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PROJECTIVE GROUPS OF DEGREE LESS THAN $4p/3$ WHERE CENTRALIZERS HAVE NORMAL SYLOW p-SUBGROUPS

BY

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ABSTRACT. This paper proves the following theorem:

Theorem 1. *Let $\overline{G}$ be a finite primitive complex projective group of degree n with a Sylow p-subgroup $\overline{P}$ of order greater than p for p prime greater than five. Let $n \not\leq p$, $n < 4p/3$, and if $p=7$, $n \leq 8$. Then $p \equiv 1 \pmod 4$, $\overline{P}$ is a trivial intersection set, and for some nonidentity element $\overline{x}$ in $\overline{G}$, $C(\overline{x})$ does not have a normal Sylow p-subgroup.*

1. **Introduction.** The main object of study in this paper is the case where a projective group $\overline{G}$ has a Sylow p-subgroup $\overline{P}$ which is a trivial intersection set and nonidentity elements of $\overline{G}$ have centralizers which have normal Sylow p-subgroups. The other possible cases were studied in earlier papers. These results are combined in Theorem 3.

2. **Notation.** If H is a subgroup of a group G, we let H^G be the normal subgroup of G generated by H. The set of nonidentity elements of H is called $H^{\#}$. For characters μ and ν of H we let $(\mu, \nu) = (1/|H|) \sum_{x \in H} \mu(x)\overline{\nu(x)}$, and we let the squared norm $\|\mu\|^2 = (\mu, \mu)$.

A representation X of a group G is called quasiprimitive if it is irreducible and its restriction to any normal subgroup of G is the sum of equivalent irreducible constituents.

3. **Projective groups where centralizers of nonidentity elements have normal Sylow p-subgroups.** We shall prove the following theorem:

Theorem 2. *Let p be a prime greater than five. Let G be a finite group with a faithful, quasiprimitive, complex representation X with character χ of dimension $n < 4p/3$, $n \neq p$. Let P be an abelian Sylow p-subgroup of G. Throughout, let $Z = Z(G)$, $C = C(P)$, $N = N(P)$, and $N_0 = \sum_{y \in P^{\#}} C(y) - Z$. Let P satisfy* [4, *Hypothesis* 4.1] (*that is, P and N_0 are trivial intersection sets and $N(N_0) = N$*). *Let $(|Z|, p) = 1$. Then $|P| \leq p$.*

By running through the classifications of groups of small degree, it can be

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seen that the hypothesis $p > 5$ of Theorem 2 is unnecessary. By [2, (4A)], the assumption $n \neq p$ is unnecessary. Also, $4p/3$ can be replaced by a number asymptotic to $\sqrt{2}\,p$ for large p. The following proof of Theorem 2 does not require the use of [7].

Proof. Let $X(G)$ be a counterexample with n minimal. Given that n is minimal, let $|G|$ be minimal. Then $|P| \geq p^2$. Suppose $X(G)$ is imprimitive on $c > 1$ subspaces and let K be the subgroup of G fixing these spaces setwise. By quasi-primitivity, K is represented faithfully by the c equivalent constituents of $\chi|_K$ of degree no larger than $2p/3$. By [4], $p^2 \nmid [K : O_p(K)]$. As $O_p(K) \trianglelefteq G$, by quasi-primitivity, $O_p(K) \subseteq Z$ and $O_p(K) = \langle 1 \rangle$. Then $p \mid [G:K] \mid c!$ and $c = n$. Then K is an abelian normal subgroup of G and $K = Z$. Then at most p divides $c!$ and $|G|$, which is a contradiction. If $X(G)$ is a subgroup of a tensor product of two smaller dimensional groups, then, unless $p = 7$ and $n = 9$, using $((p-1)/2)^2 > 4p/3 \geq n$ with [5] and using $n/2 < 2p/3 < p - 1$ with [4] we have by [6, Lemma 1] that $p^2 \nmid [G : O_p(G)]$, contrary to $O_p(G) = \langle 1 \rangle$ and $p^2 \mid |G|$. Even if $p = 7$ and $n = 9$, $X(G)$ contains $M \otimes I_3$ and $I_3 \otimes M$ where $M \simeq \mathrm{SL}(2,7)$ and a 7-element $x \in I_3 \otimes M$ has $M \otimes I_3 \subseteq C(x)$, so $C(x)$ does not have a normal Sylow 7-subgroup, contrary to P being a T.I. set. Therefore, G is strongly primitive.

Let P^G be the normal subgroup of G generated by P. By strong primitivity and [3, Theorem 51.7], $X \mid P^G$ is irreducible. If $X \mid P^G$ is imprimitive, then $X \mid P^G$ is monomial; otherwise, the p-elements generating P^G would fix all spaces of primitivity. If $X \mid P^G$ is monomial, then $\langle 1 \rangle \neq O_p(P^G) \trianglelefteq G$, contrary to $O_p(G) = \langle 1 \rangle$. As P^G also satisfies [4, Hypothesis 4.1], $G = P^G$ by minimality of $|G|$. For any nonprincipal irreducible character γ of G we write

$$\gamma|_N = \pi + \rho \tag{1}$$

where ρ is the sum of irreducible constituents of $\gamma|_N$ having P in the kernel and π is the sum of the others. As $G = P^G$, $\|\pi\| \geq 1$.

Throughout, let $t = [N : PZ]$. Let $m = \gamma(1)$ and $b = \rho(1)$. By [4, Lemma 2.11], π vanishes on $N - (N_0 \cup Z)$. As in [4, Lemma 4.1],

$$\begin{aligned}
1 = \|\gamma\|^2 &> (1/|G|)(|G|/|N|) \sum_{N_0} |\gamma(x)|^2 \\
&= -m^2/t|P| + (1/|N|) \sum_{N_0 \cup Z} (\pi(x) + \rho(x))(\overline{\pi(x)} + \overline{\rho(x)}) \\
&= -m^2/t|P| + (1/|N|)\Big[\sum_N (\pi(x)\overline{\pi(x)} + \pi(x)\overline{\rho(x)} + \rho(x)\overline{\pi(x)}) + \sum_{N_0 \cup Z} |\rho(x)|^2\Big] \qquad (2) \\
&= -m^2/t|P| + \|\pi\|^2 + (1/|N|) \sum_{N_0 \cup Z} |\rho(x)|^2 \\
&\geq -m^2/t|P| + \|\pi\|^2 + b^2/t.
\end{aligned}$$

When $\gamma = \chi$, throughout we let α and β correspond to π and ρ respectively, $q = \beta(1)$, and

$$\chi|_N = \alpha + \beta, \tag{3}$$

where we show that $n \geq p+1$, $\beta(1) \leq 1$, $\|\alpha\| = 1$, $\alpha|_P$ is a sum of $\alpha(1) = n - q$ distinct nonprincipal linear characters of P permuted transitively by N, $|P| = p^2$, P is elementary abelian, $O_{p'}(G) = Z$, and $C = PZ$.

Since $|P| \geq p^2$, $O_p(G) = \langle 1 \rangle$, and $O^2(G) \supseteq P^G = G$, by [6], we have $n > p - 1$ and $n \geq p+1$ as $n \neq p$ by hypothesis. If $t = 1$, then $N = PZ$ is abelian and by (2), $p < n \leq \|\alpha\|^2 + q < 1 + n^2/t|P| \leq 1 + (4p/3)^2/p^2 = 25/9$, which is a contradiction. Therefore, $t \geq 2$ and by (2),

$$\|\alpha\|^2 \leq 1 + n^2/t|P| \leq 1 + (4p/3)^2/2p^2 < 2,$$

and $\|\alpha\| = 1$. Also, by (2), $q = b \leq n/(|P|)^{1/2} \leq (4p/3)/p < 2$, and $q \leq 1$. More generally, in (2),

$$(\gamma|_P, 1_P) < \gamma(1)/(|P|)^{1/2}. \tag{4}$$

If $\alpha|_P$ has homogeneous constituents (Wedderburn components) of dimension e, then by summing (4) over the irreducible constituents of $\chi\bar{\chi} - 1_G$, we have

$$(n-1)e \leq [(n-q)/e]e^2 + q - 1 = ((\chi\bar{\chi} - 1_G)|_P, 1_P) < (n^2-1)/(|P|)^{1/2}. \tag{5}$$

Then $e \leq (n+1)/p < (4p/3+1)/p < 2$ and $e = 1$. As $\|\alpha\| = 1$, the distinct linear homogeneous spaces of $\alpha|_P$ are permuted transitively by N. Also, by (5), $|P| < (n+1)^2 < (4p/3+1)^2 < p^3$ and $|P| = p^2$. If P is cyclic, $[N:C] \mid |\mathrm{Aut}(P)| = p(p-1)$ and $(n-q) \mid [N:C] \mid p-1$, contrary to $n - q > p - 1$. As $\beta(1) \leq 1$, $\chi|_P$ has all distinct linear constituents.

Suppose $O_{p'}(G) \supset Z$. Then by strong primitivity, $X \mid O_{p'}(G)$ is irreducible. Replacing elements x of $X(P)$ by all unimodular scalar multiples of x, we get a group P^* of exponent p, since $(n, p) = 1$, with $X(O_{p'}(G)) \lhd P^*X(O_{p'}(G))$. By [6, Lemma 8] for $x \in P^*$ some scalar multiple y of x has all primitive pth roots of unity occurring equally often. As $(n, p) = 1$ and $\det x = \det y$, $x = y$ and trace x is rational. Then by [9],

$$p^2 = |P/P \cap Z| \leq |P^*| \leq p^{[n/(p-1)] + [n/p(p-1)] + \cdots} = p$$

which is a contradiction. If $C \supset PZ$, we may find a q-element $v \in C - PZ$ for q a prime unequal to p. By [2, proof of (3F)], since $\chi|_P$ is a sum of distinct linear constituents, $v \in O_q(G) \subseteq O_{p'}(G) = Z$, which is a contradiction.

Throughout, we write

$$\chi^2 = k1_G + \sum \gamma_i \tag{6}$$

where k equals 0 or 1 and the γ_i are irreducible nonprincipal characters of G.

We now divide the proof into parts.

(A) G/Z is simple. If $W \trianglelefteq G$ and $W \not\subseteq Z$, then $W = G$. Also, $G = G'$. Irreducible nonprincipal characters γ of G have degree greater than p. If U is a subgroup of Z and S is a subset of G, let $\bar{S}$ be the image of S in G/U. Then $\bar{N}_0$ is the similarly defined N_0 of $\bar{G}$, and $\bar{N}_0$ and $\bar{P}$ are T.I. sets whose normalizer is $\bar{N}$.

Proof. Let $Z \subset K \triangleleft G$. As $P^G = G$ and $O_{p'}(G) = Z$, we have $p^2 \nmid |K|$ and $p \mid |K|$, so K has a Sylow p-subgroup $Q = P \cap K$ of order p. As P is a T.I. set, P is a normal Sylow p-subgroup of $C(Q)$. Then P char $C(Q) \trianglelefteq N(Q)$. By a Sylow theorem $G = KN(Q)$. Then $N(Q)$ and $N(Q)/K \cap N(Q) \simeq G/K$ have normal Sylow p-subgroups. Then $O_p(G/K) = O^{p'}(G/K) = G/K$ since $G = P^G$. Then G/K is a p-group and $[G:K] = p$. As P is elementary abelian and $([N:P], p) = 1$, by complete reducibility, Q has an N/C complement R in P. Then $(R, N) = (R, P(N \cap K)) = (R, N \cap K) \subseteq R \cap K \subseteq R \cap (P \cap K) = R \cap Q = \langle 1 \rangle$. By (3) with α irreducible and $\alpha(1) \geq n - 1$, $R^\#$ consists of homologies contrary to [8] (alternatively, some element in $R^\#$ violates [1, Theorem 8, page 96] and quasiprimitivity). Therefore, G/Z is simple.

Let $W \trianglelefteq G$ and $W \not\subseteq Z$. Then W covers G/Z, $p^2 \mid |W|$, $P \subseteq W$, and $G = P^G \subseteq W$. As G' covers G/Z, $G' \not\subseteq Z$ and $G' = G$. Let γ be an irreducible nonprincipal character of G of degree less than $p + 1$ and kernel W. As $W \neq G$ and $W \trianglelefteq G$, $W \subseteq Z$. For any group M, let $i_p(M) = [M : O_p(M)]_p$. As $O^2(G/W) \supseteq O^{p'}(G/W) = G/W$, by [6], $i_p(G/W) \leq p$ if $\gamma(1) < p$. If $\gamma(1) = p$, then as P is abelian, by [2, (4A)], $i_p(G/W) \leq p$. Then by [6, Lemma 1], $p^2 = i_p(G/Z) \leq i_p(G/W) \leq p$, which is a contradiction.

Let $U \subseteq Z$. Let $\bar{x} \in \bar{G}$ centralize $\bar{y} \in \bar{P}^\#$. Then $(x, y) \in Z$ is the quotient of the commuting p-elements y^x and y. As $(|Z|, p) = 1$, $(x, y) = 1$. Then $\bar{N}_0$ is the N_0 for $\bar{G}$. As N_0 consists only of entire cosets of U, $\bar{N}_0$ is a T.I. set with normalizer $\bar{N}$. As $\bar{P}^\# \subseteq \bar{N}_0$, $\bar{P}$ also is a T.I. set with normalizer $\bar{N}$.

(B) $N/C \simeq L$ where L is a subgroup (L is fixed throughout this paper) of $\mathrm{GL}(2, p)$ of order t. We view L as a 2-dimensional matrix group over $\mathrm{GF}(p)$. Also, column vectors and row vectors correspond to elements of P and linear characters of P, respectively, on which L acts by matrix multiplication as N/C acts on P and characters of P. If rc is the matrix product of a row vector r with a column vector c, then $e^{(2\pi i/p)rc}$ equals the value of the corresponding character of P at the corresponding element of P.

Proof. As C is the kernel of the action of N on P, N/C is isomorphic to a subgroup of $\mathrm{Aut}(P) \simeq \mathrm{GL}(2, p)$. As $C = PZ$, $|L| = t$.

(C) Let θ be an irreducible character of N not having P in its kernel. Let

ϕ be a linear constituent of $\theta|_P$ and H be the subgroup of N fixing ϕ. Then $C \subseteq H \subseteq N_0 \cup Z$ and there exists a linear constituent ξ of $\theta|_H$ with $\xi|_P = \phi$ and $\theta = \xi^N$, the induced character. Also, θ vanishes on $N - (N_0 \cup Z)$.

Proof. Let θ be the character of the representation R on the space W. Let U be the homogeneous space (Wedderburn component) for R restricted to P such that $(\dim U)\phi$ is the character of the representation of P on U. Let H be the subgroup of N fixing ϕ. Then H is also the subgroup of N fixing U. Let ξ be the character of the representation of H on U. Then by Frobenius reciprocity, $(\xi^N, \theta) = (\xi, \theta|_H) \geq 1$. Since θ is irreducible and $\theta(1) = \dim W = [N:H]\dim U = [N:H]\xi(1) = \xi^N(1)$, $\theta = \xi^N$. Also, ξ is irreducible since $\xi^N = \theta$ is irreducible. Now, H/C corresponds to a p'-subgroup of the subgroup M of $\mathrm{GL}(2, p) \simeq \mathrm{Aut}(P)$ fixing a nonprincipal character of P. As M is isomorphic to a normal extension of a group of order p by a cyclic group of order $p-1$, H/C is cyclic. As elements of $C = PZ$ are represented by scalars on U and ξ is irreducible, ξ is linear. As any element of H fixes the nonprincipal linear character ϕ of P, by the permutation lemma, it also fixes a nonidentity element of P and lies in $N_0 \cup Z$. This proves (C) since ξ^N vanishes on $N - (N_0 \cup Z)$ since $N = N(N_0)$ and $H \subseteq N_0 \cup Z$.

(D) Let $N/C \simeq L \subseteq \mathrm{GL}(2, p)$ as in (B). Then either

I. $L/Z(L) \simeq A_5$ and $|L| \mid 60(p-1)$,

II. $L/Z(L) \simeq A_4$ and $|L| \mid 12(p-1)$,

III. $L/Z(L) \simeq S_4$ and $|L| \mid 24(p-1)$,

IV. L is monomial, $|L| \mid 2(p-1)^2$, and L contains a diagonal subgroup A with $[L:A] \leq 2$, or

V. L can be written as a monomial group in $\mathrm{GL}(2, p^2)$ where L contains a subgroup

$$A = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^p \end{pmatrix} \right\rangle \quad \text{for some } \zeta \in \mathrm{GF}(p^2) \text{ and } [L:A] \leq 2.$$

Here, $|L| \mid 2(p^2 - 1)$.

In cases I, II, and III, L is irreducible, $|Z(L)| \mid (p-1)$, and $-I_2 \in Z(L)$.

Proof. We may consider L to be a faithful 2-dimensional representation of a p'-group over a field of characteristic p. In the case of p'-groups, complex irreducible representations have a one to one correspondence with p-modular representations. Then L may be obtained from a finite complex p-integral 2-dimensional group by taking coefficients modulo an ideal dividing (p). By the classification in [1] of 2-dimensional complex linear groups, we have I, II, or III or L may be taken as monomial when written over a larger field. In the monomial cases there exists A, an abelian subgroup of index 1 or 2 in L. The character of the representation of A by our linear group L is a sum of two linear characters σ and τ. If σ and τ lie in $GF(p)$, we have case IV. If they do not, since $\sigma + \tau$ lies in $GF(p)$, σ and τ lie in $GF(p^2)$ and are algebraic conjugates over $GF(p)$. Then $\sigma = \tau^p$.

(E) Let θ, ϕ, and H be as in (C). Then except in case IV, $[H:C] \leq 5$. If $\theta = \alpha$, throughout this paper let $h = [H:C]$. Then $t = (n-q)h$. Suppose we are in case IV or V. Let $\theta = \alpha$, let A be as in (D), and let $M \subseteq L$ correspond to $H/C \subseteq N/C$. Then $M \cap A = \langle 1 \rangle$ and $h \leq 2$.

Proof. As in the proof of (C), H/C corresponds to the cyclic subgroup M of $L \subseteq GL(2, p)$ of order dividing $p-1$ and fixing the character ϕ. As elements of M have an eigenvalue 1, $M \cap Z\,GL(2, p) = \langle 1 \rangle$. Then since M is cyclic, in case I, II, and III, $[H:C]$ divides the order of some element in A_5, A_4, or S_4, so $[H:C] \leq 5$. In case V, I_2 is the only element of A with an eigenvalue 1, so $M \cap A = \langle 1 \rangle$ and $[H:C] \leq 2$.

Suppose $\theta = \alpha$. By (C), $n - q = \alpha(1) = [N:H] = t/h$. Suppose further that we have case IV. Let A be the diagonal subgroup of L of index at most 2. Then if $M \cap A = \langle 1 \rangle$, $[H:C] \leq 2$. Therefore, suppose that $\operatorname{diag}(\sigma, \tau)$ lies in $(M \cap A)^{\#}$. Then, as elements of M have an eigenvalue 1, σ or $\tau = 1$, say $\sigma = 1$. Then ϕ corresponds to a multiple of $(1, 0)$. Let ζ be the homomorphism from A to $GF(p)^{\#}$ with $\zeta(\operatorname{diag}(\pi, \rho)) = \pi$. Then $M \cap A$ is the kernel of ζ and $[A : M \cap A] = |\zeta(A)| \mid (p-1)$. Also, $[L:M] = [L:A][A:M \cap A]/[M : M \cap A] \mid 2(p-1)$. Therefore, $\alpha(1) = [N:H] = [L:M] \mid 2(p-1)$. Then, since $\alpha(1) \leq n < 4p/3 < 2(p-1)$, $\alpha(1) \leq p - 1$. Furthermore, $n = \alpha(1) + q \leq \alpha(1) + 1 \leq p$, contrary to (A).

(F) Let ζ and ξ be the characters of the symmetric and skew-symmetric tensors of χ^2, respectively. Then, if $-I_2 \in L$, $(n+q)/2 = (1_P, \zeta_{|P})$ and $(n-q)/2 = (1_P, \xi_{|P})$.

Proof. As multiplying a row vector by $-I_2$ corresponds to taking the complex conjugate of the corresponding character, if $-I_2 \in L$ then all complex conjugates of constituents of $\alpha_{|P}$ also occur in $\alpha_{|P}$. By linearity of the homogeneous constituents of $\chi_{|P}$ this concludes the proof of (F).

(G) $(\chi^2_{|P}, 1_P)$ equals q or n. If $(\chi^2_{|P}, 1_P) = n$ then $n - q$ is even.

Proof. If $(\chi^2_{|P}, 1_P) \neq q$, then $\alpha_{|P}$ contains a pair of complex conjugate linear characters. Then $\alpha_{|P}$ consists entirely of pairs of complex linear characters since the action of N is transitive on the linear constituents of $\alpha_{|P}$ and commutes with scalar multiplication (by -1 in particular and by elements of $GF(p)$ in general) of the linear characters of P. This proves (G) since $\chi_{|P}$ has only linear homogeneous components.

(H) Let $\gamma = \gamma_i$ be a nonprincipal constituent of χ^2. Let $\gamma_{|N} = \pi + \rho$ as in (1) with P in the kernel of ρ but not in the kernel of any constituent of π. Then $\|\pi\| = 1$.

Proof. Let $m = \gamma(1)$, $b = \rho(1)$, and $h = [H:C]$ with H being the H in (C) when $\theta = \alpha$. Let ζ and ξ be the characters of the symmetric and skew-symmetric tensors of χ^2, respectively. We may assume that $\|\pi\|^2 \geq 2$, otherwise (H) holds. By (C) irreducible characters of N without P in the kernel have degree no larger than $[N:C] = t$. Therefore, $\|\pi\|^2 \geq \{\pi(1)/t\} = \{(m-b)/t\}$ where $\{x\}$ is the smallest integer greater than or equal to x. Then fixing p and using (2), we use the following to define various functions:

$$-m^2/tp^2 + \max(2, \{(m-b)/t\}) + b^2/t < 1, \tag{7}$$

$$h(t, m, b) = -m^2/tp^2 + \max(2, m/t + b^2/t - b/t) < 1, \quad \text{and} \tag{8}$$

$$g(t, m) = -m^2/tp^2 + \max(2, m/t) < 1. \tag{9}$$

For at least one of $\mu = \zeta$ or ξ we have $\mu = s1_G + \gamma + \nu$ where ν is a sum of nonprincipal irreducible characters ψ_i of G and $s \leq k \leq 1$. By (4), $\psi_i(1) > (\psi_{i|P}, 1_P)p$. Summing this over i, we have

$$\mu(1) - s - m = \nu(1) \geq (\nu_{|P}, 1_P)p = [(\mu_{|P}, 1_P) - s - b]p.$$

Suppose that $-I_2 \in L$. Then by (F) and the above inequality, if $\mu = \zeta$ we have

$$n(n+1)/2 - s - m \geq [(n+q)/2 - s - b]p \tag{10}$$

and if $\mu = \xi$ we have

$$n(n-1)/2 - s - m \geq [(n-q)/2 - s - b]p. \tag{11}$$

In either case,

$$n(n+1)/2 - m \geq [(n-1)/2 - 1 - b]p \tag{12}$$

and

$$b \geq (n-1)/2 - 1 - n(n+1)/2p + m/p = f(n, m) \tag{13}$$

define $f(n, m)$. For fixed m, $f(n, m)$ decreases as a function of $n \geq (p-1)/2$. Since $p < n < 4p/3$ by (A),

$$b \geq f(n, m) \geq f(4p/3, m) = m/p - (2/9)p - 13/6. \tag{14}$$

We no longer assume that $-I_2 \in L$. The cases $p = 7, 11, 13, 17, 19$, and 23 were examined by computer for $n = p+1, \cdots, [4p/3]$; $q = 0$ or 1; $s = 0$ or 1; $m = p+1, \cdots, -s + n(n+1)/2$; $h = 1, \cdots, 5$; and $b = 0, \cdots, -s + (n+q)/2$. The computer ruled out cases for any of the following reasons:

1. Inequality (7) failed.
2. $n - q$ and p failed to satisfy any of the following two propositions:

P_1: $(n-q)$ divides $60(p-1)$ or $24(p-1)$,

P_2: $(n-q)$ divides $2(p-1)^2$ or $2(p^2-1)$. (This uses (D).)

3. P_1 fails and $b > 2$. (This uses (D) and (E).)
4. If ($b > 2$ or P_2 fails) and P_3 holds where P_3 is the proposition:

P_3: Inequality (10) fails and $[n(n-1)/2 - s - m < 0$ or (11) fails].

(Here, if $-I_2 \in L$, then if $\mu = \zeta$, we have (10), and if $\mu = \xi$, we have $s + m \leq n(n-1)/2$ and (11). Therefore, $-I_2 \in L$ implies P_3 fails. However, if $b > 2$ or P_2 fails, then by (D) and (E), $-I_2 \in L$.)

The only cases left by the computer satisfied one of the following:

1. $p = 7$, $n - q = 8$, and P_3 holds.
2. $p = 7$, $n - q = 9$, $b \leq 3$, $m \geq 30$, and the left side of (7) is greater than .4.
3. $p = 13$, $n - q = 16$, $b = 2$, and P_3 holds.
4. $p = 19$, $n - q = 24$, $b = 2$, and P_3 holds.

In cases 1, 3, and 4, P_3 holds, so $-I_2 \notin L$. Then we have case IV or V in (D). Let A be the diagonal subgroup of L of index at most 2 (in case V, A may have to be written in $GF(2, p^2)$). As $2 \mid b(n-q)/2 = t/2 \mid |A|$, in case V we have $-I_2 = \operatorname{diag}(-1, (-1)^p) \in A$, which is a contradiction. Therefore, we have case IV. Let D be the group of all nonsingular diagonal matrices in $GL(2, p)$. Then $A \subseteq D$ and $|D| = (p-1)^2$. In all cases 1, 3, and 4, two divides $b(n-q)/2$ and $|A|$ to at least as high a power as it divides $(p-1)^2$ and $|D|$. Then a Sylow 2-subgroup of A is a Sylow 2-subgroup of D and contains $-I_2$, which is a contradiction.

Therefore, we have case 2. As $n - q = 9 \nmid 2(p^2-1)$ and neither 60, 12, nor 24 divide $9b$, in (D) we have case IV. Then by (E), $M \cap A = \langle 1 \rangle$ and $b = 1$ or 2 where $M \subseteq L$ corresponds to $H/C \subseteq N/C$. Changing coordinates, we take

$$A = \{\operatorname{diag}(x, y) \mid x, y \in GF(7),\ x, y = 1, 2, 4\}.$$

If $b = 2$, we have $\begin{pmatrix} 0 & \psi^{-1} \\ \psi & 0 \end{pmatrix} \in M$ for some ψ. Changing coordinates by $v \longrightarrow (\operatorname{diag}(\psi, 1)v$ for $v \in \{\text{column vectors}\} \simeq P$, we take $\psi = 1$. Further changing coordinates by a scalar matrix, or by a diagonal matrix if $b = 1$, we take (1, 1) to correspond to a linear constituent of $\alpha_{|P}$. Then $\{(x, y) \mid x, y = 1, 2, 4\}$ is the set of linear constituents of $\alpha_{|P}$ and $\{(x, y) \mid x, y = 1, 2, 3, 3, 4, 5, 5, 6, 6\}$ is the set of 81 constituents of $\chi^2_{|P}$ with multiplicities. If $b = 1$, then under the action of N, the constituents of $\mu_{|P}$ lie in four orbits of length 9: O_2, O_2, O_3, and O_4; or five orbits of length 9: O_1, O_2, O_2, O_3, and O_4 if μ corresponds to the skew-symmetric or symmetric tensors, respectively, of χ^2. Here O_1 is represented by (2, 2), O_2 by (3, 3), O_3 by (2, 3), and O_4 by (3, 2). If $b = 2$, then O_3 and O_4 are joined into an orbit O_{34} of length 18. As $9 \mid |O_i|$ for all i, $9 \mid m$ and $m = 36$ or 45. If $\|\pi\|^2$
$\max(2, \{(m-b)/t\}) = \{m/9b\}$, then the right side of (2) exceeds 1.4, which is a contradiction. By (C), orbits of linear constituents of $\chi^2_{|P}$ correspond to irreduc-

ible constituents of $\chi^2_{|N}$. If $b = 2$, then O_{34} corresponds to the only possible irreducible constituent of π of degree 18. As $m \geq 36$, π has at least two irreducible constituents of degree 9 and $\|\pi\|^2 > \{m/18\}$, which is a contradiction. Therefore, $b = 1$. Then in application of (C) to any irreducible constituents of $\chi^2_{|N}$, $H = PZ = C$. If ν and μ both correspond to O_2, then by (C), ν and μ are induced from linear constituents ϕ and θ of $\nu_{|C}$ and $\mu_{|C}$, respectively, with $\phi_{|P} = \theta_{|P}$. As $\chi^2_{|Z}$ is a multiple of $\phi_{|Z}$ and $\theta_{|Z}$, $\phi_{|Z} = \theta_{|Z}$, $\phi = \theta$, and $\nu = \mu$. As $\|\pi\|^2 = \{m/9\}$, π cannot contain both constituents corresponding to O_2 as they are identical. Then $m = 36$, a case not reported by the computer as (7) fails.

We may now assume that $p \geq 29$. As $n < 4p/3$ and $t = (n - q)b \leq 5n$, $t \leq 20p/3$ and $2t \leq 40p/3 < p^2/2$. For fixed t, $g(t, m)$ is a decreasing function of m for $0 \leq m \leq 2t$, increasing for $2t \leq m \leq p^2/2$, and decreasing for $p^2/2 \leq m$. For $m \leq p^2/2$,

$$g(t, m) \geq g(t, 2t) = -(2t)^2/tp^2 + 2 = 2 - 4t/p^2 \geq 2 - 80/3p > 1. \tag{15}$$

Then by (9), $m \geq p^2/2$. Also,

$$\begin{aligned} f(4p/3, m) &= m/p - (2/9)p - 13/6 \geq p/2 - (2/9)p - 13/6 \\ &= (5/18)p - 13/6 \geq (5/18)29 - 13/6 > 1. \end{aligned} \tag{16}$$

Suppose that $-I_2 \in L$. Then (14) holds. For $b \geq 1$ and t and m fixed, $h(t, m, b)$ is an increasing function of b. By (16) we may combine (8) and (14) and define $k(t, m)$:

$$k(t, m) = h(t, m, f(4p/3, m)) \leq h(t, m, b) < 1. \tag{17}$$

As $m \geq p^2/2 > 2t$,

$$h(t, m, b) = -m^2/tp^2 + m/t + b^2/t - b/t.$$

The m^2 terms cancel in $k(t, m)$. The coefficient of m in $k(t, m)$ is

$$1/t + (2/t)(1/p)[-(2/9)p - 13/6] - (1/t)(1/p) = (5/9 - 16/3p)/t > 0.$$

Therefore, for $m \geq p^2/2$, by (16) and (15)

$$k(t, m) \geq k(t, p^2/2) \geq h(t, p^2/2, 1) = g(t, p^2/2) > 1.$$

This contradicts (17).

Therefore, we have $p \geq 29$, $m \geq p^2/2$, and $-I_2 \notin L$. By (D) we then have case IV or V. Let $x = (4p - 1)/3$. By (E), $b \leq 2$ and $t = (n - q)b \leq 2n \leq 2x$. Also, $m \leq n(n + 1)/2 \leq x(x + 1)/2 < 2p(4p + 3)/9 < p^2$. As $g(t, m)$ is decreasing in m for $m \geq p^2/2$, we have by (9),

$$1 > g(t, m) = (m/t)(1 - m/p^2)$$

$$\geq (m/2x)(1 - m/p^2) \geq [(x(x+1)/2)/2x][1 - x(x+1)/2p^2]$$

$$= (x+1)(2p^2 - x^2 - x)/8p^2 = (4p+2)(18p^2 - 16p^2 + 8p - 1 - 12p + 3)/216p^2$$

$$\geq (4(29) + 2)(1/108)(1 - 2/29) = (3186)/3132 > 1,$$

which is a contradiction.

(I) Let γ be a nonprincipal irreducible character of G with $\gamma_{|N} = \pi + \rho$ as in (1). Let $\|\pi\| = 1$ and $\pi(1) = \alpha(1)$. Then $\rho(1) \leq 1$. For the remainder of the proof, let $\omega = \gamma_1$ be a fixed nonprincipal irreducible constituent of χ^2 with $\omega_{|P}$ containing ϕ^2 where ϕ is a fixed linear constituent of $\alpha_{|P}$. Write

$$\omega_{|N} = \sigma + \tau \tag{18}$$

as in (1). Then $\|\sigma\| = 1$, $\sigma(1) = \alpha(1)$, and $q \leq \tau(1) \leq 1$.

Proof. By (4), $p\rho(1) \leq \pi(1) + \rho(1)$ and $\rho(1) \leq \pi(1)/(p-1) < 4p/3(p-1) < 2$. As ϕ^2 is a constituent of $\chi^2_{|P}$, $\omega = \gamma_1$ exists. By (H), $\|\sigma\| = 1$. As ϕ^2 and ϕ are fixed by the same subgroup H of N, by (C) we have $\sigma(1) = [N:H] = \alpha(1)$. Letting $\gamma = \omega$, we have $\tau(1) \leq 1$. If $q = 1$ and $\tau(1) = 0$, then by (A), $p < \omega(1) = \sigma(1) = \alpha(1) < \chi(1)$, contrary to (A) and minimality of $\chi(1)$ for a counterexample since by (A) the kernel of ω lies in Z.

(J) Let γ be a nonprincipal irreducible character of G with $\gamma_{|N} = \pi + \rho$ as (1). Let $\|\pi\| = 1$. Let $\gamma_{|Z}$ and $\omega_{|Z}$ be multiples of the same linear character of Z. Let $s = \sigma(1)$ and $r = \pi(1)$. Then $s\gamma - r\omega = (s\pi - r\sigma)^G$, the induced character, and vanishes outside of $N_0^G = \bigcup_{g \in G} g^{-1}N_0 g$. Also, $s\rho(1) = r\tau(1)$. Furthermore, if $\pi = \sigma$, then $\gamma = \omega$.

Proof. Suppose that $\pi \neq \sigma$. By hypothesis, $s\pi - r\sigma$ vanishes on Z. By (C), π and σ vanish on $N - (N_0 \cup Z)$ and $s\pi - r\sigma$ vanishes on $N - N_0$. As N_0 is a T.I. set, by [3, Lemma 38.15], $\|(s\pi - r\sigma)^G\|^2 = \|s\pi - r\sigma\|^2 = s^2 + r^2$. By Frobenius reciprocity,

$$((s\pi - r\sigma)^G, s\gamma - r\omega) = (s\pi - r\sigma, (s\gamma - r\omega)_{|N}) = (s\pi - r\sigma, s\pi + s\rho - r\sigma - r\tau) = s^2 + r^2.$$

Then $((s\pi - r\sigma)^G, s\gamma - r\omega)$, $\|(s\pi - r\sigma)^G\|^2$, and $\|s\gamma - r\omega\|^2$ all equal $s^2 + r^2$, so by the Cauchy-Schwarz inequality, $s\gamma - r\omega = (s\pi - r\sigma)^G$ which has support on N_0^G since $s\pi - r\sigma$ has support on N_0. As $1 \notin N_0^G$,

$$0 = s\gamma(1) - r\omega(1) = s\pi(1) + s\rho(1) - r\sigma(1) - r\tau(1) = s\rho(1) - r\tau(1).$$

Suppose that $\pi = \sigma$ and $\omega \neq \gamma$. Then $\pi(1) = \sigma(1) = \alpha(1)$ and by (I), $\rho(1) \leq 1$. Let $D\sigma$ be the determinant of the corresponding representation. As $G = G'$, γ and ω are unimodular. Then if $\tau(1) = \rho(1) = 1$, we have for $x \in N$,

$$\tau(x) = 1/(D\sigma)(x) = 1/(D\pi)(x) = \rho(x).$$

As $\tau(1)$ and $\rho(1) \leq 1$, in any event $\rho(x)\overline{\tau(x)}$ is real and nonnegative. Let $e = (1/|G|)\Sigma_{x\in N_0^G} |\gamma(x)|^2$ and $f = (1/|G|)\Sigma_{x\in N_0^G} |\omega(x)|^2$. Then since π and σ vanish off $N_0 \cup Z$ by (C), we have

$$(1/|G|) \sum_{x\in N_0^G} \gamma(x)\overline{\omega(x)} = (1/|G|)(|G|/|N|) \sum_{x\in N_0} (\pi(x) + \rho(x))(\overline{\sigma(x) + \tau(x)})$$

$$= (1/|N|) \sum_{x\in N_0\cup Z} (\pi(x)\overline{\sigma(x)} + \pi(x)\overline{\tau(x)} + \rho(x)\overline{\sigma(x)} + \rho(x)\overline{\tau(x)}) - \gamma(1)\omega(1)/p^2t$$

$$= (\pi, \sigma) + (\pi, \tau) + (\rho, \sigma) + \sum_{x\in N_0\cup Z} \rho(x)\overline{\tau(x)} - \gamma(1)\omega(1)/p^2t$$

$$\geq 1 + 0 + 0 + 0 - (t+1)(4p/3+1)/p^2t > 1/2.$$

As

$$(1/|G|) \sum_{x\in G-N_0^G} \gamma(x)\overline{\omega}(x) = -(1/|G|) \sum_{x\in N_0^G} \gamma(x)\overline{\omega}(x),$$

we have by the Cauchy-Schwarz inequality

$$1/2 < (1/|G|) \sum_{x\in N_0^G} |\gamma(x)|\,|\omega(x)|$$

$$\leq \left[(1/|G|) \sum_{x\in N_0^G} |\gamma(x)|^2 (1/|G|) \sum_{x\in N_0^G} |\omega(x)|^2\right]^{1/2} = [ef]^{1/2}$$

and

$$1/2 < (1/|G|) \sum_{x\in G-N_0^G} |\gamma(x)|\,|\omega(x)|$$

$$\leq \left[(1/|G|) \sum_{x\in G-N_0^G} |\gamma(x)|^2 (1/|G|) \sum_{x\in G-N_0^G} |\omega(x)|^2\right]^{1/2} = [(1-e)(1-f)]^{1/2}.$$

Multiplying the last two equations together, we have

$$1/4 < [ef]^{1/2}[(1-e)(1-f)]^{1/2} = [e(1-e)]^{1/2}[f(1-f)]^{1/2} \leq [1/4]^{1/2}[1/4]^{1/2} = 1/4,$$

which is a contradiction.

(K) We have $\tau(1) = q = k = 0$ and $\chi^2(x) = \chi(1)\omega(x)$ for all $x \notin N_0^G$.

Proof. By (I), $\sigma(1) = \alpha(1) = n - q$. By (H), we may apply (J) to the γ_i in (6). Letting $\gamma_i = \pi_i + \rho_i$ as in (1), $d_i = \pi_i(1)$, and $D = \Sigma\, d_i = \Sigma\, \pi_i(1) = n^2 - k - \Sigma\, \rho_i(1) = n^2 - (\chi^2_{|P}, 1_P)$, we have $\rho_i(1) = d_i\tau(1)/(n-q)$ and for $u \notin N_0^G$,

$$\gamma_i(u) = d_i\omega(u)/(n-q).$$

Summing the last two equations over i, we have

(19) $$(\chi^2_{|P}, 1_P) - k = \sum \rho_i(1) = \sum d_i\tau(1)/(n-q) = D\tau(1)/(n-q)$$

and, for $u \notin N_0^G$,

(20) $$\chi^2(u) - k = \sum \gamma_i(u) = \sum d_i\omega(u)/(n-q) = D\omega(u)/(n-q).$$

Suppose that $\tau(1) = 1$ and $N_0 \not\subseteq PZ$. Then for some $v \in P^\#$, $|N_0 \cup Z| \geq |C(V)| \geq 2|PZ|$. As $\tau(1) = 1$, $(1/|N|)\,\Sigma_{N_0 \cup Z}\,|\tau(x)|^2 = |N_0 \cup Z|/|N|$. By the second to last line of (2), for $\gamma = \omega$,

(21) $$[(4p-1)/3+1]^2/tp^2 \geq (n-q+1)^2/tp^2 \geq \omega(1)^2/tp^2 > |N_0 \cup Z|/|N| \geq 2/t.$$

Then $p < 11$, $p = 7$, $q = 0$, and $n = 9$. Even then the 2 in $2/t$ in (21) cannot be replaced by a larger integer or (21) would fail. As $N_0 \cup Z$ contains only entire cosets of PZ, $|N_0 \cup Z| = 2|PZ| = |C(v)|$. As $N(N_0) = N$, $C(v) = N_0 \cup Z$ corresponds to a normal subgroup E of order 2 of L where the element x of $E^\#$ has an eigenvalue 1. As $E \subseteq Z(L)$, by diagonalizing x, we have $L \subseteq D$ where D is the group of all nonsingular diagonal matrices in $GL(2, p)$. As $9 = (n-q)\,|\,|L|$, L contains $\{\operatorname{diag}(y, w) |\, y, w = 1, 2, 4\}$, a Sylow 3-subgroup of D. Then L contains at least five elements with an eigenvalue 1, and $|N_0 \cup Z| \geq 5|PZ|$, contrary to (21).

Still suppose that $\tau(1) = 1$. Then $N_0 \subseteq PZ$ and only p-singular elements lie in N_0 and N_0^G. There exists some $y \in N - PZ$. If $y \in N_0^G$, then $[y]_p \in P^\#$ and $y \in C([y]_p) \subseteq N_0 \cup Z \subseteq PZ$, which is a contradiction. Therefore, $y \in N - (N_0^G \cup Z)$. By (C), $\omega(y) = \tau(y)$ and $\chi(y) = \beta(y)$. Then by (20) and (G),

$$2 \geq |\beta^2(y) - k| = |\chi^2(y) - k| = D|\omega(y)|/(n-q) = D|\tau(y)|/(n-q) \geq (n^2-n)(1)/(n-q) \geq n-1,$$

which is a contradiction.

Therefore, $\tau(1) \neq 1$. By (I), $q \leq \tau(1) \leq 1$, so $q = \tau(1) = 0$. Then by (19), $((\chi^2_{|P}, 1_P) = k \leq 1$. By (G), $0 = q = (\chi^2_{|P}, 1_P) = k$. Then for $u \notin N_0^G$, by (20) $\chi^2(u) = D\omega(u)/(n-q) = n^2\omega(u)/n = \chi(1)\omega(u)$.

(L) $\chi(x) = 0$ for $x \notin Z \cup N_0^G$.

Proof. The group $\bar{G} = G/\Omega_1(O_2(Z))$ has the faithful representation Y corresponding to ω since by (A) the kernel of ω lies in Z. By (A), $Y(\bar{G})$ is a counterexample to the theorem. As $\omega(1) = \chi(1)$ by (K), by minimality of $|G|$, $|\Omega_1([Z]_2)| = 1$ and $(|Z|, 2) = 1$. Let ζ be a primitive $|G|$th root of unity, Q be the rationals, and $K = Q(\zeta)$. Let μ be the automorphism of K taking $[\zeta]_{2'}$ and $[\zeta]_2$ (that is, the 2-part of ζ where $\zeta = [\zeta]_{2'}[\zeta]_2$) to $([\zeta]_{2'})^2$ and $[\zeta]_2$, respectively. Then $\omega_{|Z} = \chi^\mu_{|Z}$ and by (J), for $u \notin N_0^G$, $\omega(u) = \chi^\mu(u)$. Then by (K), for $u \notin N_0^G$,

$$\chi^2(u) = \chi(1)\chi^{\mu}(u). \tag{22}$$

Let $u \notin N_0^G$ and $\chi(u) \neq 0$, and let ν be any automorphism of K. There exists an integer e such that $\zeta^{\nu} = \zeta^{e}$ and $(e, |G|) = 1$. Then $u^e \notin N_0^G$, otherwise, u^e, $u \notin Z$, $\langle u \rangle = \langle u^e \rangle$ centralizes a p-singular element and $u \in N_0^G$. Then by (22),

$$(\chi^{\nu}(u))^2 = (\chi(u^e))^2 = \chi(1)\chi^{\mu}(u^e) = \chi(1)\chi^{\mu\nu}(u). \tag{23}$$

As we let ν run over all automorphisms of K, $\mu\nu$ runs over all automorphisms of K. Taking the product of (23) as ν ranges over all automorphisms of K,

$$\left(\prod_{\operatorname{aut} K} \chi^{\nu}(u)\right)^2 = \chi(1)^{|\operatorname{aut} K|} \prod_{\operatorname{aut} K} \chi^{\nu}(u) \quad \text{and} \tag{24}$$
$$\prod_{\operatorname{aut} K} \chi^{\nu}(u) = \chi(1)^{|\operatorname{aut} K|}.$$

As $|\chi^{\nu}(u)| \leq \chi(1)$ for all $\nu \in \operatorname{aut} K$, by (24) this is always equality. Then $|\chi(u)| = \chi(1)$ and $u \in Z$.

We are now able to complete the proof of the theorem. By (K), $q = 0$, and $\chi_N = \alpha$, so by (L),

$$1 = \|\chi\|^2 = (1/|G|) \sum_{x \in G} |\chi(x)|^2$$
$$= (1/|G|)\left[\sum_{x \in Z} |\chi(x)|^2 + \sum_{x \in N_0^G} |\chi(x)|^2\right] = (|Z|/|G|)n^2 + (1/|G|)(|G|/|N|) \sum_{x \in N_0} |\chi(x)|^2$$
$$< (|Z|/|N|)n^2 + (1/|N|) \sum_{x \in N_0} |\chi(x)|^2 = (1/|N|) \sum_{x \in Z \cup N_0} |\chi(x)|^2 \leq \|\alpha\|^2 = 1,$$

which is a contradiction.

4. **The case** $p \equiv -1 \pmod 4$. The following theorem combines Theorem 2 and [7, Theorem 4].

Theorem 3. *Let p be prime greater than 5 with $p \equiv -1 \pmod 4$. Let G be a finite group with a faithful, quasiprimitive, complex representation X with character χ of dimension $n < 4p/3$, $n \neq p$, and if $p = 7$, $n \leq 8$. Then $p^2 \nmid |G/Z(G)|$.*

The hypothesis $p > 5$ of Theorem 3 is unnecessary by the classification of 2-dimensional groups in the case $p = 3$. Given Theorem 2 it is easier to prove the combination Theorem 3 and [7, Theorems 2, 3, and 4] than it is to prove [7, Theorems 2, 3, and 4] since the stronger induction hypothesis allowed in proving the former combination eliminates some of the cases that had to be studied in the proof of [7, Theorems 2, 3, and 4]. We now prove Theorem 3.

Proof. We may replace elements x of $X(G)$ by all unimodular scalar multiples of x. This does not affect quasiprimitivity or change $G/Z(G)$ within isomorphism. Therefore, we may assume that $X(G)$ is unimodular. Let P be a Sylow p-subgroup of G. Then P is abelian, otherwise some element in $(P' \cap Z(P))^{\#}$ contradicts [1, Theorem 8, p. 96] and quasiprimitivity. Then as $(n, p) = 1$, $(|Z(G)|, p) = 1$. We may assume that $X(G)$ is a counterexample to our theorem, so $|P| > p$. Then by [7, Theorem 4], P is a T.I. set and there exists a subgroup $E \subseteq G$ with $F = C(E)$ (possibly equal to G) having the following properties: $P \subseteq F$ and $X | F$ has a strongly primitive constituent Y of degree larger than p. Also, Y is faithful on P and $Y(P) \subseteq Y(F)$ satisfy [4, Hypothesis 4.1]. Furthermore, all homogeneous subspaces of $Y(P)$ are linear and $C_{Y(F)}(Y(P)) = Y(P)Z(Y(F))$. Also, $(|Z(Y(F))|, p) = 1$. Finally, $N_{Y(F)}(Y(P))$ acts transitively on the nonprincipal homogeneous subspaces of $Y(P)$.

By the above, Y has degree unequal to p. Y is quasiprimitive, $|Y(P)| = |P| > p$, $(|Z(Y(F))|, p) = 1$, and $Y(P) \subseteq Y(F)$ satisfy [4, Hypothesis 4.1]. Therefore, $Y(F)$ contradicts Theorem 2. This completes the proof.

We now prove Theorem 1 from the abstract. Let G be a linear group corresponding to $\overline{G}$ satisfying the hypothesis of Theorem 1. By [7, Theorem 1, (b)], $\overline{P}$ is a trivial intersection set. By Theorem 3, $p \equiv 1 \pmod 4$. By replacing elements x of G by all unimodular scalar multiples, we may assume that G is unimodular. As $p \nmid n$, $(|Z(G)|, p) = 1$. Suppose that for all $\overline{x} \in \overline{G}^{\#}$, $C(\overline{x})$ has a normal Sylow p-subgroup. As in the proof of Theorem 3, P is abelian. Suppose that N_0 is defined as in Theorem 2, and for some $y, g \in G$, $y \in N_0 \cap N_0^g$. Let H be the preimage of $C(\overline{y})$. Then $C(y) \subseteq H$ and $C(\overline{y})$ and H have normal Sylow p-subgroups, so $C(y)$ has a normal Sylow p-subgroup Q which we may assume is contained in a Sylow p-subgroup P^h of G. Now, y centralizes some nonidentity elements u and v of P and P^g, respectively. As $u, v \in Q \subseteq P^h$ and P is a T.I. set, $P = P^h = P^g$ and P satisfies [4, Hypothesis 4.1]. Then by Theorem 2, $|P| \leq p$, which is a contradiction.

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