ set $g = g_2$. On image λ set $g\lambda(y, t) = g_2\lambda(y, \theta(\begin{smallmatrix}0 & 2\\ 1 & 2\end{smallmatrix})(t))$. Extend g to Q by using the cone structures on the cells $B(t)$ to have g take (27) each $\phi(N(v_i, L^2))$ conewise into $B(v_i)$, (28) each $\phi(N(u_k, L^2))$ conewise into $B(u_k)$, and (29) each $\phi(N(b(\sigma_j), L^2))$ conewise into $B(\sigma_j)$ so that if u_k or $v_i < \sigma_j$ then $g\phi(b(u_k * b(\sigma_j)))$ or $g\phi(b(v_i * b(\sigma_j)))$ belongs to $\mathrm{Int}\,(B(u_k) \cap B(\sigma_j))$ or $\mathrm{Int}\,(B(v_i) \cap B(\sigma_j))$. From conditions on g_2 and the $B(t)$'s and from condition (26), g is a μ-approximation to f.

Set $N = N(2) \cup \bigcup B(\sigma_j) \cup \bigcup B(u_k)$. Then N is a pwl 3-manifold which contains a neighborhood of $g(Q)$, and the sum of N and $\mathrm{Cl}(N(3) - N(2)) \cup \bigcup F(\sigma_j)$ is a closed, polyhedral neighborhood of $f(Q)$. Let (T, ψ) be a triangulation of S such that, for each $t \in T$, $\lambda(\psi(N(t, T)) \times [0, 1]) \subseteq O(s)$ for some $s \in L$. Let T^2 be the second barycentric subdivision of T. For $i = 0, \cdots, 3$ set

$$N_i = \mathrm{Cl}\,(N - \bigcup \{\lambda((\psi(N(b(t), T^2) \cap S_1) \times [0, 1)): t \text{ is a simplex of } T \text{ of dimension} \geq 3 - i\}).$$

Then $N_3 = g(Q) \cup \bigcup B(v_i) \cup \bigcup B(\sigma_j) \cup \bigcup B(u_k)$. Set $N_4 = g(Q) \cup \bigcup B(v_i) \cup \bigcup B(u_k)$ and $N_5 = g(Q)$. Now $N = N_0 \searrow N_1 \searrow \cdots \searrow N_5$, and each of these collapses is a μ_1-collapse which is admissible with respect to Bd M. From our application of Corollary 3.5, N is a μ-regular neighborhood of $g(Q)$.

For the cubes-with-handles $\{H_i\}$ take $\{F(v_i)\} \cup \{F(\sigma_j)\} \cup \{G_k: G_k \cap C(v_i) = \emptyset$ for every $v_i\}$. Then $(N, g, \{H_i\})$ is a canonical neighborhood system for (M, Q, f, μ).

8. Existence and uniqueness of canonical neighborhoods.

Theorem 8.1. *Suppose $Q_a < Q$ are polyhedra, M is a pwl 3-manifold, f is a closed embedding of (Q, Q_a) into $(M, \mathrm{Bd}\, M)$, and $\mu \in C(M, f(Q))$.*

Then there is a canonical neighborhood system $(g, N, \{H_i\})$ for (M, Q, f, μ).

Proof. The central idea in the proof is to make repeated applications of the splitting theorems in §6 to transform f into an embedding of a polyhedron for which Theorem 7.3 applies and then to use properties of special maps and reducing systems to recover a canonical neighborhood system for $f(Q)$. We will construct a tower of polyhedra and maps indicated in Diagram 8.1. This tower will be commutative with respect to solid lines. The Q's denote polyhedra, the i's pwl inclusions or embeddings, the r's pwl maps, the f's topological embeddings, the g's pwl approximations to the f's, the π's special maps, and H a pwl homeomorphism approximating π.

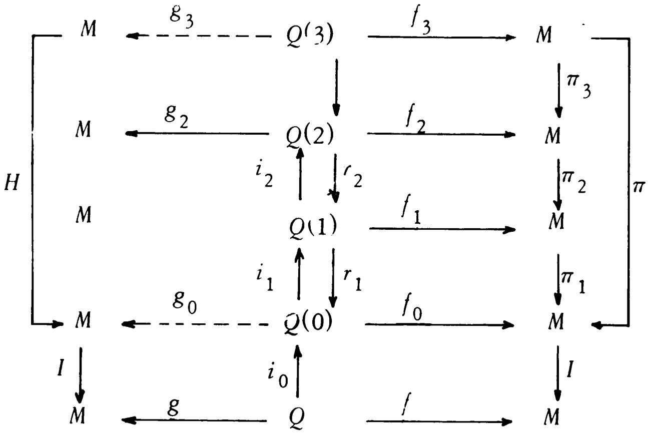

Diagram 8.1

Step 1. *First story of tower.* Let $\mu_1 \in C(M, f(Q))$ correspond to μ_1 in Corollary 3.5 for the substitution $(M \to M, f(Q) \to Y, \mu \to \mu, 10 \to k)$. Because $f(Q_a)$ is tame in Bd M there is some pwl structure on Bd M making $f \mid Q_a$ pwl. Let $i_0\colon Q \to E^7$ and j_0: Bd $M \to E^7$ be closed embeddings of Q and Bd M onto polyhedra in E^7 such that (1) $i_0^{-1} j_0(\text{Bd } M) = Q_a$, (2) $j_0^{-1} i_0 \mid Q_a = f \mid Q_a$, and (3) i_0 is pwl. By Lemma 3.6 there is a regular neighborhood $Q_a(0)$ of $i_0(Q_a)$ in $j_0(\text{Bd } M)$ such that $j_0^{-1}(Q_a(0))$ is a μ_1-regular neighborhood of $f(Q_a)$ with an appropriate pwl structure put on Bd M. Set $Q(0) = i_0(Q) \cup Q_a(0)$, $P_a(0) = Q_a(0)$, and let $P(0)$ denote the set of points of $Q(0)$ where the local dimension of $Q(0)$ exceeds one. Define $f_0\colon Q(0) \to M$ by $f_0 = f i_0^{-1}$ on $i_0(Q)$ and $f_0 = j_0^{-1}$ on $Q_a(0)$.

Choose $\mu_2 \in C(M, f_0(Q(0)))$ so that $\mu_2 \le \mu_1$ and so that if $f_0'\colon (Q(0), Q_a(0)) \to (M, \text{Bd } M)$ is a pwl μ_2-approximation to f_0 then $f_0'(Q_a(0))$ is μ_1-regular neighborhood of $f_0' i_0(Q_a)$ in Bd M. Theorem 7.2 of [10] together with the two dimensional approximation theorems in [1], [2], [22] can be used to show that such a μ_2 exists. Use Lemma 2.1 to find μ_3 and $\mu_4 \in C(M, f_0(Q(0)))$ so that (4) $(M, \mu_2/6, 2\mu_3, 2)$ and (5) $(M, \mu_3/3, \mu_4, 4)$ have Property S. Let $L(0)$ be a rectilinear complex in E^7 such that (6) $|L(0)| = Q(0)$, (7) there are full subcomplexes $K(0)$ and $K_a(0)$ whose carriers are respectively $P(0)$ and $P_a(0)$, and (8) for each $t \in L(0)$, $f_0(N(t, L(0)))$ is a μ_4-set. Choose open μ_4-sets $O(t)$ for the simplexes t of $L(0)$ such that, for each $t \in L(0)$, $f_0(N(t, L(0))) \subseteq O(t)$ and $s < t \Rightarrow O(s) \subseteq O(t)$.

Step 2. *Second story of tower.* Let $\{\Delta_i\}$ denote the 2-simplexes of $L(0)$ whose links are empty. Let $\{y_i\}$ be a collection of mutually exclusive points in $E^7 - Q(0)$ so that (9) every finite subset of $\{y_i\}$ is joinable to every finite subcomplex of $L(0)$ and (10) $\rho(y_i, b(\Delta_i)) < 1/i$ for each i. Set $K(1) = K(0) \cup \bigcup y_i * \Delta_i$, $L(1) = K(1) \cup L(0)$, $K_a(1) = K_a(0)$, $P(1) = |K(1)|$, $Q(1) = |L(1)|$, and $Q_a(1) = P_a(1) = P_a(0)$. Let i_1 denote the inclusion of $Q(0)$ into $Q(1)$, and let r_1 denote the retraction of $Q(1)$ onto $Q(0)$ which is affine on each simplex of $L(1)$ and sends each y_i to $b(\Delta_i)$.

Let $\{U(\Delta_i)\}$ be a collection of mutually exclusive open sets in M such that (11) each $U(\Delta_i) \cap f_0(Q(0)) = f_0(\text{Int } \Delta_i)$ and (12) $U(\Delta_i) \subseteq O(t)$ provided $\Delta_i \cap t \neq \emptyset$. By applying Theorem 6.3 for each Δ_i we find that there is a closed embedding $f_1: (Q(1), Q_a(1)) \to (M, \text{Bd } M)$ and there is a special map π_1 for $(M, f_0, f_1, r_1, \mu_4)$ which is the identity on $(M - \bigcup U(\Delta_i)) \cup \text{Bd } M$.

Step 3. *Third story of tower.* Let $\{\sigma_j\}$ denote the 1-simplexes of $K(1)$ whose links are not connected. Set $L(2, 1) = (L(1), L_1(1))^2$ the relative second barycentric subdivision, and let $K(2, 1)$ and $K_a(2, 1)$ denote the appropriate subcomplexes corresponding to $K(1)$ and $K_a(1)$. Let $\{U(\sigma_j)\}$ be a collection of mutually exclusive open sets in M such that (13) each $U(\sigma_j) \cap f_1(Q(1)) = f_1(\text{st}^\circ(\sigma_j, L(2, 1)))$, and (14) each $U(\sigma_j) \subseteq O(t)$ provided $\sigma_j \cap t \neq \emptyset$.

Repeated applications of Theorem 6.3 show that there is a complex pair $(L(2), L_a(2))$ and there is a pwl retraction $r_2: |L(2)| \to Q(1)$ such that $(L(2), L_a(2), r_2)$ reduces $(L(2, 1), L_a(2, 1))$ at the 1-simplexes $\{\sigma_j\}$, and there is an embedding f_2 of $(|L(2)|, |L_a(2)|)$ into $(M, \text{Bd } M)$ and there is a special map π_2 for $(M, f_1, f_2, r_2, \mu_4)$ which is the identity on $(\text{Bd } M) \cup (M - \bigcup U(\sigma_j))$. Set $Q(2) = |L(2)|$, $Q_a(2) = P_a(2) = P_a(1)$, let $P(2)$ denote the set of points of $Q(2)$ where the local dimension exceeds one, and let $K(2)$ and $K_a(2)$ denote the subcomplexes of $L(2)$ whose carrier s are $P(2)$ and $P_a(2)$. Let i_2 denote the inclusion of $Q(1)$ into $Q(2)$.

Step 4. *Fourth story of tower.* Let $L^2(2)$ be a second derived subdivision of $L(2)$ inducing a subdivision $L'(0)$ of $L(0)$ for which $r_1 r_2: L^2(2) \to L'(0)$ is simplicial. Let $\{v_k\}$ denote the vertices v of $K(2)$ for which $\text{lk}(v, L(2))$ fails to be a 2-cell or a 2-sphere, and let $\{u_k\}$ denote the vertices of $L_0(2) - K_0(2)$. Both these sets are contained in $L_0(0)$. From invariance of domain it follows that each component of each $\text{lk}(v_k, L^2(2))$ is either a point or a punctured 2-sphere. Let $\{U(v_k)\}$ be mutually exclusive open sets in M such that (15) each $f_2(Q(2)) \cap U(v_k) = f_2(st^\circ(v_k), L^2(2))$, and (16) each $U(v_k) \subseteq O(v_k)$.

An application of Lemma 6.3 like the previous one shows that there are (17) complexes and polyhedra $L(3)$, $L_a(3)$, $Q(3) = |L(3)|$, and $Q_a(3) = |L_a(3)|$, (18) a simplicial map r_3 from $L(3)$ to $L^2(2)$ such that $(L(3), L_a(3), r_3)$ reduces $(L^2(2), L^2_a(2))$ at the vertices $\{v_k\}$, (19) an embedding $f_3: (Q(3), Q_a(3)) \to (M, \text{Bd } M)$ such that each $f_3^{-1}(\text{Cl } U(v_k)) = r_3^{-1}(N(v_k, L^2(2)))$, and (20) a special map π_3 for $(M, f_2, f_3, r_3, \mu_4)$ which is the identity on $(\text{Bd } M) \cup (M - \bigcup U(v_k))$. Let $P(3)$ denote the subpolyhedron of $Q(3)$ consisting of the points where the local dimension of $Q(3)$ exceeds one, set $P_a(3) = Q_a(3)$, and let $K(3)$ and $K_a(3)$ denote the full subcomplexes of $L(3)$ whose carriers are $P(3)$ and $P_a(3)$.

Notice that $Q(3)$ is the sum of a graph and the pinched 3-manifold $P(3)$, and that $K(3)$ fails to be a combinatorial 3-manifold only at certain vertices w in $(K_0(3) - K_{a0}(3)) \cap r_3^{-1}(\{v_k\})$ where at each such w every component of $\mathrm{lk}(w, K(3))$ is a disk. Notice also that (21) each $(r_1r_2r_3)^{-1}(|\mathrm{lk}(v_k, L'(0))|)$ is a surface because $r_1r_2: L^2(2) \to L'(0)$ is simplicial.

Step 5. *A neighborhood for* $f(Q)$. Set $\pi = \pi_1\pi_2\pi_3$ and $r = r_1r_2$. For each simplex t of $L(3)$, $\pi f_3(N(t, L(3))) \subseteq O(s)$ provided $rr_3(t) \subseteq s$. From Theorem 7.3 there is a canonical neighborhood system $(g_3, N(3), \{G_i\})$ for $(M, Q(3), f_3, \mu_4)$. From the proof of Theorem 7.3 we may suppose that (22) for each $t \in L(3)$, $\pi g_3(N(t, L(3))) \subseteq O(s)$ provided $rr_3(t) \subseteq s$ and (23) there are mutually exclusive pwl 3-manifolds $B(t)$ in the sets $(\mathrm{Int}\, M) \cap \pi^{-1}(O(t))$ for $t \in \{u_k\} \cup \{v_k\}$ so that each $B(t)$ is a regular neighborhood of $g_3r^{-1}(N(b(t), L^2(2)))$ modulo $g_3(Q(3) - N(b(t), L(3)))$ in M and $N(3) = N_0(3) \searrow \cdots \searrow N_4(3) = g_3(Q(3)) \cup \bigcup B(v_k) \cup B(u_k)$ by a sequence of four μ_4-collapses admissible with respect to Bd M where each component of each $\mathrm{Cl}(N_{i-1}(3) - N_i(3))$ is contained in some $\pi^{-1}(O(s))$. Condition (21) enables us to achieve condition (23). From Lemma 5.2 and [16] we see that each $B(t)$ is a pwl 3-cell.

Now $g_3r_3^{-1} \mid g_3^{-1}(\mathrm{Cl}(M - \bigcup B(v_k)))$ is a pwl embedding.. Use the cone structures on the $B(v_k)$'s to extend this restriction to a pwl embedding g_2 of $(Q(2), Q_a(2))$ into $(M, \mathrm{Bd}\, M)$ so that each $g_2^{-1}(\mathrm{Int}\, B(v_k)) = \mathrm{st}^\circ(v_k, L^2(2))$. Set $g_0 = g_2i_2i_1$. Set $N_5(3) = g_2(Q(2))$. Let $L'''(0)$ be the second barycentric subdivision of $L'(0)$, the subdivision mentioned at the beginning of step 4. For each $t \in L'(0)$, $g_2r^{-1}(N(b(t), L'''(0))) \subseteq \pi^{-1}(O(s))$ for some s. Set $N_{5+i}(3) = g_0(Q(0)) \cup \bigcup \{g_2r^{-1}(N(b(t), L'''(0))): t$ is a simplex of dimension $\leq 2 - i$ in $L'(0)\}$. Then $N_4(3) \searrow N_5(3) \searrow \cdots \searrow N_8(3) = g_0(Q(0))$, each of these collapses is admissible with respect to Bd M, and each component of each $\mathrm{Cl}(N_{i-1}(3) - N_i(3))$ is contained in some $\pi^{-1}(O(s))$.

Because π is the composition of three special μ_4-maps it is a special μ_3-map. Further, $\pi(N(3) \cup \bigcup G_i)$ contains a neighborhood of $f(Q)$. Thus from [1], [2], [22] there is a pwl $2\mu_3$-homeomorphism H of M onto itself which approximates π so closely that (24) $H(N(3) \cup \bigcup G_i)$ contains a neighborhood of $f(Q)$, and (25) each component of each $H(\mathrm{Cl}(N_{i-1}(3) - N(3)))$ is contained in $O(s)$ for some s. Set $g = Hg_0i_0$, $N = H(N(3))$, and $\{H_i\} = \{H(G_i)\}$. We claim that $(g, N, \{H_i\})$ is a canonical neighborhood system for (M, Q, f, μ). First g_0 is a μ_4-approximation to f_0 and H is a $2\mu_3$-map so Hg_0 is a μ_2-approximation to f_0. Thus g is a μ-approximation to f. Furthermore by Lemma 3.6 and the conditions on μ_2, if we set $N_i = H(N_i(3))$ then $N = N_0 \searrow \cdots \searrow N_8 \searrow N_9 \searrow N_{10} = g(Q)$ by a sequence of 10 μ_1-collapses which are admissible with respect to Bd M. From Corollary 3.5, N is a μ-regular neighborhood of $g(Q)$. Similar considerations show that each H_i is a μ-set.

Except for the distribution of cubes-with-handles canonical neighborhoods are unique. This statement is given meaning by the following theorem on μ-regular neighborhoods.

Theorem 8.2. *Suppose $Q_a < Q$ are polyhedra, M is a pwl 3-manifold, f is a closed embedding of (Q, Q_a) into $(M, \mathrm{Bd}\, M)$, and $\mu \in C(M, f(Q))$.*

There is a $\mu_1 \in C(M, f(Q))$ such that if g_1 and g_2 are pwl embeddings of (Q, Q_a) into $(M, \mathrm{Bd}\, M)$ which μ_1-approximate f, and if N_1 and N_2 are μ_1-regular neighborhoods of $g_1(Q)$ and $g_2(Q)$ respectively, then there is a pwl μ-homeomorphism h of $(N_1, N_1 \cap \mathrm{Bd}\, M)$ onto $(N_2, N_2 \cap \mathrm{Bd}\, M)$ which extends $g_2 g_1^{-1}$.

Proof. Use Lemma 2.1 and [11, Theorem 3.1] to find μ_2, μ_3, and $\mu_1 \in C(M, f(Q))$ so that $(M, \mu/3, 2\mu_2, 4)$, $(M, \mu_2/2, 2\mu_3, 3)$, and $(M, \mu_3, \mu_1, 2)$ have Property S, and so that if f' and f'' are pwl μ_1-approximations to f which take (Q, Q_a) into $(M, \mathrm{Bd}\, M)$, then there is a pwl extension of $f''(f')^{-1}$ which takes a polyhedral neighborhood pair $(R', R' \cap \mathrm{Bd}\, M)$ of $(f'(Q), f'(Q_a))$ onto a polyhedral neighborhood pair $(R'', R'' \cap \mathrm{Bd}\, M)$ of $(f''(Q), f''(Q_a))$.

Let g_1, g_2, N_1, and N_2 be given as in the hypothesis. Let h_0 be the promised pwl extension of $g_2 g_1^{-1}$ which takes a neighborhood pair $(R_1, R_1 \cap \mathrm{Bd}\, M)$ onto a neighborhood pair $(R_2, R_2 \cap \mathrm{Bd}\, M)$. Note that $g_2 g_1^{-1}$ is a μ_3-map; thus by Lemma 3.6 there are μ_1-regular neighborhoods N_1' of $g_1(Q)$ in R_1 and $N_2' = h_0(N_1')$ of $g_2(Q)$ in R_2 such that $h_0 \mid N_1'$ is a $2\mu_3$-map. Furthermore by Lemma 3.1 there are pwl μ_2-homeomorphisms h_1 of $(N_1, N_1 \cap \mathrm{Bd}\, M)$ onto $(N_1', N_1' \cap \mathrm{Bd}\, M)$ and h_2 of $(N_2', N_2' \cap \mathrm{Bd}\, M)$ onto $(N_2, N_2 \cap \mathrm{Bd}\, M)$ which are the identity on $f_1(Q)$ and $f_2(Q)$ respectively. Thus $h_2 h_0 h_1$ is a μ-map and a pwl extension of $g_2^{-1} g_1$ which takes $(N_1, N_1 \cap \mathrm{Bd}\, M)$ onto $(N_2, N_2 \cap \mathrm{Bd}\, M)$.

9. **Canonical neighborhoods for surfaces.** In this section we exhibit an alternate characterization of canonical neighborhoods for surfaces which is more in the spirit of McMillan's work.

Theorem 9.1. *Suppose S is a polyhedral surface, $R \subseteq \mathrm{Bd}\, S$ is either a polyhedral 1-manifold or the empty set, M is a pwl 3-manifold, and f is a closed embedding of (S, R) into $(M, \mathrm{Bd}\, M)$. Suppose that $\mu \in C(M, f(S))$.*

If $f((\mathrm{Int}\, S) \cup (\mathrm{Int}\, R))$ is two sided in some open subset of M, then there is a pwl homeomorphism g of $(S \times [0, 1], R \times [0, 1])$ into $(M, \mathrm{Bd}\, M)$ and there is a discrete collection of polyhedral cubes-with-handles $\{H_i\}$ in M such that

(1) *for each $x \in S$, and each $t \in [0, 1]$, $\rho(f(x), g(x, t)) \leq \mu(f(x))$,*

(2) *each H_i is a μ-set in $\mathrm{Int}\, M$ which intersects $g(S \times [0, 1])$ in a disk, and*

(3) *$g(S \times [0, 1]) \cup \bigcup H_i$ contains a neighborhood of $f(S)$.*

Proof. From §8 it is sufficient to show that for each $\mu \in C(M, f(Q))$ there is some pwl homeomorphism g of $(S \times [0, 1], R \times [0, 1])$ into $(M, \mathrm{Bd}\, M)$ which satisfies condition (1) and there is some pwl embedding f_0: $(S, R) \to (M, \text{Bd } M)$ which μ-approximates f so that $g(S \times [0, 1])$ is a μ-regular neighborhood of $f_0(S)$.

Let μ_1 correspond to M, μ, $f(Q)$, and $k = 5$ in Corollary 3.5. Let $\mu_2 \in C(M, f(\text{Int } S \cup \text{Int } R), f((\text{Bd } S) - R))$ be such that $\mu_2 \leq \mu_1$. From Theorem 6.1, the definition of special map, and [2], there is a locally pwl homeomorphism g': $((\text{Int } S \cup \text{Int } R) \times [0, 1], (\text{Int } R) \times [0, 1]) \to (M, \text{Bd } M)$ so that, for each $y \in \text{Int } S \cup \text{Int } R$ and each $t \in [0, 1]$, $\rho(f(y), g'(y, t)) \leq \mu_2 f(y)$ and $g'(y \times [0, 1])$ is a μ_2-set.

Let λ: $((\text{Bd } S) - \text{Int } R) \times [0, 3] \to S$ be a proper pwl collar on $(\text{Bd } S) - \text{Int } R$ in S so small so that if $g: S \times [0, 1] \to M$ is defined by $g \mid (S - \text{image } \lambda) \times [0, 1] = g' \mid (S - \text{image } \lambda) \times [0, 1]$ and $g(\lambda(y, s), t) = g'(\lambda(y, \theta(\begin{smallmatrix} 0 & 3 \\ 1 & 3 \end{smallmatrix})(s)), t)$, then each $g(y \times [0, 1])$ is a μ_1-set, for each $(y, t) \in S \times [0, 1]$, $\rho(g(y, t), f(y)) \leq \mu_1(f(y))$, and if f_0 denotes the pwl embedding $f_0(y) = g'(y, \frac{1}{2})$ for $y \notin$ image λ and $f_0\lambda(y, t) = g'(\lambda(y, \theta(\begin{smallmatrix} 0 & 3 \\ 2 & 3 \end{smallmatrix})(t)), \frac{1}{2})$, then f_0 μ_1-approximates f. Now $g(S \times [0, 1]) \searrow g(S \times \frac{1}{2})$ by a sequence of 3 μ_1-collapses admissible with respect to Bd M and $g(S \times \frac{1}{2}) \searrow f_0(S)$ by two μ_1-collapses admissible with respect to Bd M. These five collapses are essentially those introduced in the proofs of Theorems 3.6 and 7.3. From Corollary 3.5, $g(S \times [0, 1])$ is a μ-regular neighborhood of $f_0(S)$.

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