

SOME REMARKS CONCERNING THE VARIETIES GENERATED BY THE DIAMOND AND THE PENTAGON⁽¹⁾

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ABSTRACT. In 1945 M. P. Schützenberger exhibited two identities. He asserted that one provided an equational base for the diamond M_3 and the other a base for the pentagon N_5 . Recently Ralph McKenzie produced another equational base for N_5 . In the present paper the authors modify McKenzie's idea to verify Schützenberger's assertion for M_3 . They also show Schützenberger's claim about N_5 is false.

Introduction. In this note we make some observations based on the preceding paper [2] by R. McKenzie. In §1 we modify the ideas in §2 of McKenzie's paper to obtain analogous results for $\mathfrak{O}M_3$, the variety of lattices generated by the diamond. In particular, we provide a proof of the result announced by Schützenberger [3] that $\mathfrak{O}M_3$ is characterized by the single identity:

$$\begin{aligned} \alpha. \quad x \cdot (y + z \cdot (u + v)) &= x \cdot (y + zu) + x \cdot (y + zv) + xz \cdot (u + v) \\ &\quad + xu \cdot (z + yv) + xv \cdot (z + yu). \end{aligned}$$

This fact also follows from the much stronger results of Jónsson [1]; however, our proof of this result, like McKenzie's proof that certain identities characterize $\mathfrak{O}N_5$, is model-theoretic in nature while Jónsson's results involve deeper lattice theoretic techniques.

In the article cited above, Schützenberger also asserted without proof that the variety $\mathfrak{O}N_5$ generated by the pentagon is characterized by the identity:

$$\begin{aligned} \beta. \quad x \cdot [y + z \cdot (u + v)] &= x \cdot (xy + zu) + x \cdot (y + xzu) + x \cdot (xy + zv) \\ &\quad + x \cdot (y + xzv) + xz \cdot (xzu + v) + xz \cdot (u + xzv). \end{aligned}$$

In §2 of our note we observe that β holds in some lattice not contained in $\mathfrak{O}N_5$; thus β does not characterize $\mathfrak{O}N_5$. Equational bases for $\mathfrak{O}N_5$ have been found by McKenzie (see [2]).

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stimulating conversations concerning the questions considered here and in particular Professor McKenzie for making available an early version of his results.

1. Following McKenzie, a special term of type one (an ST1) is any lattice term of the form $\rho \cdot (\sigma + \tau)$ where ρ, σ, τ are each products of variables. A term in the dual form is called an ST2. McKenzie proved that for any ST1 ν and ST2 ϕ the inclusion $\nu \leq \phi$ (called a special inclusion) either holds in every lattice or else implies (modulo lattice theory Λ) the modular law. We will modify McKenzie's proof to show the following.

Lemma 1.1. *Every special inclusion ϵ either holds in all modular lattices or else implies (modulo Λ) the distributive law.*

If Θ is any equational theory we write $\sigma \leq_{\Theta} \tau$ instead of $\sigma \leq \tau \in \Theta$ and $\sigma \sim_{\Theta} \tau$ in place of $\sigma = \tau \in \Theta$. The equational theory of modular lattices is denoted by \mathbf{M} ; distributive lattices by Δ . The following is an analogue of McKenzie's Lemma 2.2.

Lemma 1.2. *For each term σ , there are finite, nonempty sets of terms F_1 and F_2 such that*

- (i) F_1 consist of ST1's ν satisfying $\nu \leq_{\mathbf{M}} \sigma$; moreover $\sigma = \Sigma F_1 \in \Theta_l[\alpha]$;
- (ii) F_2 consist of ST2's ϕ satisfying $\sigma \leq_{\mathbf{M}} \phi$; moreover $\sigma = \Pi F_2 \in \Theta_l[\alpha]$.

As an application of Lemmas 1.1 and 1.2 we prove α characterizes \mathbf{UM}_3 .

Theorem 1.3. $\Theta_{\mathbf{M}_3} = \Theta_l[\alpha]$.

As an elementary application of the model-theoretic ideas to be used in the proof of 1.3 we first give a simple proof of the following well-known theorem.

Theorem 1.4. *The variety of lattices generated by the two element chain is the class of all lattices satisfying the distributive law $x \cdot (y + z) = xy + xz$.*

Proof. Clearly the two element chain satisfies the distributive law. Since the distributive law is self-dual, it is easily seen that from this law and Λ every term is equivalent to a term $\Sigma \sigma_i$ where σ_i is a product of variables and also to a term $\Pi \tau_j$ where τ_j is a sum of variables. Thus, every lattice inclusion $\nu \leq \phi$ is equivalent to a conjunction of inclusions of the form $\sigma \leq \tau$ where σ is a product and τ a sum of variables. Now, an inclusion $\sigma \leq \tau$ of this form will either hold in every lattice or fail in the two element chain depending upon whether or not some variable occurring in σ also occurs in τ . Thus, every identity holding in the two element chain follows from the distributive law and Λ .

In view of McKenzie's Lemma 2.1, our Lemma 1.1 and the above proof every special inclusion is either a lattice identity or equivalent to the modular law, distributive law or $x = y$. This fact was also observed independently by McKenzie.

Proof of Theorem 1.3. From Lemma 1.2 it follows that modulo $\Theta_l[\alpha]$ every equation satisfied by the diamond M_3 is equivalent to a conjunction of special inclusions. Such a special inclusion does not imply the distributive law; thus, by Lemma 1.1 it belongs to \mathbf{M} . It is easily seen (and we will prove later) that α implies the modular law; thus, $\Theta M_3 \subseteq \Theta_l[\alpha]$.

It remains to show that α holds in M_3 which we do directly. Suppose the variables x, y, z, u , and v are, respectively, assigned to elements x', y', z', u' and v' in M_3 . The right (resp. left) side of α is then assigned to an element RS (resp. LS) in M_3 . By modularity each summand on the right side of α is contained in LS; hence it suffices to show LS is always a sum of elements on the right. Whenever u' and v' are comparable or one of $x', y', z' \in \{0, 1\}$, $LS \leq RS$ is easily checked. Now assume u' and v' are incomparable and $x', y', z' \notin \{0, 1\}$. Since $u' + v' = 1$, it suffices to show

$$\begin{aligned} x' \cdot (y' + z') &= x' \cdot (y' + z'u') + x' \cdot (y' + z'v') \\ &\quad + x'z' + x'u'(z' + y'v') + x'v' \cdot (z' + y'u'). \end{aligned}$$

Moreover, we may assume that x', y', z' are mutually incomparable; for if y', z' are comparable then $x' \cdot (y' + z') = x' \cdot (y' + z'u') + x'z'$ and, if x' is comparable with y' or z' , then

$$x' \cdot (y' + z') = x'y' + x'z' = x' \cdot (y' + z'u') + x'z'$$

by modularity. Thus, there are only three remaining cases: $z' = u', z' = v', \{x', y'\} = \{u', v'\}$. In the first two cases $x' \cdot (y' + z') = x' \cdot (y' + z'u') + x' \cdot (y' + z'v')$ while in the last case $x' \cdot (y' + z') = x'u' \cdot (z' + y'v') + x'v' \cdot (z' + y'u')$. Hence, in every case $LS \leq RS$ completing the proof that $\alpha \in \Theta M_3$.

Proof of Lemma 1.1. Let ϵ be any special inclusion

$$\rho \cdot (\sigma + \tau) \leq \phi + \chi \cdot \psi$$

which fails in some modular lattice. For any term π , let π^0 denote the set of all variables occurring in π . We wish to show that $\Delta \leq \Theta_l[\epsilon]$. The proof is exactly the same as the proof of Lemma 2.1 in McKenzie's paper except for the last case where the sets $\rho^0 \cap \phi^0, \rho^0 \cap \psi^0, \tau^0 \cap \phi^0, \tau^0 \cap \chi^0$, and $\sigma^0 \cap \psi^0$ are empty while the sets $\rho^0 \cap \chi^0, \tau^0 \cap \psi^0$, and $\sigma^0 \cap \phi^0$ are nonempty. Suppose that, in addition, $\sigma^0 \cap \chi^0 \neq 0$. The assumption that various sets of variables are nonempty implies (modulo Δ) the inclusions $\sigma \leq \chi, \rho \leq \chi, \tau \leq \psi$, and $\sigma \leq \phi$. Hence, \mathbf{M} implies $\rho \cdot (\sigma + \tau) \leq \chi \cdot (\chi \cdot \phi + \psi) = \chi \cdot \phi + \chi \cdot \psi \leq \phi + \chi \cdot \psi$ contrary to our assumption that $\epsilon \notin \mathbf{M}$. Hence, $\sigma^0 \cap \chi^0 = 0$ and in this last case the following five sets are pairwise disjoint:

$$\begin{aligned} &\rho^0 \cap \chi^0, \quad \sigma^0 \cap \phi^0, \quad \tau^0 \cap \psi^0, \\ (1) \quad &(\rho^0 - \chi^0) \cup (\tau^0 - \psi^0) \cup (\sigma^0 - \phi^0), \\ &(\phi^0 - \sigma^0) \cup (\psi^0 - \tau^0) \cup (\chi^0 - \rho^0). \end{aligned}$$

Choose three distinct variables v_0, v_1, v_2 not occurring in ϵ . Replacing the variables in the five sets listed in (1), respectively, by $v_0, v_1, v_2, v_0 + v_1 + v_2, v_0 \cdot v_1 \cdot v_2$ and all remaining variables by v_0 , we see that $v_0 \cdot (v_1 + v_2) \leq v_1 + v_0 \cdot v_2$ belongs to $\Theta_l[\epsilon]$. Since this inclusion fails in both M_3 and N_5 , it implies the distributive law; thus, $\Lambda \subseteq \Theta_l[\epsilon]$ as desired.

Before proceeding with the proof of Lemma 1.2 we need to derive several consequences of α , namely, $\alpha_1 - \alpha_4$.

$$\begin{aligned} \alpha_1. & x \cdot (y + u + v) = x \cdot (y + u) + x \cdot (y + v) + x \cdot (u + v), \\ \alpha_2. & xy + zu = (x + zu) \cdot (y + zu) \cdot (z + xy) \cdot (u + xy), \\ \alpha_3 & (= \alpha^d). x + y \cdot (z + uv) = (x + y \cdot (z + u)) \cdot (x + y \cdot (z + v)) \cdot (x + z + uv) \cdot \\ & (x + u + z \cdot (y + v)) \cdot (x + v + z \cdot (y + u)), \\ \alpha_4 & (= \alpha_2^d). (x + y) \cdot (z + u) = x \cdot (z + u) + y \cdot (z + u) + z \cdot (x + y) + u \cdot (x + y). \end{aligned}$$

For the remainder of this section we let $\Theta = \Theta_l[\alpha]$. We first observe that the modular law belongs to Θ since

$$x \cdot (z + yx) \sim_{\Lambda} xx \cdot (z + (yx) \cdot x) \leq_{\Theta} x \cdot (yx + z \cdot (x + x)) \leq_{\Lambda} yx + zx.$$

Actually, the modular law is equivalent (modulo Λ) to the inclusion $RS_{\alpha} \leq LS_{\alpha}$.

Substituting $u + v$ for z in α gives $\alpha_1 \in \Theta$. Obviously $LS_{\alpha_2} \leq_{\Lambda} RS_{\alpha_2}$. Now,

$$\begin{aligned} RS_{\alpha_2} & \sim_M [(y + zu) \cdot z + xy] \cdot [(x + zu) \cdot u + xy], \\ & \sim_M (xy + zu + yz) \cdot (xy + zu + ux), \\ & \sim_M LS_{\alpha_2} + ux \cdot (xy + zu + yz), \end{aligned}$$

and

$$ux \cdot (xy + zu + yz) \sim_{\alpha_1} ux \cdot (xy + zu) + ux \cdot (xy + yz) + ux \cdot (zu + yz)$$

where each of these three terms are obviously $\leq_{\Lambda} LS_{\alpha_2}$. Thus $\alpha_2 \in \Theta$. By duality $\alpha_4 \in \Theta$ once we have shown $\alpha_3 \in \Theta$.

It is easily seen that $LS_{\alpha_3} \leq_M RS_{\alpha_3}$; to illustrate we check the fourth factor on the right side: $LS_{\alpha_3} \leq_{\Lambda} x + (y + uv) \cdot (z + uv) \sim_M x + uv + z \cdot (y + uv) \leq_{\Lambda} x + u + z(y + v)$. It now remains to show $RS_{\alpha_3} \leq_{\Theta} LS_{\alpha_3}$. Let γ be the product of the last four terms on the right side of α_3 . Then

$$RS_{\alpha_3} \sim_{\Theta} \gamma \cdot (x + yz) + \gamma \cdot (x + yu) + \gamma y \cdot (z + u) + yz \cdot (y + xu) + \gamma u \cdot (y + xz).$$

Clearly $\gamma \cdot (x + yz) \leq LS_{\alpha_3}$; we consider each of the remaining terms separately.

Case 1.

$$\begin{aligned} \gamma u \cdot (y + xz) & \sim_{\Lambda} u \cdot (y + xz) \cdot (x + y(z + v)) \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u)) \\ & \leq_{\Theta} LS_{\alpha_3} + \delta_1 + \delta_2 + \delta_3 + \delta_4 \end{aligned}$$

where

$$\begin{aligned}\delta_1 &= u \cdot (y + xz) \cdot (x + yv) \cdot (x + z + uv), \\ \delta_2 &= uy \cdot (z + v) \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u)), \\ \delta_3 &= u \cdot (y + xz) \cdot z \cdot (y + xv) \cdot (x + v + z(y + u)) \leq_M LS_{\alpha_3}, \\ \delta_4 &= u \cdot (y + xz) \cdot v \leq_{\theta} x + y \cdot (z + uv) = LS_{\alpha_3}.\end{aligned}$$

Now, $\delta_1 \sim_M u \cdot (yv + x \cdot (xy + xz)) \cdot (x + z + uv) \leq_{\theta} LS_{\alpha_3} + \delta_{11} + \delta_{12}$ where $\delta_{11} = u \cdot (yv + xy) \cdot (x + z + uv)$ and $\delta_{12} = u \cdot (yv + xz) \cdot (x + z + uv)$. It is easily seen that each $\delta_{1i} \leq_{\theta} LS_{\alpha_3}$ by applying α_1 to δ_{1i} and then M and α to each summand not obviously $\leq LS_{\alpha_3}$. Hence, $\delta_1 \leq_{\theta} LS_{\alpha_3}$. Also,

$$\begin{aligned}\delta_2 &\sim_M uy \cdot ((z + uv) + x \cdot (z + v)) \cdot (x + v + z \cdot (y + u)) \\ &\leq_{\theta} LS_{\alpha_3} + yu \cdot (xv + z + uv) (x + v + z \cdot (y + u)) \\ &\sim_M LS_{\alpha_3} + yu \cdot (z + v(x + uv)) (x + v + z \cdot (y + u)) \\ &\leq_{\theta} LS_{\alpha_3} + yu \cdot (z + vx) \cdot (x + v + z \cdot (y + u)) \\ &\sim_M LS_{\alpha_3} + yu \cdot (z \cdot (y + u) + (v + x)(z + vx)) \\ &\leq_{\theta} LS_{\alpha_3} + yu(z \cdot (y + u) + vx) + yu \cdot (v + x)(z + vx).\end{aligned}$$

Each of these two summands are easily seen to be $\leq_{\theta} LS_{\alpha_3}$ by applying α and M where appropriate. Thus, $\delta_2 \leq_{\theta} LS_{\alpha_3}$ and hence $yu(y + xz) \leq_{\theta} LS_{\alpha_3}$.

Case 2.

$$\begin{aligned}yu \cdot (z + u) &\sim_M y \cdot (z + uv + x \cdot (z + u)) \cdot (x + y \cdot (z + v)) \\ &\quad \cdot (x + u + z \cdot (y + v)) (x + v + z(y + u)) \\ &\leq_{\theta} LS_{\alpha_3} + \delta_1 + \delta_2\end{aligned}$$

where

$$\delta_1 = y \cdot (z + uv + xu) \cdot (x + y(z + v)) \cdot (x + u + z \cdot (y + v)) (x + v + z \cdot (y + u))$$

and

$$\delta_2 = yu(x + z) \cdot (x + y(z + v)) (x + v + z \cdot (y + u)) \leq_{\Delta} yu \cdot (y + xz) \leq_{\theta} LS_{\alpha_3}.$$

Now,

$$\begin{aligned}\delta_1 &\sim_M y \cdot (z + u \cdot (x + uv)) (x + y \cdot (z + v)) \\ &\quad \cdot (x + u + z \cdot (y + v)) (x + v + z \cdot (y + u)) \\ &\leq_{\theta} LS_{\alpha_3} + \delta_{11} + \delta_{12}\end{aligned}$$

where

$$\delta_{11} = y \cdot (z + ux) \cdot (x + y \cdot (z + uv)) \cdot (x + u + z \cdot (y + v)) \cdot (x + v + z \cdot (y + u))$$

and

$$\delta_{12} = yu \cdot (x + uv) \cdot (x + y \cdot (z + v)) \leq_{\mathbf{A}} yu \cdot (y + xz) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}.$$

Now,

$$\delta_{11} \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + \delta_{111} + \delta_{112} + \delta_{113}$$

where

$$\delta_{111} = y \cdot (z + ux) \cdot (x + yv) \cdot (x + u + z \cdot (y + v)),$$

$$\delta_{112} = y(z + ux) \cdot (z + v) \cdot (x + u + z(y + v)) \cdot (x + v + z(y + u)),$$

$$\delta_{113} = yv \cdot (z + ux) \cdot (x + u + z \cdot (y + v)) \leq_{\mathbf{A}} \delta_{111}.$$

Now,

$$\begin{aligned} \delta_{111} &\sim_{\mathbf{M}} y \cdot (ux + z \cdot (x + yv)) \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + yv \cdot (z + ux) \cdot (x + u + z \cdot (y + v)) \end{aligned}$$

where the last term $\leq_{\mathbf{A}} yv \cdot (y + xz) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}$ by Case 1 with u and v permuted. A similar argument shows $\delta_{112} \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}$. Hence, $\gamma y \cdot (z + u) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}$.

Case 3.

$$\begin{aligned} yz \cdot (y + xu) &\sim_{\mathbf{A}} z \cdot (y + xu) \cdot (x + y \cdot (z + v)) \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + \delta_1 + \delta_2 + \delta_3 \end{aligned}$$

where $\delta_1 = z \cdot (y + xu) \cdot (x + yv) \cdot (x + u + z \cdot (y + v))$, $\delta_2 = z \cdot (y + xu) \cdot (y + xv)$, and $\delta_3 = z \cdot (y + xu) \cdot v \cdot (y + zx) \cdot (x + u + z \cdot (y + v)) \leq_{\mathbf{M}} \text{LS}_{\alpha_3}$.

Now,

$$\begin{aligned} \delta_1 &\sim_{\mathbf{M}} z \cdot (yv + x \cdot (y + xu)) \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + z \cdot (yv + xu) \cdot (x + u + z \cdot (y + v)) \\ &\sim_{\mathbf{M}} \text{LS}_{\alpha_3} + z \cdot (xu + yv \cdot (x + u + z \cdot (y + v))) \\ &\leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + \delta_{11} + \delta_{12} + \delta_{13} \end{aligned}$$

where $\delta_{11} = z \cdot (xu + yv \cdot (x + u)) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + zu \cdot (yv + xu) \leq_{\mathbf{M}} \text{LS}_{\alpha_3}$, $\delta_{12} = z \cdot (x + u) \cdot (yv + xuz \cdot (y + v)) \leq_{\mathbf{M}} \text{LS}_{\alpha_3}$, and $\delta_{13} = z \cdot (y + v) \cdot (yv + xu) \sim_{\mathbf{M}} z \cdot (yv + xu \cdot (y + v)) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + z \cdot (yv + xvu) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}$ since $z \cdot (yv + xuv) \leq_{\mathbf{\Theta}} x + yv(z + uv)$. Hence, $\delta_1 \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}$. Now,

$$\delta_2 \sim_{\mathbf{M}} z \cdot (y + xu \cdot (y + xv)) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3} + z \cdot (y + xuv) \leq_{\mathbf{\Theta}} \text{LS}_{\alpha_3}$$

as with δ_{13} . Hence, $\gamma z \cdot (y + xu) \leq_{\Theta} \text{LS}_{a_3}$.

Case 4.

$$\begin{aligned} \gamma \cdot (x + yu) &\sim_{\Lambda} (x + yu) \cdot (x + \gamma \cdot (z + v)) \\ &\quad \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u)) \\ &\leq_{\Theta} \text{LS}_{a_3} + \delta_1 + \delta_2 + \delta_3 + \delta_4 \end{aligned}$$

where $\delta_1 = (x + yu) \cdot (x + yv) \cdot (x + z + uv)$, $\delta_2 = \gamma \cdot (x + yu) (z + v) \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u))$, $\delta_3 = z \cdot (x + yu) (y + xv) \cdot (x + v + z \cdot (y + u))$, $\delta_4 = v \cdot (x + yu) (y + xz) \cdot (x + z + uv)$. Now,

$$\delta_1 \sim_M (x + yu) \cdot (x + yv \cdot (x + z + uv)) \sim_M x + yv(x + yu) \cdot (x + z + uv) \leq_{\Lambda} x + \delta_4$$

and $\delta_i \leq_{\Theta} \text{LS}_{a_3}$ (for $i = 2, 3, 4$) follows from Cases 2, 3, and 1 respectively with variables u and v interchanged. Hence, $\gamma \cdot (x + yu) \leq_{\Theta} \text{LS}_{a_3}$ completing the proof that $\alpha_3 \in \Theta_I[\alpha]$.

Proof of Lemma 1.2. We only prove (i); (ii) follows by duality. The proof is only a slight modification of McKenzie's proof of Lemma 2.2. Let F (resp. G), with or without subscripts, always denote a finite, nonempty set of ST1's (resp. ST1's and terms of the type $\zeta + \sigma$, where ζ and σ are products of variables).

First, note that given F , we can find a G with $\Sigma G \leq_M \Sigma F$ and $x \cdot \Sigma F \sim_{\Theta} \Sigma \{x \cdot \phi : \phi \in G\}$. Indeed, by α_1 ,

$$x \cdot \Sigma F \sim_{\Theta} \Sigma \{x \cdot (\phi_0 + \phi_1) : \phi_0, \phi_1 \in F\}$$

and each $\phi_0 + \phi_1 \leq_M \Sigma F$. By several applications of α : if $\phi_K = \zeta_K \cdot (\sigma_{K0} + \sigma_{K1})$, then (where κ, λ, δ range over $\{0, 1\}$)

$$\begin{aligned} (2) \quad x \cdot (\phi_0 + \phi_1) &\sim_{\Theta} \sum_{\kappa, \lambda} x \cdot (\zeta_0 \cdot \sigma_{0\kappa} + \zeta_1 \cdot \sigma_{1\lambda}) + \sum_{\kappa} x \cdot \zeta_{\kappa} \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1}) \\ &\quad + x \cdot \zeta_0 \cdot \sigma_{10} \cdot \sigma_{11} \cdot (\sigma_{00} + \sigma_{01}) \\ &\quad + \sum_{\kappa, \lambda, \delta} x \cdot \sigma_{\kappa \lambda} \cdot (\zeta_{\kappa} + \zeta_{1-\kappa} \cdot \sigma_{\kappa 1-\lambda} \cdot \sigma_{1-\kappa \delta}) \\ &\quad + \sum_{\kappa, \lambda} x \cdot \sigma_{1\kappa} \sigma_{0\lambda} \cdot (\sigma_{1\ 1-\kappa} \cdot \zeta_0 + \zeta_1 \cdot \sigma_{0\ 1-\lambda}). \end{aligned}$$

G is a subset of F together with the above five types of ϕ 's associated with pairs ϕ_0, ϕ_1 in F . We need only check that for each of the types of elements ϕ described, $\phi \leq_M \Sigma F$. For ϕ a member of F or one of the first three types above it is obvious that $\phi \leq_M \Sigma F$. Suppose $\phi = \sigma_{1\kappa} \cdot \sigma_{0\lambda} \cdot (\sigma_{1\ 1-\kappa} \cdot \zeta_0 + \zeta_1 \cdot \sigma_{0\ 1-\lambda})$, i.e., is of the fifth type. Then

$$\phi \leq_{\Lambda} (\sigma_{00} + \sigma_{01}) \cdot (\sigma_{1\ 1-\kappa} \cdot \zeta_0 + \zeta_1 \cdot (\sigma_{00} + \sigma_{01})) \leq_M \sigma_{1\ 1-\kappa} \cdot \zeta_0 \cdot (\sigma_{00} + \sigma_{01}) + \zeta_1.$$

Therefore,

$$\begin{aligned} \phi \leq_M \sigma_{1\kappa} \cdot (\zeta_1 + \phi_0 \cdot \sigma_{1-\kappa}) &\leq_A (\sigma_{10} + \sigma_{11}) \cdot (\zeta_1 + \phi_0 \cdot (\sigma_{10} + \sigma_{11})) \\ &\leq_M \phi_1 + \phi_0 \leq \sum F. \end{aligned}$$

Thus, $\phi \leq_M \sum F$. For $\phi = \sigma_{\kappa\lambda} \cdot (\zeta_\kappa + \zeta_{1-\kappa} \cdot \sigma_{\kappa 1-\lambda} \cdot \sigma_{1-\kappa})$ of the fourth type a similar argument shows

$$\phi \leq_M \zeta_\kappa \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1}) + \zeta_{1-\kappa} \cdot \sigma_{1-\kappa} \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1}) \leq \phi_\kappa + \phi_{1-\kappa} \leq \sum F.$$

This completes our preliminary remarks. (i) is proved by induction on the formation of terms. The only nontrivial part of the argument is the passage over products: assume that $\sigma_\kappa \sim_\Theta \sum F_\kappa$ and $\sum F_\kappa \leq_M \sigma_\kappa$ (for $\kappa = 0, 1$) and consider $\sigma_0 \cdot \sigma_1$. By the above we have sets G_0, G_1 such that

$$\sigma_0 \cdot \sigma_1 \sim_\Theta \sum \{\sigma_0 \cdot \phi_1 : \phi_1 \in G_1\} \sim_\Theta \sum \{\phi_0 \cdot \phi_1 : \phi_1 \in G_1, \phi_0 \in G_0\}$$

where each term $\phi_0 \cdot \phi_1 \leq_M \sigma_0 \cdot \sigma_1$. Thus, it only remains to consider the simple terms $\phi_0 \cdot \phi_1$. Suppose $\phi_\kappa = \zeta_\kappa \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1})$ for $\kappa = 0, 1$. We apply α_4, α_1 and then α repeatedly as in the construction of G above to see that $\phi_0 \cdot \phi_1$ is equivalent modulo Θ to sums of ST1's of the type that occur in (2). Repeating our previous argument shows that, for each such ST1 ν , $\nu \leq_M \phi_0 \cdot \phi_1$ and hence $\nu \leq_M \sigma_0 \cdot \sigma_1$ as desired. This completes the proof of Lemma 1.2.

The following corollary which is an easy consequence of Lemmas 1.1 and 1.2 was suggested to us by Professor McKenzie.

Corollary 1.5. *An inclusion $\sigma \leq \tau$ is valid in M_3 iff every special inclusion $\nu \leq \phi$, for which $\nu \leq_M \sigma$ and $\tau \leq_M \phi$, is provable in M .*

2. We will now show that Schützenberger's identity β does not characterize \mathfrak{UN}_5 . Since β is rather complicated it is useful to observe that it is equivalent to the following two identities:

$$\beta_1. x \cdot (y + z) = x \cdot (xy + z) + x \cdot (y + xz),$$

$$\beta_2. x \cdot (w + y \cdot (u + v)) = xy \cdot (u + v) + x \cdot (yu + w) + x \cdot (yv + w).$$

The identity β_1 is just McKenzie's η_3 ; β_2 is the dual of η_7 .

Suppose Q_3 is the lattice given in Figure 1. Observe that Q_3 is subdirectly irreducible, self-dual and has a nontrivial automorphism ϕ . Our results are based on the following lemma.

Lemma 2.1. β_2 holds in Q_3 .

As a consequence of this lemma, Q_3 gives a counterexample to Schützenberger's claim.

Theorem 2.2. *The identity β holds in Q_3 but Q_3 is not a member of \mathfrak{UN}_5 .*

Proof. That $Q_3 \notin \mathfrak{UN}_5$ is mentioned in McKenzie's paper. For a short direct proof we need only observe that McKenzie's identity η_1 , $x \cdot (y + u) \cdot (y + v) \leq x \cdot (y + w) + xu + xv$, fails in Q_3 when f, e, g and b are assigned to x, y, u , and v

respectively. The fact that $\beta \in \Theta Q_3$ follows from Lemma 2.1, the self-duality of Q_3 and the remark that $\beta_1 \in \Theta_1[\beta_2^d]$. To see that β_1 is a consequence of Λ and β_2^d , $x + w(y + uv) = (x + y + uv) \cdot (x + w \cdot (y + u)) \cdot (x + w \cdot (y + v))$, we first observe

$$(1) \quad w \cdot (x + y) \leq_{\beta_2^d} x + w \cdot (y + xw).$$

This follows from β_2^d by setting $u = x$ and $v = w$. Thus,

$$x \cdot (xy + z) + x \cdot (y + xz) \sim_{\beta_2^d} [x(xy + z) + y] \cdot x \cdot (y + z) \sim_{\beta_2^d} x \cdot (y + z)$$

where the last equality holds by (1).

Proof of Lemma 2.1. Suppose the elements x', y', w', u' , and v' in Q_3 are substituted for the variables x, y, w, u and v in β_2 respectively. The left (similarly, the right) side of β_2 is assigned the value LS (similarly RS). It is obvious that $RS \leq LS$; thus, it suffices to show LS is always equal to a sum of the values on the right. This is obviously true if either $w' \geq y'$ or $w' \geq u' + v'$ or u' and v' are comparable. We assume

$$(2) \quad w' \not\geq y', w' \not\geq u' + v', \text{ and } u' \text{ and } v' \text{ are incomparable.}$$

In view of the automorphism ϕ and the fact that $LS \leq RS$ whenever $w' \in \{0, 1\}$, it is enough to show $LS \leq RS$ whenever $w' \in \{a, f, g, e, c\}$. If w' is incomparable with $u' + v'$, then either $w' = a$ and $u' + v' = b$ or $w' = g$ and $u' + v' \in \{e, b\}$. In the first case either $u' = b$ or $v' = b$ so $LS = x' = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$; in the second, if $u' + v', y' \in \{1, b, b\}$ then $LS = x' = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$ and if either $u' + v' = e$ or $y' \in \{a, f, e, d\}$, $LS = x'a = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$. Hence, we may assume

$$(3) \quad w' < u' + v'.$$

If $w' < y'$, then $y' \cdot (u' + v') \geq w'$ so $LS = x'y' \cdot (u' + v')$; hence, we may also assume

$$(4) \quad w' \text{ and } y' \text{ are incomparable.}$$

From (2), (3), (4) it remains to consider four cases

$$(6.1) \quad w' = a, y' \in \{b, b\}, u' + v' = 1,$$

$$(6.2) \quad w' \in \{f, e\}, y' = g, u' + v' \in \{1, a, b\},$$

$$(6.3) \quad w' = g, y' \in \{f, e, b, b, d\}, u' + v' \in \{1, a\},$$

$$(6.4) \quad w' = c, y' \in \{d, b\}, u' + v' \in \{1, a, b, e\}.$$

To illustrate, we consider (6.4). If $u' + v' \in \{a, e\}$, either u' or v' belong to $\{d, e, f\}$; thus, $LS = x' \cdot e = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$. On the other hand, if $u' + v' \in \{1, b\}$, either u' or v' belongs to $\{b, b\}$; so either $y' \cdot u' = y'$ or $y' \cdot v' = y'$. Thus $LS = x' \cdot (w' + y') = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$ as desired. The

other cases are, likewise, easily checked. We conclude that $LS \leq RS$, and hence β_2 , always holds in \mathcal{Q}_3 .

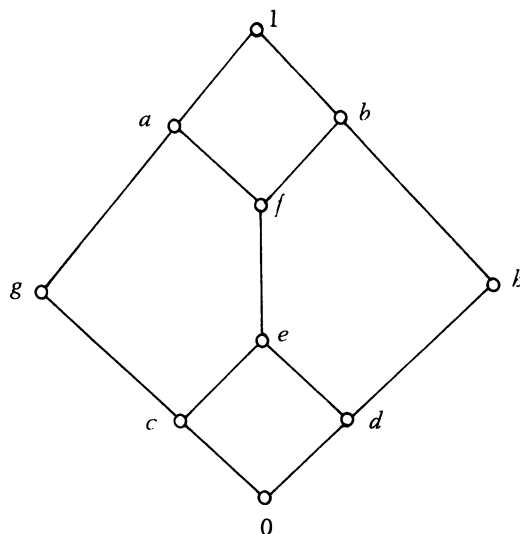


Figure 1

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