

AVERAGING OPERATORS IN $C(S)$ AND LOWER SEMICONTINUOUS SECTIONS OF CONTINUOUS MAPS

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ABSTRACT. For certain kinds of compact Hausdorff spaces S , necessary and sufficient topological conditions are provided for determining if there exists a norm 1 projection of $C(S)$ onto any given separable selfadjoint subalgebra A , the conditions being in terms of the decomposition that A induces on S . In addition, for arbitrary S and selfadjoint closed subalgebra A of $C(S)$, some results on lower bounds for the norms of projections of $C(S)$ onto A are obtained. An example is given which shows that the greatest lower bound of the projection norms need not be attained.

1. Introduction. Let S and T be compact Hausdorff spaces and let ϕ be a continuous map of S onto T . By a (regular) averaging operator for ϕ , we mean a continuous linear transformation u (of norm 1) from $C(S)$ to $C(T)$ satisfying $u(g\phi) = g$ for all $g \in C(T)$, where $C(X)$ denotes the Banach space of all bounded real or complex continuous functions on X with the sup norm. If $\phi^\circ: C(T) \rightarrow C(S)$ is the isometric embedding which takes $g \in C(T)$ into $g\phi$, then the above condition on u can be written $u\phi^\circ = \text{id}_{C(T)}$, and it is clear that ϕ admits a (regular) averaging operator u if and only if there is a continuous projection P (of norm 1) of $C(S)$ onto its subalgebra $\phi^\circ C(T)$, where u and P are related by $P = \phi^\circ u$.

In the first part of this paper, we answer a question raised by Pełczyński [9, Problem 3, p. 65] concerning the existence of regular averaging operators. Let $F(S)$ denote the collection of all closed nonempty subsets of S . A mapping $\Phi: T \rightarrow F(S)$ is called a lower semicontinuous (l.s.c.) section of ϕ if (i) $\Phi(t) \subset \phi^{-1}(t)$ for all $t \in T$, and (ii) Φ is l.s.c., i.e., the set $\{t \in T: V \cap \Phi(t) \neq \emptyset\}$ is open for every open $V \subset S$; a continuous section of ϕ is a continuous map $\theta: T \rightarrow S$ such that $\theta(t) \in \phi^{-1}(t)$ for all $t \in T$ (i.e., a single-valued l.s.c. section of ϕ). Following Pełczyński [9], a compact Hausdorff space which is the continu-

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ous image of a Cantor space $\{0, 1\}^m$, for some cardinal m , by a map which admits a regular averaging operator, is called a *Milutin space*. (It can be shown, for example, that any product of compact metric spaces is a Milutin space [9, Theorem 5.6, p. 32].) We show that if S is a Milutin space and T is metrizable, then ϕ admits a regular averaging operator if and only if ϕ admits a l.s.c. section; if, in addition, T is totally disconnected, then ϕ admits a regular averaging operator if and only if ϕ admits a continuous section. Necessary and sufficient conditions are then given for an arbitrary continuous surjection to admit a l.s.c. section in terms involving the decomposition $\{\phi^{-1}(t)\}_{t \in T}$ and its limit sets (relative to the finite topology of $F(S)$ defined in §2).

In the remainder of the paper, we consider the question of lower bounds for the norms of averaging operators. From one basic proposition, we obtain, as corollaries, an ultimate form of Arens' " $3 - 2/n$ lower bound theorem" [2, Theorem 3.1] and a generalization of Pełczyński's theorem on maps of Cantor type [9, Proposition 9.8]. Obtainable as well is a theorem of Amir [1, Theorem 1] on lower bounds for the norms of projections of $B(S, \Sigma)$ onto $C(S)$, for fields Σ containing a base for the topology of S (see Remark 5.7). Another consequence of this lemma is a proof of the existence of an uncomplemented selfadjoint subalgebra of $C[0, 1]$ which is isometrically and algebraically isomorphic to $C[0, 1]$ (equivalently, there is a continuous map of the unit interval onto itself which admits no averaging operator). We close with an example which solves Problem 4 of [9] and answers some other questions as well: namely, we exhibit a continuous surjection which admits an averaging operator of norm $< 1 + \epsilon$, for each $\epsilon > 0$, but admits no regular averaging operator.

2. Preliminaries. For a compact Hausdorff space X , let $M(X)$ denote the Banach space of all (real or complex) finite regular Borel measures on X with the norm $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the total variation of μ . By the Riesz representation theorem, we can identify $M(X)$ with $C(X)^*$, the dual of $C(X)$, and for $\mu \in M(X)$, $f \in C(X)$ we shall write both $\mu(f)$ and $\int f d\mu$ interchangeably. The *probability measures* on X are the positive measures in $M(X)$ of norm 1. For $\mu \in M(X)$, the *support* of μ , or *support* μ , is the set of points $x \in X$ for which $|\mu|(V) \neq 0$ for every neighborhood V of x ; it is the smallest closed subset K of X for which $|\mu|(K) = \|\mu\|$. The *unit point mass* at $x \in X$ is denoted by ρ_x and is that measure in $M(X)$ satisfying $\rho_x(f) = f(x)$ for all $f \in C(X)$. The *weak* topology* in $M(X)$ is the weakest topology on $M(X)$ for which all the functions in $C(X)$, as linear functionals on $M(X)$, are continuous. Hence, a net of measures μ_α converges to a measure μ weak* in $M(X)$ if and only if $\mu_\alpha(f) \rightarrow \mu(f)$ for all $f \in C(X)$. The map $x \rightarrow \rho_x$ is a homeomorphism of X into the space $M(X)$ endowed with the weak* topology.

Much of the following lemma is proved in [9, Proposition 4.1, p. 22]. The remainder follows easily from the definitions except that in (ii), part (a), we use the fact that $\int g\phi d\mu = \int g d(\mu\phi^{-1})$ for $g \in C(T)$, $\mu \in M(S)$ (see p. 163 of [5]).

Lemma 2.1. *Let S, T be compact Hausdorff spaces and $\phi: S \rightarrow T$ a continuous surjection. Let u be a linear transformation from $C(S)$ to the vector space of scalar-valued functions on T . For $t \in T$, define μ_t on $C(S)$ by $\mu_t(f) = u(f)(t)$.*

(i) *u is a continuous linear operator from $C(S)$ to $C(T)$ if and only if $t \mapsto \mu_t$ is a continuous map from T to the space $M(S)$ endowed with the weak* topology; moreover, $\|u\| = \sup \{\|\mu_t\| : t \in T\}$.*

(ii) *If u is a continuous linear operator from $C(S)$ to $C(T)$, then*

(a) *u is an averaging operator for ϕ if and only if $(\phi^\circ)^*(\mu_t) = \rho_t$ for all $t \in T$ (where $(\phi^\circ)^*$ is the operator dual to ϕ°), hence, if and only if $\mu_t\phi^{-1} = \rho_t$ for all $t \in T$.*

(b) *u is a regular averaging operator for ϕ if and only if μ_t is a probability measure on S with support $\mu_t \subset \phi^{-1}(t)$ for all $t \in T$.*

Note that if $u: C(S) \rightarrow C(T)$ is a continuous linear operator, the functional μ_t defined in Lemma 2.1 is precisely $u^*(\rho_t)$, where u^* is the operator dual to u .

Corollary 2.2. *If u is an averaging operator for $\phi: S \rightarrow T$ and $\mu_t = u^*(\rho_t)$ for $t \in T$, then $\mu_t(\phi^{-1}(t)) = 1$ for all $t \in T$.*

Lemma 2.3. *Let Q, S, T be compact Hausdorff spaces and let $\psi: Q \rightarrow S$, $\phi: S \rightarrow T$ be continuous maps with $\phi\psi$ onto.*

(i) *If w is a (regular) averaging operator for $\phi\psi$, then $w\psi^\circ$ is a (regular) averaging operator for ϕ .*

(ii) *If ψ, ϕ are both onto with (regular) averaging operators v, u respectively, then uv is a (regular) averaging operator for $\phi\psi$.*

Proof. (i) $(w\psi^\circ)\phi^\circ = w(\phi\psi)^\circ = \text{id}_{C(T)}$.

(ii) $(uv)(\phi\psi)^\circ = u(v\psi^\circ)\phi^\circ = u\phi^\circ = \text{id}_{C(T)}$.

For a topological space X , let $F(X)$ denote the collection of all closed nonempty subsets of X . The *finite topology* in $F(X)$ is the topology generated by the sets of the form $\langle U_1, \dots, U_n \rangle = \{A \in F(X) : A \subset \bigcup_{i=1}^n U_i \text{ and } A \cap U_i \neq \emptyset \text{ for } i = 1, \dots, n\}$, where $\{U_1, \dots, U_n\}$ is a finite collection of open sets in X . For a detailed account of the finite topology, which includes the content of the following lemma, see [6].

Lemma 2.4. *If X is compact Hausdorff, then so is $F(X)$.*

Given any net A_α in $F(X)$, let $\liminf A_\alpha$ (resp., $\limsup A_\alpha$) denote the

set of points $x \in X$ for which $V \cap A_\alpha$ is eventually (resp., frequently) nonempty for every neighborhood V of x . Clearly, $\liminf A_\alpha \subset \limsup A_\alpha$, both are closed subsets of X , and, if X is compact, $\limsup A_\alpha \neq \emptyset$. The following lemma is an easy consequence of the definitions.

Lemma 2.5. *If X is compact Hausdorff and A_α is a net in $F(X)$, then $A_\alpha \rightarrow A$ in the finite topology of $F(X)$ if and only if $\limsup A_\alpha = A = \liminf A_\alpha$.*

We define $F^*(X)$ to be $F(X)$ with the empty set \emptyset adjoined as an isolated point. The topology of $F^*(X)$, when $F(X)$ is given the finite topology, is also called the finite topology. Moreover, a mapping Φ from a topological space S to $F^*(X)$ is again called *lower semicontinuous (l.s.c.)* if $\{s \in S: V \cap \Phi(s) \neq \emptyset\}$ is open for every open $V \subset X$.

Lemma 2.6. *Let X, Y, Z be compact Hausdorff spaces.*

(i) *If $\phi: X \rightarrow Y$ is continuous and $\Phi: Y \rightarrow F^*(Z)$ is l.s.c., then the mapping from X to $F^*(Z)$ given by $x \rightarrow \Phi(\phi(x))$ is l.s.c.*

(ii) *If $\Phi: X \rightarrow F^*(Y)$ is l.s.c. and $\phi: Y \rightarrow Z$ is continuous, then the mapping from X to $F^*(Z)$ given by $x \rightarrow \phi(\Phi(x))$ is l.s.c.*

(iii) *If $\Phi_1: X \rightarrow F^*(Y)$ and $\Phi_2: Y \rightarrow F^*(Z)$ are l.s.c., then the mapping Φ from X to $F^*(Z)$ given by $\Phi(x) = (\bigcup \{\Phi_2(y): y \in \Phi_1(x)\})^-$ is l.s.c.*

Proof. Since any map $\theta: X \rightarrow Y$ is continuous if and only if the set-valued map $x \rightarrow \{\theta(x)\}$ is l.s.c., it suffices to prove (iii).

Now, for any set-valued map Θ from a space S to the subsets of a space T , and any subset W of T , let $W(\Theta)$ denote the set $\{s \in S: W \cap \Theta(s) \neq \emptyset\}$. If we define the set-valued map $\Phi_2\Phi_1$ on X by $\Phi_2\Phi_1(x) = \bigcup \{\Phi_2(y): y \in \Phi_1(x)\}$, then it follows from our definitions that for any subset V of Z , $V(\Phi_2\Phi_1) = [V(\Phi_2)](\Phi_1)$. Hence, $V(\Phi_2\Phi_1)$ is open whenever V is open since Φ_2 and Φ_1 are l.s.c. Therefore, Φ is l.s.c. since $V(\Phi) = V(\Phi_2\Phi_1)$ whenever V is open.

3. Existence theorems for regular averaging operators. Throughout this section, S and T are compact Hausdorff spaces and ϕ is a continuous map of S onto T . We derive some necessary and sufficient conditions that ϕ admit a regular averaging operator.

Proposition 3.1. *The map $\mu \rightarrow \text{support } \mu$ from $M(S)$ to $F^*(S)$ is l.s.c. relative to the weak* topology of $M(S)$.*

Proof. Suppose not. Then there is an open set $V \subset S$, a measure $\mu_0 \in M(S)$, and a net μ_α in $M(S)$ which converges weak* to μ_0 such that $V \cap \text{support } \mu_0 \neq \emptyset$ and $V \cap \text{support } \mu_\alpha = \emptyset$ for all α . Choose a compact set $K \subset V$ such that $|\mu_0|(K) > \frac{1}{2}|\mu_0|(V)$ and choose $b' \in C(K)$ such that $\|b'\| \leq 1$ and $\int_K b' d\mu_0 > \frac{1}{2}|\mu_0|(V)$.

Extend b' to $b \in C(S)$ with $\|b\| \leq 1$ and $b = 0$ outside V . Then

$$|\mu_0(b)| = \left| \int_K b' d\mu_0 + \int_{V \setminus K} b d\mu_0 \right| > \frac{1}{2} |\mu_0|(V) - |\mu_0|(V \setminus K) > 0.$$

But we also have $\mu_0(b) = 0$ since $\mu_\alpha(b) \rightarrow \mu_0(b)$ and $b = 0$ on support μ_α for all α . Therefore, assuming the proposition is false leads to a contradiction.

Corollary 3.2. *For any continuous linear operator $u: C(S) \rightarrow C(T)$, the map $t \rightarrow \text{support } u^*(\rho_t)$ from T to $F^*(S)$ is l.s.c.*

Proof. This follows from Proposition 3.1, Lemma 2.6 (i), and Lemma 2.1 (i).

Corollary 3.3. *If u is a regular averaging operator for ϕ , then $t \rightarrow \text{support } u^*(\rho_t)$ is a l.s.c. section of ϕ .*

Proof. By Corollary 3.2 and Lemma 2.1.

Theorem 3.4. *A necessary condition that the continuous surjection $\phi: S \rightarrow T$ admit a regular averaging operator is that ϕ admit a l.s.c. section. If S is a Milutin space and T is metrizable, then this condition is also sufficient.*

Proof. In view of Corollary 3.3, only the second statement requires verification.

Let Φ be a l.s.c. section of ϕ .

Case 1. Suppose that S is metrizable. By a theorem due to Milutin and Pełczyński (see [9, Theorem 5.6, p. 32] and [4]), there is a continuous map ψ of the Cantor set K onto T which admits a regular averaging operator. By Lemma 2.6(i), the map from K to $F(S)$ which takes $x \in K$ into $\Phi(\psi(x))$ is l.s.c. Hence, by a selection theorem of Michael [7, Theorem 2], there is a continuous map $\theta: K \rightarrow S$ such that $\theta(x) \in \Phi(\psi(x))$ for all $x \in K$. Since $\Phi(\psi(x)) \subset \phi^{-1}(\psi(x))$ for all $x \in K$, it follows that $\psi = \phi\theta$. But ψ admits a regular averaging operator and so, by Lemma 2.3(i), so does ϕ .

Case 2. Suppose now that S is an arbitrary Milutin space. Then, for some index set A , there is a continuous surjection $\phi_1: D^A \rightarrow S$ which admits a regular averaging operator, where $D = \{0, 1\}$. Since T is metrizable, the map $\psi = \phi\phi_1: D^A \rightarrow T$ depends on at most countably many coordinates (see [10, Theorem 4]) i.e., there is a countable subset N of A such that if $\pi: D^A \rightarrow D^N$ is the natural projection map, then $\psi(x) = \psi(y)$ whenever $\pi(x) = \pi(y)$. Therefore, if $\theta: D^N \rightarrow D^A$ is any continuous section of π , then $\psi = \psi\theta\pi$. Now, by Corollary 3.3, ϕ_1 admits a l.s.c. section $\Phi_1: S \rightarrow F(D^A)$ and if we define $\Phi_2: T \rightarrow F(D^A)$ by $\Phi_2(t) = (\bigcup \{\Phi_1(s): s \in \Phi(t)\})^-$ and $\Phi_3: T \rightarrow F(D^N)$ by $\Phi_3(t) = \pi(\Phi_2(t))$, then, by Lemma 2.6, Φ_2 and Φ_3 are both l.s.c. Moreover, if $t \in T$ and $y \in \Phi_3(t)$, then $y = \pi(x)$ for some $x \in \Phi_2(t)$, and since

$$\Phi_2(t) \subset (\bigcup \{\phi_1^{-1}(s): s \in \phi^{-1}(t)\})^- = \phi_1^{-1}\phi^{-1}(t) = \psi^{-1}(t),$$

we have

$$\psi\theta(y) = \psi\theta\pi(x) = \psi(x) = t$$

so that Φ_3 is a l.s.c. section of $\psi\theta$. But D^N is metrizable. Hence, Case 1 applies and $\psi\theta = \phi\phi_1\theta$ admits a regular averaging operator. By Lemma 2.3(i), then, ϕ also admits a regular averaging operator.

Corollary 3.5. *Any continuous open surjection of a Milutin space onto a compact metric space admits a regular averaging operator.*

Proof. For any surjection $\phi: S \rightarrow T$ and any subset V of S , $\phi(V) = \{t \in T: V \cap \phi^{-1}(t) \neq \emptyset\}$. Therefore, ϕ is open if and only if $t \rightarrow \phi^{-1}(t)$ is l.s.c.

Remark 3.6. The proof of Theorem 3.4, for the case when S is metrizable, could also have been carried out by using an argument due to Michael (see [8] where Corollary 3.5 is proved for an open map between metric spaces). In brief, it involves applying a selection theorem of Michael to the mapping $t \rightarrow \overline{\text{co}} \Phi(t)$, where Φ is the l.s.c. section of ϕ and $\overline{\text{co}} \Phi(t)$ is the closed convex hull of $\Phi(t)$, regarding S as embedded in $M(S)$ with the weak* topology (via the map $s \rightarrow \rho_s$). Using the same argument and a selection theorem of Corson and Lindenstrauss, as is done in [3, Theorem 2.6], it can be shown that *Theorem 3.4 also holds when S is a compact subset of $C_0(\Gamma)$ for some discrete space Γ , where $C_0(\Gamma)$ is the collection of all scalar-valued functions f on Γ which vanish at infinity (i.e., for every $\epsilon > 0$, $|f(\gamma)| \geq \epsilon$ for only finitely many $\gamma \in \Gamma$), endowed with the topology of pointwise convergence. In [3, Example 5], Corson and Lindenstrauss also show that, for general S , Corollary 3.5 may fail to hold. In §6 we give an example of a continuous surjection which admits a l.s.c. section and admits an averaging operator of norm $< 1 + \epsilon$, for each $\epsilon > 0$, but does not admit a regular averaging operator.*

For arbitrary S and T , if ϕ admits a continuous section $\sigma: T \rightarrow S$, then $\phi\sigma = \text{id}_T$ and $\sigma^\circ \phi^\circ = \text{id}_{C(T)}$. Hence, σ° is a regular averaging operator for ϕ . However, as Corollary 3.5 indicates, ϕ may admit a regular averaging operator but not admit a continuous section (for example, the map $z \rightarrow z^2$ of the unit circle onto itself is a continuous open surjection which admits no continuous section). In certain situations, this cannot happen.

Theorem 3.7. *If S is a Milutin space and T is metrizable and totally disconnected, then ϕ admits a regular averaging operator if and only if ϕ admits a continuous section.*

Proof. Let ϕ admit a regular averaging operator.

Case 1. Suppose that S is metrizable. Then, by Theorem 3.4, ϕ admits a l.s.c. section Φ which, in turn, by a theorem of Michael [7, Theorem 2], admits a

selection, i.e. a continuous function $\sigma: T \rightarrow S$ such that $\sigma(t) \in \Phi(t)$ for all $t \in T$. Since $\Phi(t) \subset \phi^{-1}(t)$ for all $t \in T$, σ is a continuous section of ϕ .

Case 2. Suppose now that S is an arbitrary Milutin space. Then, as in the Proof of Theorem 3.4 (Case 2), there is a continuous surjection $\phi_1: D^A \rightarrow S$ which admits a regular averaging operator, for some index set A , and a countable subset N of A such that if $\pi: D^A \rightarrow D^N$ is the canonical projection map and $\theta: D^N \rightarrow D^A$ is any continuous section of π , then $\phi\phi_1 = \phi\phi_1\theta\pi$. By Lemma 2.3(ii), $\phi\phi_1$ admits a regular averaging operator. Since $\phi\phi_1 = \phi\phi_1\theta\pi$, it follows, then, from Lemma 2.3(i) that $\phi\phi_1\theta$ admits a regular averaging operator. But $\phi\phi_1\theta$ maps D^N onto T and D^N is metrizable so that, by Case 1, $\phi\phi_1\theta$ admits a continuous section σ . Hence, $\phi\phi_1\theta\sigma = \text{id}_T$ and so $\phi_1\theta\sigma$ is a continuous section of ϕ .

Remark 3.8. For the special case when S is metrizable and T is countable, Theorem 2 of [4] furnishes an alternate proof of Theorem 3.7. Under these circumstances, using the notation of [4, Theorem 2], $\bigcup_{t \in T} N_t$ is countable, so for some $x \in I$, $\theta(\cdot, x)$ is continuous at t for all $t \in T$ and hence is a continuous section of ϕ .

4. An existence theorem for l.s.c. sections. As above, let S and T be compact Hausdorff spaces and $\phi: S \rightarrow T$ a continuous surjection. We derive necessary and sufficient conditions in topological terms that ϕ admit a l.s.c. section.

Definition 4.1. Let \mathfrak{m} be the topological weight of T , i.e., \mathfrak{m} is the smallest cardinal number which is the cardinal number of a base for the topology of T , and let Ω be the first ordinal whose cardinal number exceeds \mathfrak{m} . For any collection $\Gamma \subset F^*(S)$, let $\text{cl } \Gamma$ be the closure of Γ relative to the finite topology of $F^*(S)$. Now, for $t \in T$, set

$$\Lambda_\phi^{(0)}(t) = \phi^{-1}(t).$$

Then $\Lambda_\phi^{(0)}(T) \subset F^*(S)$ and $\text{cl } \Lambda_\phi^{(0)}(T)$ consists of all those sets $K \in F^*(S)$ such that $K = \lim_\alpha \phi^{-1}(t_\alpha)$ (in the finite topology) for some net t_α in T . It is clear, in fact, that t_α must converge to some $t_0 \in T$ and $K \subset \phi^{-1}(t_0)$, and that this is also the case whenever $K = \lim_\alpha K_\alpha$, where $K_\alpha \subset \phi^{-1}(t_\alpha)$, for some net t_α in T . Hence, for $t \in T$, $\text{cl } \Lambda_\phi^{(0)}(T) \cap F^*(\phi^{-1}(t))$ consists of all those closed sets $K \subset \phi^{-1}(t)$ for which $K = \lim_\alpha \phi^{-1}(t_\alpha)$ for some net $t_\alpha \rightarrow t$. Let $\Lambda_\phi^{(1)}(t)$ be the intersection of all these sets, i.e.,

$$\Lambda_\phi^{(1)}(t) = \bigcap (\text{cl } \Lambda_\phi^{(0)}(T) \cap F^*(\phi^{-1}(t))).$$

Then, $\phi^{-1}(t) \supset \Lambda_\phi^{(1)}(t) \in F^*(S)$. By induction, define $\Lambda_\phi^{(\alpha)}$, for $\alpha \leq \Omega$, by

$$\Lambda_\phi^{(\alpha)}(t) = \bigcap (\text{cl } \Lambda_\phi^{(\alpha-1)}(T) \cap F^*(\phi^{-1}(t))) \quad \text{if } \alpha - 1 \text{ exists,}$$

$$\Lambda_\phi^{(\alpha)}(t) = \bigcap_{\beta < \alpha} \Lambda_\phi^{(\beta)}(t) \quad \text{if } \alpha \text{ is a limit ordinal,}$$

and let $\Lambda_\phi^{(\Omega)}$ be denoted by Λ_ϕ .

It is clear that, for all $t \in T$, each $\Lambda_\phi^{(\alpha)}(t)$ is a closed (possibly empty) subset of $\phi^{-1}(t)$ and, if $\beta < \alpha$, then $\Lambda_\phi^{(\alpha)}(t) \subset \Lambda_\phi^{(\beta)}(t)$.

Proposition 4.2. (i) *The map $\Lambda_\phi: T \rightarrow F^*(S)$ is l.s.c.*

(ii) *If Φ is any l.s.c. section of ϕ , then $\Phi(t) \subset \Lambda_\phi(t)$ for all $t \in T$.*

Proof. Denote Λ_ϕ and $\Lambda_\phi^{(\alpha)}$ by Λ and $\Lambda^{(\alpha)}$ respectively.

(i) Since T has topological weight \mathfrak{m} , every point $t \in T$ has a neighborhood base $B(t)$ with $\text{card } B(t) \leq \mathfrak{m}$. If we suppose that Λ is not l.s.c., then there is a $t \in T$ and a closed neighborhood V of a point $s \in \Lambda(t)$ such that, for every $B \in B(t)$, there is a $t_B \in B$ with $V \cap \Lambda(t_B) = \emptyset$. Also, \mathfrak{m} must be infinite and Ω , therefore, is a limit ordinal. By a standard compactness argument, we get that, for every $B \in B(t)$, there is an ordinal $\alpha_B < \Omega$ such that $V \cap \Lambda^{(\alpha_B)}(t_B) = \emptyset$. Now, $\text{card } B(t) \leq \mathfrak{m}$ and $\text{card } \alpha_B \leq \mathfrak{m}$ for every $B \in B(t)$, since every $\alpha_B < \Omega$, so that, if $\alpha = \sup \{\alpha_B: B \in B(t)\}$, then $\text{card } \alpha \leq \mathfrak{m}$ and hence, $\alpha < \Omega$. Since $V \cap \Lambda^{(\alpha)}(t_B) = \emptyset$ for every $B \in B(t)$, it follows that $s \notin K = \limsup \Lambda^{(\alpha)}(t_B)$. But $\Lambda^{(\alpha+1)}(t) \subset K$, since $t_B \rightarrow t$ and some subnet of $\langle \Lambda^{(\alpha)}(t_B) \rangle_{B \in B(t)}$ converges. This is a contradiction, since $s \in \Lambda(t) \subset \Lambda^{(\alpha+1)}(t)$.

(ii) From the definition of $\Lambda(t)$, it clearly suffices to show that, for any $\alpha < \Omega$, if $\Phi(t) \subset \Lambda^{(\alpha)}(t)$ for all $t \in T$, then $\Phi(t) \subset \Lambda^{(\alpha+1)}(t)$ for all $t \in T$. Suppose then that $\Phi(t) \subset \Lambda^{(\alpha)}(t)$ for all $t \in T$ and let $s \in \Phi(t_0)$, where $t_0 \in T$ is arbitrary. Let $X \in \text{cl } \Lambda^{(\alpha)}(T) \cap F^*(\phi^{-1}(t_0))$. Then $X = \lim \Lambda^{(\alpha)}(t_\gamma)$ for some net $t_\gamma \rightarrow t_0$. Since Φ is l.s.c., every neighborhood of s eventually meets every $\Phi(t_\gamma)$. But $\Phi(t_\gamma) \subset \Lambda^{(\alpha)}(t_\gamma)$ for all γ . Therefore, $s \in X$. It follows that $\Phi(t_0) \subset \Lambda^{(\alpha+1)}(t_0)$ for any $t_0 \in T$.

Corollary 4.3. *ϕ admits a l.s.c. section if and only if $\Lambda_\phi(t) \neq \emptyset$ for all $t \in T$.*

Corollary 4.4. *If S is a Milutin space (or a compact subset of $C_0(\Gamma)$ for some discrete Γ) and T is metrizable, then ϕ admits a regular averaging operator if and only if $\Lambda_\phi(t) \neq \emptyset$ for all $t \in T$.*

Proof. By Theorem 3.4, Remark 3.6, and Corollary 4.3.

5. Lower bounds for the norms of averaging operators. In this section we see what can be said about lower bounds for $\|u\|$, where u is an averaging operator for a fixed continuous surjection $\phi: S \rightarrow T$ between compact Hausdorff spaces. Our results are related in kind to previous work of Amir, Arens, and Pełczyński. The following proposition is basic to the discussion.

Proposition 5.1. *If u is an averaging operator for ϕ and if, for $t \in T$, we let $\mu_t = u^*(\rho_t)$ and $R_t = \|\mu_t\| - |\mu_t|(\phi^{-1}(t))$, then for any net $t_\alpha \rightarrow t_0$ in T , we have*

$$\liminf R_{t_\alpha} \geq 1 + \|\mu_{t_0}\| - 2|\mu_{t_0}|(\limsup \phi^{-1}(t_\alpha)).$$

Proof. Let $S_0 = \limsup \phi^{-1}(t_\alpha)$, $m = |\mu_{t_0}|(S_0)$, and $M = \|\mu_{t_0}\|$. Then S_0 is a closed nonempty subset of $\phi^{-1}(t_0)$ and $\phi^{-1}(t_\alpha)$ is eventually contained in each neighborhood of S_0 .

Let $\epsilon > 0$ be arbitrary. By the regularity of μ_{t_0} , there is a compact set $K \subset S \setminus S_0$ such that $|\mu_{t_0}|(K) > M - m - \epsilon$.

Let V be an open neighborhood of S_0 with $\bar{V} \cap K = \emptyset$ and let $W \subset V$ be a closed neighborhood of S_0 . Then letting $K' = S \setminus K$, we have $|\mu_{t_0}|(K' \setminus S_0) < \epsilon$ and $|\mu_{t_0}|(\bar{V}) < m + \epsilon$.

Choose $b_0 \in C(K)$ with $\|b_0\| \leq 1$ and

$$\int_K b_0 d\mu_{t_0} > M - m - \epsilon.$$

Extend b_0 to $b_1 \in C(S)$ with $\|b_1\| \leq 1$ and $b_1|_{\bar{V}} = 0$. Then

$$|\mu_{t_0}(b_1)| = \left| \int_K b_0 d\mu_{t_0} + \int_{K' \setminus \bar{V}} b_1 d\mu_{t_0} \right| > M - m - 2\epsilon.$$

Choose $b_2 \in C(S)$ such that $0 \leq b_2 \leq 1$, $b_2 = 1$ on W , and $b_2 = 0$ on $S \setminus V$.

Then

$$|\mu_{t_0}(b_2)| \leq |\mu_{t_0}|(b_2) \leq |\mu_{t_0}|(V) < m + \epsilon.$$

Now, eventually we have $\phi^{-1}(t_\alpha) \subset W$, $|\mu_{t_\alpha}(b_1)| > M - m - 2\epsilon$, and $|\mu_{t_\alpha}(b_2)| < m + \epsilon$. By Corollary 2.2, $\mu_{t_\alpha}(\phi^{-1}(t_\alpha)) = 1$, so that, if we let

$$c_\alpha = \int_{V \setminus \phi^{-1}(t_\alpha)} b_2 d\mu_{t_\alpha},$$

then we eventually have $|1 + c_\alpha| < m + \epsilon$ and, hence, eventually,

$$|\mu_{t_\alpha}|(V \setminus \phi^{-1}(t_\alpha)) \geq \int_{V \setminus \phi^{-1}(t_\alpha)} b_2 d|\mu_{t_\alpha}| \geq |c_\alpha| > 1 - m - \epsilon,$$

and

$$|\mu_{t_\alpha}|(S \setminus V) \geq \int |b_1| d|\mu_{t_\alpha}| \geq |\mu_{t_\alpha}(b_1)| > M - m - 2\epsilon.$$

Therefore, we eventually have

$$\|\mu_{t_\alpha}\| > |\mu_{t_\alpha}|(\phi^{-1}(t_\alpha)) + 1 - m - \epsilon + M - m - 2\epsilon,$$

i.e. $R_{t_\alpha} > 1 + M - 2m - 3\epsilon$, which proves the proposition.

In order to properly state the corollaries which follow from the above proposition, we require some definitions.

Definition 5.2. For the continuous surjection $\phi: S \rightarrow T$, let

$$p(\phi) = \inf \{\|u\|: u \text{ is an averaging operator for } \phi\}.$$

Hence, ϕ admits no averaging operator if and only if $p(\phi) = \infty$.

Definition 5.3. For $\Gamma \subset F(S)$ and $t \in T$, let

$$\delta(\Gamma) = \sup \{n: \Gamma \text{ contains } n \text{ disjoint sets, } 0 \leq n < \infty\}$$

and

$$N(t; \Gamma) = \delta(\text{cl } \Gamma \cap F(\phi^{-1}(t))),$$

where $\text{cl } \Gamma$ is the closure of Γ relative to the finite topology of $F(S)$. Now let $\Delta_\phi = \{\phi^{-1}(t): t \in T\}$ and $N_\phi(t) = N(t; \Delta_\phi)$. Finally, for any positive integer k and sequence of integers $n_1, \dots, n_k \geq 1$, define $\Delta_\phi^{(k)}(n_1, \dots, n_k)$ inductively as follows:

$$\Delta_\phi^{(1)}(n_1) = \{\phi^{-1}(t): N(t; \Delta_\phi) \geq n_1\},$$

$$\Delta_\phi^{(k)}(n_1, \dots, n_k) = \{\phi^{-1}(t): N(t; \Delta_\phi^{(k-1)}(n_1, \dots, n_{k-1})) \geq n_k\} \quad \text{if } k \geq 2.$$

Corollary 5.4. *If $\Delta_\phi^{(k)}(n_1, \dots, n_k) \neq \emptyset$, then $p(\phi) \geq 1 + 2 \sum_{i=1}^k (1 - 1/n_i)$.*

Proof. Suppose that u is an averaging operator for ϕ and $\mu_t = u^*(\rho_t)$ for $t \in T$. Now, if $\Delta_\phi^{(j)}(n_1, \dots, n_j) \neq \emptyset$, then for some $t_0 \in T$ there is a net $t_\alpha \rightarrow t_0$ in T such that each $\phi^{-1}(t_\alpha) \in \Delta_\phi^{(j-1)}(n_1, \dots, n_{j-1})$ (or Δ_ϕ , if $j = 1$) and

$$|\mu_{t_0}|(\limsup \phi^{-1}(t_\alpha)) \leq (1/n_j) |\mu_{t_0}|(\phi^{-1}(t_0)).$$

By Proposition 5.1 and Corollary 2.2,

$$\liminf R_{t_\alpha} \geq 1 + R_{t_0} + (1 - 2/n_j) |\mu_{t_0}|(\phi^{-1}(t_0)) \geq R_{t_0} + 2(1 - 1/n_j).$$

Since $\Delta_\phi^{(k)}(n_1, \dots, n_k) \neq \emptyset$, it follows that for arbitrary $\epsilon > 0$ there are points t_0, t_1, \dots, t_k in T such that, for $0 \leq i \leq k-1$, $R_{t_{i+1}} > R_{t_i} + 2(1 - 1/n_{k-i}) - \epsilon/k$. Summing over i , we get

$$R_{t_k} > R_{t_0} + 2 \sum_{i=1}^k \left(1 - \frac{1}{n_i}\right) - \epsilon.$$

Hence

$$\|u\| \geq \|\mu_{t_k}\| \geq 1 + R_{t_k} > 1 + 2 \sum_{i=1}^k \left(1 - \frac{1}{n_i}\right) - \epsilon$$

which completes the proof.

Corollary 5.5. *If $\Delta_\phi^{(n)}(2, 2, \dots, 2) \neq \emptyset$ for all n , then ϕ admits no averaging operator.*

Corollary 5.6. $p(\phi) \geq \sup \{3 - 2/N_\phi(t): t \in T\}$.

Proof. Since we know that $p(\phi) \geq 1$, it suffices to show that, if $N_\phi(t) \geq n \geq 2$ for some $t \in T$, then $p(\phi) \geq 3 - 2/n$. But, if $N_\phi(t) \geq n \geq 2$, then $\Delta_\phi^{(1)}(n) \neq \emptyset$.

Hence, the result follows from Corollary 5.4.

Remark 5.7. (i) The numbers $1 + 2\sum_{i=1}^k (1 - 1/n_i)$ appear as well in a paper of Amir [1, Theorem 1] wherein a lower bound is found for the norms of projections of $B(S, \Sigma)$ onto $C(S)$, where Σ is a field of subsets of S which contains a base for the topology of S and $B(S, \Sigma)$ is the uniform closure in $l_\infty(S)$ of the subspace spanned by the characteristic functions of members of Σ . If we identify $B(S, \Sigma)$ with $C(M)$, where M is the maximal ideal space of $B(S, \Sigma)$, then the projections correspond to averaging operators for the natural map of M onto S (essentially, the restriction map $B(S, \Sigma)^* \rightarrow C(S)^*$ restricted to M) and Amir's lower bound theorem can be obtained by applying Corollary 5.4. (ii) Another result obtainable from Corollary 5.4 is Pełczyński's theorem on maps of Cantor type [9, Proposition 9.8, p. 49], since it is easily shown that if $\phi: S \rightarrow T$ is of Cantor type and $F_\phi^{[m]} \neq \emptyset$ (see p. 49 of [9] for definitions), then $\Delta_\phi^{(m-1)}(2, 2, \dots, 2) \neq \emptyset$ so that $p(\phi) \geq 1 + 2\sum_{i=1}^{m-1} 1/2 = m$. (iii) For compact spaces, Corollary 5.5 considerably generalizes a related theorem of Arens [2, Theorem 3.1]. (iv) Corollary 5.5 applies, for example, to the Cantor map from the Cantor set K onto the unit interval I , i.e., the map taking $\sum_{i=1}^\infty 2\xi_i/3^i$ onto $\sum_{i=1}^\infty \xi_i/2^i$, for every sequence $\xi_i \in \{0, 1\}$ (see [9, Corollary 9.12]), and to an example given by Arens [2, Theorem 3.5]. It also applies to the map of the unit interval onto itself which extends the Cantor map and is constant on the disjoint intervals of $I \setminus K$. We can therefore state the following.

Corollary 5.8. *There is a continuous map of the unit interval onto itself which admits no averaging operator.*

6. An example. In this section we show that there exist compact Hausdorff spaces S and T and a continuous surjection $\phi: S \rightarrow T$ such that ϕ admits an averaging operator of norm $1 + 2/m$, for every integer $m \geq 2$, but ϕ admits no regular averaging operator, i.e., the greatest lower bound $p(\phi) = 1$, but it is not attained.

Let N be the positive integers and for $m \in N$, let $J_m = \{0, 1, \dots, m-1\}$. For $m \geq 2$, let S_m be that subset of the plane consisting of all points of the form $(0, 2^m + j)$, where $j \in J_m$, or of the form $(1/n, 2^m + j)$, where $n \in N$, $j \in J_m$, and $j \not\equiv n$ modulo m . Let $T = \{0\} \cup \{1/n: n \in N\}$ and let $\phi_m: S_m \rightarrow T$ be projection onto the first coordinate. Then S_m and T are compact and ϕ_m is a continuous surjection. Note also that S_m and S_n are disjoint if $m \neq n$.

Define $u_m: C(S_m) \rightarrow C(T)$ as follows. If $f \in C(S_m)$, let

$$u_m(f)(0) = \frac{1}{m} \sum_{j \in J_m} f(0, 2^m + j)$$

and, for $n \in N$, if $n \equiv k$ modulo m , where $k \in J_m$, let

$$u_m(f)\left(\frac{1}{n}\right) = \frac{1}{m}f(0, 2^m + k) + \sum_{j \neq k} \left(\frac{1}{m-1}f\left(\frac{1}{n}, 2^m + j\right) - \frac{1}{m(m-1)}f(0, 2^m + j) \right).$$

It is easily checked that u_m is an averaging operator for ϕ_m and that $\|u_m\| = 1 + 2/m$.

For each $k \in J_m$, let

$$L_k^m = \{(0, 2^m + j): k \neq j \in J_m\}.$$

Then $\text{cl} \{\phi_m^{-1}(t)\}_{t \in T} \cap F^*(\phi_m^{-1}(0))$ consists precisely of $\phi_m^{-1}(0)$ and the sets L_k^m , $k \in J_m$.

It shall follow from what is proved below that ϕ_m admits no regular averaging operator. However, this fact can be seen directly either by Corollary 4.4, since $\Lambda_{\phi_m}(0) = \bigcap_{k \in J_m} L_k^m = \emptyset$, or by Theorem 3.7, since ϕ_m clearly admits no continuous section. In fact, it is not too difficult to show that $p(\phi_m) = 1 + 2/m$.

We define S and ϕ as follows. Let $X = \bigcup_{m \geq 2} S_m$ and let $S = \beta(X)$, the Stone-Čech compactification of X . Let $\phi: S \rightarrow T$ be the unique continuous map such that $\phi|X$ is projection onto the first coordinate, i.e., $\phi|S_m = \phi_m$ for $m \geq 2$.

Then, for each $m \geq 2$, the map $f \rightarrow u_m(f|S_m)$, $f \in C(S)$, is clearly an averaging operator for ϕ of norm $1 + 2/m$.

It remains to show that ϕ admits no regular averaging operator. Note that this cannot be derived from Theorem 3.4 since ϕ does admit a l.s.c. section. Indeed, it is clear that $\Lambda_\phi(t) = \phi^{-1}(t)$, if $t \neq 0$, whereas

$$\emptyset \neq \Lambda_\phi(0) \subset S \setminus X.$$

The latter statement can be seen as follows. Every set $K \neq \phi^{-1}(0)$ in $\mathfrak{L} = \text{cl} \{\phi^{-1}(t)\}_{t \in T} \cap F^*(\phi^{-1}(0))$ has the property that $K \cap X = \bigcup_{m \geq 2} L_{k_m}^m$ for some sequence $\langle k_m \rangle_{m \geq 2}$. Therefore, for every finite collection of limit sets K_1, \dots, K_r in \mathfrak{L} , $\bigcap_{i=1}^r K_i \neq \emptyset$, since, in fact $\bigcap_{i=1}^r K_i \cap S_m \neq \emptyset$ if $m > r$ (because $\bigcap_{i=1}^r L_{k_i}^m = \{(0, 2^m + j): j \neq k_1, \dots, k_r\}$). By compactness, then, $\Lambda_\phi^{(1)}(0) = \bigcap_{\phi} \mathfrak{L} \neq \emptyset$. But since $\Lambda_\phi(t) = \phi^{-1}(t)$ for $t \neq 0$, it follows that $\Lambda_\phi(0) = \Lambda_\phi^{(1)}(0)$. Hence, $\Lambda_\phi(0) \neq \emptyset$. The fact that $\Lambda_\phi(0) \subset S \setminus X$ follows from the observation that, for every $m \geq 2$ and $k \in J_m$, there is a limit set $K \in \mathfrak{L}$ with the property that $K \cap S_m = L_k^m$. Hence, $\Lambda_\phi(0) \cap S_m = \bigcap_{k \in J_m} L_k^m = \emptyset$.

Suppose that ϕ admits a regular averaging operator u . Then, by Lemma 2.1, $t \rightarrow \mu_t = u^*(\rho_t)$ is a weak* continuous map from T to $M(S)$ and each μ_t is a probability measure with support $\mu_t \subset \phi^{-1}(t)$. Moreover, by Proposition 4.2 (ii), Corollary 3.3, and our remarks above,

$$\text{support } \mu_0 \subset \phi^{-1}(0) \setminus X.$$

For each $r \geq 2$, let $X_r = \bigcup_{i \geq r} S_i$ and $Y_r = \overline{X_r}$. Let $\mu_n = \mu_{1/n}$, for $n \in N$.

Claim 1. For any sequence of integers $r_n \rightarrow \infty$, $\mu_n(Y_{r_n}) \rightarrow 0$.

Suppose not. Then for some sequence $r_n \rightarrow \infty$, there exists $\epsilon > 0$ and a subsequence $\langle r_n' \rangle_{n \in N}$ such that $\mu_n(Y_{r_n'}) \geq \epsilon$ for all n . Let σ be any infinite subset of $\{n'\}_{n \in N}$ whose complement in $\{n'\}_{n \in N}$ is also infinite and let

$$K_\sigma = \left(\bigcup_{j \in \sigma} \left(Y_{r_j} \cap \phi^{-1} \left(\frac{1}{j} \right) \right) \right)^-.$$

Then, for any $j \in \sigma$, $\mu_j(K_\sigma) = \mu_j(Y_{r_j}) \geq \epsilon$. Now, if $H_\sigma = \bigcup_{j \in \sigma} (X_{r_j} \cap \phi^{-1}(1/j))$, then, since $r_n \rightarrow \infty$, H_σ is both open and closed in X . Therefore, $K_\sigma = \overline{H_\sigma}$ is both open and closed in $S = \beta(X)$ so that $\mu_n(K_\sigma) \rightarrow \mu_0(K_\sigma)$. But this is impossible since $\mu_n(K_\sigma) \geq \epsilon$ if $n' \in \sigma$, whereas $\mu_n(K_\sigma) = 0$ if $n' \notin \sigma$. This proves Claim 1.

Now, for $n \in N$, let $A_n = \{r: \mu_n(Y_r) \geq 1/2\}$, and let the sequence r_n be defined by

$$\begin{aligned} r_n &= \max A_n \quad \text{if } A_n \text{ is bounded,} \\ r_n &= n \quad \text{otherwise.} \end{aligned}$$

Claim 2. $r_n \rightarrow \infty$.

Suppose not. Then there exists $r \geq 2$ and a subsequence $\langle r_n' \rangle$ of $\langle r_n \rangle$ such that $r_n' < r$ for all n . Therefore, we eventually have $\mu_n(Y_r) < 1/2$ and, hence, $\mu_n(K) \geq 1/2$, where $K = \bigcup_{i < r} S_i$. But K is both open and closed in S . Hence, $\mu_0(K) \geq 1/2$. However, this is impossible since $K \subset X$ and support μ_0 is disjoint from X .

Therefore, $r_n \rightarrow \infty$ and $\mu_n(Y_{r_n}) \geq 1/2$ for all n , which contradicts Claim 1. Hence ϕ does not admit a regular averaging operator.

Added in proof. A better result can be obtained in Proposition 5.1. The author was made aware of this by calculations of the same sort which appear in the paper *Norm reduction of averaging operators*, by H. B. Cohen, M. A. Labbe, and J. Wolfe, soon to appear. With reference to the proof of Proposition 5.1, if we use the fact that

$$\begin{aligned} \lim |c_\alpha| &= \lim \left| \int b_2 d\mu_{t_\alpha} - 1 \right| = \left| \int b_2 d\mu_{t_0} - 1 \right| \\ &= \left| \mu_{t_0}(S_0) + \int_{V \setminus S_0} b_2 d\mu_{t_0} - 1 \right|, \end{aligned}$$

we easily obtain the sharper inequality

$$\liminf R_{t_\alpha} \geq \|\mu_{t_0}\| + |1 - \mu_{t_0}(S_0)| - |\mu_{t_0}(S_0)|$$

or, letting $A_0 = \phi^{-1}(t_0) \setminus S_0$,

$$\liminf R_{t_\alpha} \geq R_{t_0} + |\mu_{t_0}(A_0)| + |\mu_{t_0}|(A_0)$$

which, incidentally, shows that the function $t \rightarrow R_t$ is lower semicontinuous.

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