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TENSOR PRODUCTS OF GROUP ALGEBRAS

BY

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ABSTRACT. Let C be a commutative Banach algebra. A commutative Banach algebra A is a Banach C-algebra if A is a Banach C-module and $c \cdot (aa') = (c \cdot a)a'$ for all $c \in C$, $a, a' \in A$. If $A_1, \cdots, A_n$ are commutative Banach C-algebras, then the C-tensor product $A_1 \otimes_C \cdots \otimes_C A_n \equiv D$ is defined and is a commutative Banach C-algebra. The maximal ideal space $\mathfrak{M}_D$ of D is identified with a closed subset of $\mathfrak{M}A_1 \times \cdots \times \mathfrak{M}A_n$ in a natural fashion, yielding a generalization of the Gelbaum-Tomiyama characterization of the maximal ideal space of $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$.

If $C = L^1(K)$ and $A_i = L^1(G_i)$, for LCA groups K and G_i, $i = 1, \cdots, n$, then the $L^1(K)$-tensor product D of $L^1(G_1), \cdots, L^1(G_n)$ is uniquely written in the form $D = N \oplus D_e$, where N and D_e are closed ideals in D, $L^1(K) \cdot N = \{0\}$, and D_e is the essential part of D, i.e. $D_e = L^1(K) \cdot D$. Moreover, if $D_e \neq \{0\}$, then D_e is isometrically $L^1(K)$-isomorphic to $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$, where $G_1, \cdots, G_n$ is a K-tensor product of $G_1, \cdots, G_n$ with respect to naturally induced actions of K on $G_1, \cdots, G_n$. The above theorems are a significant generalization of the work of Gelbaum and Natzitz in characterizing tensor products of group algebras, since here the algebra actions are *arbitrary*. The Cohen theory of homomorphisms of group algebras is required to characterize the algebra actions between group algebras. Finally, the space of multipliers $\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H))$ is characterized for all instances of algebra actions of $L^1(K)$ on $L^1(G)$ and $L^1(H)$, generalizing the known result when $K = G = H$ and the module action is given by convolution.

If A_1 and A_2 are Banach modules over a Banach algebra C, then Rieffel [18] has defined and systematically studied the C-tensor product, $A_1 \otimes_C A_2$, of A_1 and A_2. If A_1 and A_2, or more generally $A_1, \cdots, A_n$, are commutative Banach C-algebras for a commutative Banach algebra C, then the C-tensor product of $A_1, \cdots, A_n$, $A_1 \otimes_C \cdots \otimes_C A_n$, is naturally a commutative Banach C-algebra. In this paper we study this tensor algebra, characterize its structure

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space, and analyze it in the situation when $A_1 = L^1(G_1), \cdots, A_n = L^1(G_n)$, and $C = L^1(K)$ for locally compact Abelian groups $G_1, \cdots, G_n$, and K.

In a paper prior to Rieffel [18], Gelbaum [4] introduced a C-tensor product of commutative Banach C-algebras. We show that although this tensor product is not equivalent to the tensor product of Rieffel, the two tensor products are closely related (Theorem 3.1) and, in fact, when C has a bounded approximate identity, the tensor algebra of Gelbaum is the essential part of the tensor algebra of Rieffel.

In his studies of tensor products Gelbaum [4] was motivated by the following interesting example. If K and G are LCA groups and θ: $K \longrightarrow G$ is a continuous homomorphism, then $L^1(G)$ is naturally a Banach $L^1(K)$-algebra under *θ-convolution*:

$$c *_\theta a(x) = \int_K c(k)a(x - \theta(k))\,dk, \qquad x \in G,\ c \in L^1(K),\ a \in L^1(G).$$

Gelbaum posed the following question. If $L^1(G)$ and $L^1(H)$ are the $L^1(K)$-algebras whose actions are induced by the continuous homomorphism θ: $K \longrightarrow G$ and ψ: $K \longrightarrow H$, is the $L^1(K)$-tensor product of $L^1(G)$ and $L^1(H)$ the L^1-group algebra of a LCA group? Gelbaum [4] and Natzitz [14] obtained partial results to this question.

In this paper we analyze the most general situation, that of characterizing $L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n)$ in *all* instances of (isometric) $L^1(K)$-algebra actions on $L^1(G_1), \cdots, L^1(G_n)$. The main result (Theorem 6.5) is briefly paraphrased by stating that the essential part of $L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n)$ is a closed ideal and direct summand, and (when nonzero) it is isometrically $L^1(K)$-isomorphic to $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$ where $G_1 \otimes_K \cdots \otimes_K G_n$ is the LCA-group and K-tensor product of $G_1, \cdots, G_n$ with respect to induced actions of K on $G_1, \cdots, G_n$.

The complexity of the general problem itself is perhaps best reflected in the three deep results required to prove the main result. First, the complete Cohen theory [1] of homomorphisms of group algebras is needed to characterize the algebra actions of $L^1(K)$ on $L^1(G)$. Secondly, the result of Gilbert [6], showing that closed subsets of the coset ring of the dual group are sets of spectral synthesis, is used to show that the tensor product of group algebras over a group algebra is *always* a strongly semisimple Banach algebra. Finally, the fundamental result of Grothendieck [7] (and Johnson [11]) asserting that the projective tensor product of L^1-group algebras is the L^1-group algebra of the direct product of groups is the starting point of our analysis; in fact, our work is evidently a generalization of this fundamental fact.

In §§4 and 6 we introduce the notions of LCA K-groups and their K-tensor products. The purpose here is to exhibit the almost functorial interrelation between actions of one LCA group on another and actions of the respective group algebra on the group algebra, and between K-group tensor products and group $L^1(K)$-algebra tensor products.

Finally, in §7 we characterize the space $\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H))$ of generalized multipliers for arbitrary $L^1(K)$-algebra actions on $L^1(G)$ and $L^1(H)$ where $L^\infty(H)$ is the $L^1(K)$-module under the adjoint action.

1. **Banach C-algebras and C-tensor products.** Presented in this section are the basic definitions and facts about commutative Banach C-algebras and their C-tensor products. As will be readily apparent to the reader, we have borrowed freely from the paper of Rieffel [18] concerning tensor products of Banach modules, and consequently, familiarity with [18] should facilitate reading this paper.

Throughout this section C will denote an arbitrary but fixed commutative Banach algebra. It is assumed throughout that we are dealing with complex spaces and all terms such as Banach space, Banach algebra, and the like should be interpreted in the context of the complex field of scalars.

Definition 1.1. A commutative *Banach C-algebra* is a commutative Banach algebra A together with a continuous, complex bilinear map $C \times A \ni (c, a) \longrightarrow c \cdot a \in A$ such that

(i) $(cc') \cdot a = c \cdot (c' \cdot a)$,

(ii) $c \cdot (aa') = (c \cdot a)a' = a(c \cdot a')$,

for all $c, c' \in C$, $a, a' \in A$. (It is easily seen that the second equality in (ii) follows from the first by the commutativity of A.) Note that the continuity of the action of C on A is equivalent to the existence of a nonnegative constant χ such that

(iii) $\|c \cdot a\|_A \leq \chi \|c\|_C \|a\|_A$, $c \in C$, $a \in A$.

The least such $\chi \geq 0$ satisfying (iii) is called the *norm* of the action of C on A (and of course this is the bilinear norm of the bilinear map $(c, a) \longrightarrow c \cdot a$). If χ can be chosen to be one in (iii) then we say the action of C on A is *isometric* (following [18]).

In [18] Banach modules over such general spaces as sets, groups, etc., have been treated. We could equally as well have defined Banach C-algebras in this spirit. However, since our basic concern is with algebras over Banach algebras, there is insufficient reason for undertaking this generality. The reader can easily supply the necessary changes.

Examples of algebra actions of one commutative Banach algebra on another are in abundance. Of particular concern in this paper are the algebra actions of one group algebra on another. In [12] this author has characterized these actions completely, and for the sake of completeness we summarize these results in §4.

As a final remark, note that since we consider modules over commutative rings, it is unnecessary to distinguish between left and right module actions;

indeed, every right action can be viewed as a left action in the obvious fashion, and vice versa. Therefore, in making references to [18], qualification of left and right actions is not needed.

Definition 1.2. If A is a Banach C-algebra (or C-module), the closed linear subspace of A spanned by $C \cdot A = \{c \cdot a \mid c \in C, a \in A\}$ is called the *essential part* of A and is denoted by A_e. A is *essential* if $A_e = A$.

If C has a bounded approximate identity, then from Hewitt's factorization theorem [10, Theorem 32.22] the essential part, A_e, of the Banach C-algebra A is actually $C \cdot A$. This fact together with the ring structure on the modules and the definition of an algebra action yield the evident but important

Proposition 1.3. *If C has a bounded approximate identity and if A is a Banach C-algebra, then A_e is a closed C-ideal in A, A_e is an essential Banach C-algebra, and A_e contains any C-subalgebra of A which is essential.*

(*Note.* This is merely the analog of [18, Proposition 3.6], now in the context of Banach C-algebras.)

We now consider tensor products of Banach C-algebras. The definition parallels that in [18] for tensor products of Banach modules, and the existence is established by the same construction; but for the sake of completeness we include the details. Also, since we will have occasion to speak of tensor products of more than two Banach C-algebras we find it convenient to base our discussion in the setting of tensor products of any finite number of Banach C-algebras.

Definition 1.4. Let $A_1, \cdots, A_n$ be Banach C-algebras. A continuous complex n-linear map ϕ of $A_1 \times \cdots \times A_n$ into a Banach space V is *C-balanced* if

$$\phi(a_1, \cdots, c \cdot a_i, \cdots, a_n) = \phi(a_1, \cdots, c \cdot a_j, \cdots, a_n) \qquad (1 \le i, j \le n),$$

for all $c \in C$, $a_k \in A_k$, $k = 1, \cdots, n$. If V is a Banach algebra and $\phi(a_1b_1, \cdots, a_nb_n) = \phi(a_1, \cdots, a_n)\phi(b_1, \cdots, b_n)$ for all $a_k, b_k \in A_k$, $k = 1, \cdots, n$, then ϕ is said to be *multiplicative*.

Definition 1.5. Let $A_1, \cdots, A_n$ be commutative Banach C-algebras. A C-tensor product of $A_1, \cdots, A_n$ is a pair $(D; \rho)$ consisting of a commutative Banach C-algebra D and a continuous, complex and C-n-linear, multiplicative map ρ of $A_1 \times \cdots \times A_n$ into D satisfying the following universal mapping property. If ϕ is a continuous, complex n-linear, C-balanced map of $A_1 \times \cdots \times A_n$ into a Banach space V, then there is a unique continuous linear transformation $\Phi: D \to V$ such that $\phi = \Phi \circ \rho$ and $\|\phi\| = \|\Phi\|$ (where $\|\phi\|$ denotes the multilinear norm of ϕ). If ϕ is also C-n-linear when V is a Banach C-module, then Φ is also C-homogeneous, and, if ϕ is also multiplicative when V is a Banach algebra, then Φ is also multiplicative (as a map between Banach algebras).

The existence of C-tensor products of Banach C-algebras is obtained by the same construction as that used for the tensor product of Banach modules [18]. Namely, if $A_1, \cdots, A_n$ are commutative Banach C-modules, let $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$ denote the commutative Banach tensor algebra and projective tensor product of $A_1, \cdots, A_n$ (where γ denotes the Schatten [21] greatest cross norm). $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$ is naturally a Banach C-algebra, when $c \cdot (a_1 \otimes \cdots \otimes a_n) = (c \cdot a_1) \otimes \cdots \otimes a_n$ implements the action of C on $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$. Let J denote the closed linear subspace generated in $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$ by the set of all elements of the form $(a_1 \otimes \cdots \otimes c \cdot a_i \otimes \cdots \otimes a_n) - (a_1 \otimes \cdots \otimes c \cdot a_j \otimes \cdots \otimes a_n)$ where $1 \le i, j \le n$ and $c \in C$, $a_k \in A_k$, $k = 1, \cdots, n$. Since $A_1, \cdots, A_n$ are C-algebras, the generators of J are closed under action by C and multiplication by fundamental tensors of $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$. Consequently, J is (naturally) a closed C-ideal in $A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$. We define $A_1 \otimes_C \cdots \otimes_C A_n = A_1 \otimes_\gamma \cdots \otimes_\gamma A_n/J$, the quotient commutative Banach C-algebra with the quotient norm, and define $\rho: A_1 \times \cdots \times A_n \to A_1 \otimes_C \cdots \otimes_C A_n$ by

$$\rho(a_1, \cdots, a_n) = a_1 \otimes \cdots \otimes a_n/J, \qquad (a_1, \cdots, a_n) \in A_1 \times \cdots \times A_n.$$

Now, given $a_k \in A_k$, $k = 1, \cdots, n$, we will write $a_1 \otimes_C \cdots \otimes_C a_n$ (called a fundamental tensor) for the element $a_1 \otimes \cdots \otimes a_n/J$ of $A_1 \otimes_C \cdots \otimes_C A_n$, or simply $a_1 \otimes \cdots \otimes a_n$ when there is no confusion.

The next proposition is the parallel of [18, Theorem 2.3], now in the context of tensor products of Banach C-algebras.

Proposition 1.6. *$(A_1 \otimes_C \cdots \otimes_C A_n; \rho)$ is a C-tensor product of the commutative Banach C-algebras $A_1, \cdots, A_n$.*

Proof. Since $(A_1 \otimes_C \cdots \otimes_C A_n; \rho)$ is a C-tensor product of $A_1, \cdots, A_n$ as Banach C-modules [18, Theorem 2.3], every continuous, complex n-linear, C-balanced [resp. C-n-linear] map ϕ of $A_1 \times \cdots \times A_n$ into a Banach space V factors through $A_1 \otimes_C \cdots \otimes_C A_n$ to yield a continuous linear [resp. C-linear] transformation Φ such that $\Phi \circ \rho = \phi$ and $\|\phi\| = \|\Phi\|$. If ϕ is also multiplicative, then Φ is also multiplicative on products of fundamental tensors and therefore by the linearity and continuity of Φ and the fact that the fundamental tensors linearly generate $A_1 \otimes_C \cdots \otimes_C A_n$, we have Φ is multiplicative on the tensor algebra $A_1 \otimes_C \cdots \otimes_C A_n$ into the Banach algebra V.

It is of interest to obtain the existence of a tensor product of $A_1, \cdots, A_n$ by a different approach, namely as the quotient of a topological function algebra on $A_1 \times \cdots \times A_n$. The construction we offer parallels the construction by Gelbaum [3] for tensor products of Banach algebras over the scalar field.

Let $\mathfrak{F}(A_1, \cdots, A_n)$ denote the complex linear function space consisting of all complex valued functions f on $A_1 \times \cdots \times A_n$ such that

(1) $f(a_1, \cdots, a_n) = 0$ if $a_i = 0 \in A_i$ for some i, $1 \leq i \leq n$,

(2) $|||f||| = \Sigma_{A_1 \times \cdots \times A_n} |f(a_1, \cdots, a_n)| \, \|a_1\| \cdots \|a_n\| < \infty$.

Now, $\mathfrak{F}(A_1, \cdots, A_n)$ is a Banach space of functions on $A_1 \times \cdots \times A_n$ under the norm $|||\cdot|||$, and is a commutative Banach algebra under convolution: if $f, g \in \mathfrak{F}(A_1, \cdots, A_n)$, then the convolution of f and g, $f * g$, is the element in $\mathfrak{F}(A_1, \cdots, A_n)$ given by

$$f * g(a_1, \cdots, a_n) = \sum_{b_k b_k' = a_k;\, 1 \leq k \leq n} f(b_1, \cdots, b_n) g(b_1', \cdots, b_n'),$$

if $\|a_1\| \cdots \|a_n\| > 0$ and a_k is factorable for $k = 1, \cdots, n$;

$= 0$, otherwise.

The convergence of the above series can be easily shown (e.g., [3, p. 131]).

For each $(a_1, \cdots, a_n) \in A_1 \times \cdots \times A_n$ let $\chi_{a_1, \cdots a_n}$ denote the fundamental functional and element of $\mathfrak{F}(A_1, \cdots, A_n)$ such that $\chi_{a_1, \cdots, a_n} \equiv 0$ if $\|a_1\| \cdots \|a_n\| = 0$, and, $\chi_{a_1, \cdots, a_n}$ is the characteristic function of the one point set $\{(a_1, \cdots, a_n)\} \subseteq A_1 \times \cdots \times A_n$ if $\|a_1\| \cdots \|a_n\| > 0$. Now, each $f \in \mathfrak{F}(A_1, \cdots, A_n)$ can be expressed as the series

$$f = \sum_{A_1 \times \cdots \times A_n} f(a_1, \cdots, a_n) \chi_{a_1, \cdots, a_n}$$

where there are at most countably many nonzero terms in the sum and the series converges absolutely in norm to f. We also note that

$$\chi_{a_1, \cdots, a_n} * \chi_{b_1, \cdots, b_n} = \chi_{a_1 b_1, \cdots, a_n b_n}$$

for all $a_k, b_k \in A_k$, $1 \leq k \leq n$. Finally, we make $\mathfrak{F}(A_1, \cdots, A_n)$ into a Banach C-algebra as follows. For each $c \in C$ and $f \in \mathfrak{F}(A_1, \cdots, A_n)$ define the action of c on f by

$$c \cdot f = \sum_{A_1 \times \cdots \times A_n} f(a_1, \cdots, a_n) \chi_{c \cdot a_1, a_2, \cdots, a_n}.$$

(Since the series converges absolutely and $\mathfrak{F}(A_1, \cdots, A_n)$ is complete, $c \cdot f$ is a well-defined element of $\mathfrak{F}(A_1, \cdots, A_n)$.)

Let $\mathfrak{J}$ denote the closed linear subspace generated in $\mathfrak{F}(A_1, \cdots, A_n)$ by the family of functions

(i) $\chi_{a_1, \cdots, a_k + b_k, \cdots, a_n} - \chi_{a_1, \cdots, a_k, \cdots, a_n} - \chi_{a_1, \cdots, b_k, \cdots, a_n}$,

(ii) $\lambda \chi_{a_1, \cdots, a_k, \cdots, a_n} - \chi_{a_1, \cdots, \lambda a_k, \cdots, a_n}$,

(iii) $\chi_{a_1, \cdots, c \cdot a_i, \cdots, a_n} - \chi_{a_1, \cdots, c \cdot a_j, \cdots, a_n}$,

where $1 \leq i, j, k \leq n$, $c \in C$, $a_k, b_k \in A_k$, and λ is complex. It is seen immediately that the generators of $\mathfrak{J}$ are closed under convolution by the fundamental functionals in $\mathfrak{F}(A_1, \cdots, A_n)$ and also by the action of C, and therefore, $\mathfrak{I}$ is naturally a closed C-ideal in $\mathfrak{F}(A_1, \cdots, A_n)$.

Let $\mathfrak{D}$ denote the quotient commutative Banach C-algebra $\mathfrak{D} = \mathfrak{F}(A_1, \cdots, A_n)/\mathfrak{I}$ with the quotient norm and define the map δ: $A_1 \times \cdots \times A_n \ni (a_1, \cdots, a_n) \to \chi_{a_1, \cdots, a_n}/\mathfrak{I} \in \mathfrak{D}$. With the above notations we obtain

Proposition 1.8. $(\mathfrak{D}; \delta)$ *is a C-tensor product for the commutative Banach C-algebras $A_1, \cdots, A_n$, and there is an isometric C-algebra isomorphism of $A_1 \otimes_C \cdots \otimes_C A_n$ onto $\mathfrak{D}$ which carries $a_1 \otimes \cdots \otimes a_n$ to $\chi_{a_1, \cdots, a_n}/\mathfrak{I}$.*

Proof. It is evident that δ is a continuous, complex- and C-n-linear, and multiplicative map of $A_1 \times \cdots \times A_n$ into $\mathfrak{D}$. If ϕ is a continuous, complex n-linear, and C-balanced [resp. C-n-linear] map of $A_1 \times \cdots \times A_n$ into a Banach space [resp. C-module] V, define the map $\Phi^{\sim}$ of $\mathfrak{F}(A_1, \cdots, A_n)$ into V by

$$\Phi^{\sim}(f) = \sum_{A_1 \times \cdots \times A_n} f(a_1, \cdots, a_n)\phi(a_1, \cdots, a_n), \qquad f \in \mathfrak{F}(A_1, \cdots, A_n).$$

Then for each $f \in \mathfrak{F}(A_1, \cdots, A_n)$,

$$\|\Phi^{\sim}(f)\|_V \le \sum_{A_1 \times \cdots \times A_n} |f(a_1, \cdots, a_n)| \, \|\phi\| \, \|a_1\| \cdots \|a_n\| = \|\phi\| \, |||f|||,$$

and $\Phi^{\sim}$ is a bounded linear [resp. C-linear] transformation such that $\|\Phi^{\sim}\| \le \|\phi\|$. Since $\Phi^{\sim}(\chi_{a_1, \cdots, a_n}) = \phi(a_1, \cdots, a_n)$ and hence $\|\phi(a_1, \cdots, a_n)\|_V \le \|\Phi^{\sim}\| \|a_1\| \cdots \|a_n\|$, we have $\|\phi\| \le \|\Phi^{\sim}\|$. Thus, $\|\phi\| = \|\Phi^{\sim}\|$. Since ϕ is complex n-linear and C-balanced, $\Phi^{\sim}$ annihilates the generators of $\mathfrak{I}$, and therefore, Ker $\Phi^{\sim} \supseteq \mathfrak{I}$. $\Phi^{\sim}$ induces a bounded linear [resp. C-linear] map Φ: $\mathfrak{D} \to V$ such that $\Phi \circ \delta = \phi$ and $\|\Phi\| = \|\Phi^{\sim}\|$ $(= \|\phi\|)$. Since the linear span of the image of δ is dense in $\mathfrak{D}$, Φ is the unique bounded linear transformation factoring ϕ through $\mathfrak{D}$.

If V is a Banach algebra and ϕ is also multiplicative, then Φ maps the product of two elementary generators in $\mathfrak{D}$ to the product in V of their respective images. The continuity and linearity of Φ imply Φ is a multiplicative transformation of $\mathfrak{D}$ into V. Finally, the isomorphism between $\mathfrak{D}$ and $A_1 \otimes_C \cdots \otimes_C A_n$ follows from the next proposition which is a parallel of [18, Theorem 2.5].

Proposition 1.9. *If $(D; \rho)$ and $(D'; \rho')$ are C-tensor products for the commutative Banach C-algebras $A_1, \cdots, A_n$, then there is an isometric C-algebra isomorphism, J, of D onto D' such that $J \circ \rho = \rho'$.*

The commutativity and unrestricted associativity of tensor products of C-algebras have natural extensions in the context of Banach C-algebras.

Proposition 1.10. *If $A_1, \cdots, A_n$ are commutative Banach C-algebras and σ is a permutation of $\{1, \cdots, n\}$, then there is a natural isometric C-algebra isomorphism*

$$A_1 \otimes_C A_2 \otimes_C \cdots \otimes_C A_n \cong A_{\sigma(1)} \otimes_C A_{\sigma(2)} \otimes_C \cdots \otimes_C A_{\sigma(n)}$$

which carries $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ to $a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(n)}$.

Proposition 1.11. *If $A_1, \cdots, A_n$ are commutative Banach C-algebras and $1 \le i < n$, then there is a natural isometric C-algebra isomorphism*

$$(A_1 \otimes_C \cdots \otimes_C A_i) \otimes_C (A_{i+1} \otimes_C \cdots \otimes_C A_n) \cong A_1 \otimes_C \cdots \otimes_C A_n$$

which carries $(a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$ to $a_1 \otimes \cdots \otimes a_n$.

In conclusion to this section we collect certain facts to be applied later in our discussions of tensor products of group algebras. Although the facts we list are essentially restatements of those in [18] we find it convenient to have the results recorded.

The first lemma is essentially [18, Theorem 4.4] but now presented in the context of Banach C-algebras.

Lemma 1.12. *Let C be a commutative Banach algebra with bounded approximate identity and let A be a commutative Banach C-algebra. Viewing C as a C-algebra, so that $C \otimes_C A$ is defined, there is a natural bicontinuous C-algebra isomorphism $C \otimes_C A \cong A_e$ which carries $c \otimes a$ to $c \cdot a$. If C has an approximate identity of norm one and if A is an isometric C-algebra, then the isomorphism is isometric.*

Proof. The lemma follows immediately from Theorem 4.4 and Proposition 3.1 of [18] once it is observed that they have analogs in the context of Banach C-algebras.

Corollary 1.13. *If C is a commutative Banach algebra with bounded [resp., norm one] approximate identity and if A_k is a commutative [and isometric] Banach C-algebra, $k = 1, \cdots, n$, then there is a natural bicontinuous [resp., isometric] C-algebra isomorphism $C \otimes_C A_1 \otimes_C \cdots \otimes_C A_n \cong (A_1 \otimes_C \cdots \otimes_C A_n)_e$ which carries $c \otimes a_1 \otimes \cdots \otimes a_n$ to $c \cdot (a_1 \otimes \cdots \otimes a_n)$.*

2. **Related tensor products considered by earlier authors.** A few years earlier than the appearance of Rieffel's paper [18], Gelbaum [4] considered a variant of the C-tensor product of Banach C-algebras. It may be of interest to those familiar with [4] to put it into the perspective of the C-tensor product we have discussed in §1.

We begin with a brief description of the setting considered by Gelbaum in [4]. Let A and B be commutative Banach C-algebras. Let $\mathfrak{F}_C(A, B)$ denote the commutative Banach algebra and vector-valued function space on $A \times B$ consisting of all C-valued functions f on $A \times B$ such that $f(a, 0) = f(0, b) = 0$ for all $(a, b) \in A \times B$ and

$$\gamma_1(f) \equiv \sum_{A\times B} \|f(a, b)\|_C \|a\|_A \|b\|_B < \infty,$$

where multiplication is given by convolution (the same formula occuring in §1 for defining convolution in $\mathfrak{F}(A, B)$ can be used here). Now, $\mathfrak{F}_C(A, B)$ naturally becomes a Banach C-algebra under pointwise multiplication by C. We will denote an elementary function in $\mathfrak{F}_C(A, B)$ by $c\chi_{a,b}$ where $c \in C$, $a \in A$, $b \in B$. Let $\mathfrak{I}_C$ denote the closed C-ideal and closed linear subspace generated in $\mathfrak{F}_C(A, B)$ by all elements of the form

(i) $c\chi_{a+a',b} - c\chi_{a,b} - c\chi_{a',b}$, (ii) $c\chi_{a,b+b'} - c\chi_{a,b} - c\chi_{a,b'}$,

(iii) $\lambda c\chi_{a,b} - c\chi_{\lambda a,b}$, (iv) $\lambda c\chi_{a,b} - c\chi_{a,\lambda b}$,

(v) $cc'\chi_{a,b} - c'\chi_{c\cdot a,b}$, (vi) $cc'\chi_{a,b} - c'\chi_{a,c\cdot b}$,

where $a, a' \in A$, $b, b' \in B$, $c, c' \in C$, and λ is a complex scalar.

Let $\mathfrak{D}$ denote the quotient algebra and commutative Banach C-algebra $\mathfrak{D} \equiv \mathfrak{F}_C(A, B)/\mathfrak{I}_C$ with the quotient norm induced by γ_1. $\mathfrak{D}$ is the tensor product of A and B over C considered by Gelbaum in [4], [5], and subsequently by Natzitz [14] and Lardy [13].

The precise connection between $\mathfrak{D}$ and the C-tensor product of A and B defined in §1 is given by

Theorem 2.1. *If A and B are commutative Banach C-algebras let $\mathfrak{D} = \mathfrak{F}_C(A, B)/\mathfrak{I}_C$ be the commutative Banach C-algebra defined above. Then $\mathfrak{D}$ is a C-tensor product for C, A, and B, and there is an isometric C-algebra isomorphism $\mathfrak{D} \cong C \otimes_C A \otimes_C B$ which carries $c\chi_{a,b}/\mathfrak{I}_C$ to $c \otimes a \otimes b$.*

Proof. Let σ: $C \times A \times B \to \mathfrak{D}$ denote the continuous, complex and C-trilinear, multiplicative map defined by

$$\sigma(c, a, b) = c\chi_{a,b}/\mathfrak{I}_C, \qquad (c, a, b) \in C \times A \times B.$$

We must show $(\mathfrak{D}; \sigma)$ is a C-tensor product for C, A, and B. Since the arguments are similar to those used in the proof of Proposition 1.8 we merely sketch the proof. Let ϕ be any continuous, complex trilinear, C-balanced map of $C \times A \times B$ into a Banach space V. Define $\Phi^{\sim}: \mathfrak{F}_C(A, B) \to V$ by

$$\Phi^{\sim}(f) = \sum_{A\times B} \phi(f(a, b), a, b), \qquad f \in \mathfrak{F}_C(A, B).$$

Then $\Phi^{\sim}$ is a bounded linear transformation of norm $\|\phi\|$ and Ker $\Phi^{\sim} \supseteq \mathfrak{J}_C$. $\Phi^{\sim}$ induces a bounded linear transformation Φ: $\mathfrak{D} \longrightarrow V$ of norm $\|\Phi^{\sim}\| = \|\phi\|$ such that $\Phi \circ \sigma = \phi$. If ϕ is C-trilinear, then Φ is C-homogeneous, and if ϕ is multiplicative, then Φ is multiplicative. Thus, $\mathfrak{D}$ is a C-tensor product of C, A, and B and, by Proposition 1.9, it is isometrically C-algebra isomorphic to $C \otimes_C A \otimes_C B$.

Corollary 2.2. *If C has a bounded approximate identity and if A and B are Banach C-algebras, then there is a natural bicontinuous C-algebra isomorphism $\mathfrak{D} \cong (A \otimes_C B)_e$ which carries $c\chi_{a,b}/\mathfrak{J}_C$ to $c \cdot (a \otimes b)$, and, if C has an approximate identity of norm one and if the actions of C on A and B are isometric, then the above isomorphism is isometric.*

Proof. Apply Theorem 2.1 and Corollary 1.13.

3. **The structure space of C-tensor products.** Throughout this section C will denote a commutative Banach algebra. In general, if A is a commutative Banach algebra, we will denote by $\mathfrak{M}_A$ and $\mathfrak{M}_A^0$ the spaces of *all nonzero* and *all* multiplicative linear functionals on A, respectively; both are topologized with the weak* topology as subsets of the unit ball of the Banach space dual of A. We adopt the standard terminology of calling $\mathfrak{M}_A$ the structure space of A. The Gelfand transform of an $a \in A$ will be denoted by $\hat{a}$ and we consider $\hat{a}$ as simultaneously defined on $\mathfrak{M}_A$ and $\mathfrak{M}_A^0$: $\hat{a}(\phi) = \langle a, \phi\rangle$, $\phi \in \mathfrak{M}_A^0$. The principal aim of this section is to characterize the structure space of the tensor algebra $A_1 \otimes_C \cdots \otimes_C A_n$ in terms of the structure spaces of $A_1, \cdots, A_n$.

Now, in [4] Gelbaum has characterized the structure space of the commutative Banach algebra $\mathfrak{D} = \mathfrak{F}_C(A, B)/\mathfrak{J}_C$, or equivalently $C \otimes_C A \otimes_C B$. Consequently, the characterization we obtain should contain his as a special case. Our proof is different from [4], but we adopt the usage of the adjoint maps (between the respective structure spaces) induced by the C-algebra actions. This concept is made explicit in the next lemma which is a slight generalization of [4, Lemmas 1, 2, §2].

Lemma 3.1. *If the commutative Banach algebra A is algebraically a C-algebra, then there is a continuous map μ: $\mathfrak{M}_A \longrightarrow \mathfrak{M}_C^0$ such that $[c \cdot a]\hat{\ }(\phi) = \hat{c}(\mu(\phi))\hat{a}(\phi)$, $\phi \in \mathfrak{M}_A$, for all $c \in C$ and $a \in A$.*

Proof. Let $\phi \in \mathfrak{M}_A$, and suppose $a \in A$ and $\hat{a}(\phi) \neq 0$. The mapping $C \ni c \longrightarrow [c \cdot a]\hat{\ }(\phi)/\hat{a}(\phi)$ is a multiplicative linear functional on C and since A is a C-algebra an easy computation shows this multiplicative linear functional is independent of those a in A such that $\hat{a}(\phi) \neq 0$; we denote this functional by $\mu(\phi)$ and the map $\mu\colon \mathfrak{M}_A \to \mathfrak{M}_C^0$ is defined. Since every regular ideal in A is a C-ideal—indeed, if I is regular and $u \in A$ is the identity modulo I, then $c \cdot i = [c \cdot i - u(c \cdot i)] + (c \cdot u)i \in I$ for all $c \in C$, $i \in I$—we have

$$[c \cdot a]\hat{}(\phi) = \hat{c}(\mu(\phi))\hat{a}(\phi), \qquad \phi \in \mathfrak{M}_A,$$

for *all* $a \in A$, and $c \in C$. To show the continuity of μ, let $\{\phi_\alpha\}$ be a net in $\mathfrak{M}_A$ converging weak* to $\phi \in \mathfrak{M}_A$. Choose $a \in A$ such that $\hat{a}(\phi) = 1$. Now, for some α_0 and all $\alpha \succ \alpha_0$, $\hat{a}(\phi_\alpha) \neq 0$. Then

$$\hat{c}(\mu(\phi_\alpha)) = [c \cdot a]\hat{}(\phi_\alpha)/\hat{a}(\phi_\alpha) \to [c \cdot a]\hat{}(\phi)/\hat{a}(\phi) = \hat{c}(\mu(\phi)),$$

and hence $\mu(\phi_\alpha)$ converges weak* to $\mu(\phi)$. Thus, μ is continuous.

Definition 3.2. If A is a commutative Banach C-algebra, the continuous map $\mu\colon \mathfrak{M}_A \to \mathfrak{M}_C^0$, such that $[c \cdot a]\hat{} = \hat{c} \circ \mu\, \hat{a}$ for all $(c, a) \in C \times A$, is called the *adjoint map of the action of C on A*.

Theorem 3.3. *Let $A_1, \cdots, A_n$ be commutative Banach C-algebras and let $\mu_k\colon \mathfrak{M}_{A_k} \to \mathfrak{M}_C^0$, $k = 1, 2, \cdots, n$, be the respective adjoint maps. Let $D = A_1 \otimes_C \cdots \otimes_C A_n$. There is a homeomorphism τ of $\mathfrak{M}_D$ onto a closed subset, $\tau(\mathfrak{M}_D)$, of $\mathfrak{M}_{A_1} \times \cdots \times \mathfrak{M}_{A_n}$ and $\tau(\mathfrak{M}_D) = (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n^0)$ where Δ_n^0 is the diagonal subset of $\mathfrak{M}_C^0 \times \cdots \times \mathfrak{M}_C^0$. Moreover, if $\phi \in \mathfrak{M}_D$ and $\tau(\phi) = (\phi_{A_1}, \cdots, \phi_{A_n})$, then for every $z \in D$ and representative $\sum_m \lambda_m(a_{1,m} \otimes \cdots \otimes a_{n,m})$ of z we have*

$$\hat{z}(\phi) = \sum_m \lambda_m \hat{a}_{1,m}(\phi_{A_1}) \cdots \hat{a}_{n,m}(\phi_{A_n}).$$

Proof. Let $E = A_1 \otimes_\gamma \cdots \otimes_\gamma A_n$ and let J be the closed C-ideal and closed linear subspace generated in E by the elements

$$a_1 \otimes \cdots \otimes c \cdot a_i \otimes \cdots \otimes a_n - a_1 \otimes \cdots \otimes c \cdot a_j \otimes \cdots \otimes a_n$$

for all $1 \le i, j \le n$, $a_k \in A_k$, $k = 1, \cdots, n$, $c \in C$, so that $D = E/J$. Now, Gelbaum [2, §2, Theorem 2] has shown that there is a homeomorphism η of $\mathfrak{M}_E$ onto $\mathfrak{M}_{A_1} \times \cdots \times \mathfrak{M}_{A_n}$ such that if $\phi_E \in \mathfrak{M}_E$ and $\eta(\phi_E) = (\phi_{A_1}, \cdots, \phi_{A_n})$, then for each $x \in E$ and representative $\sum_m \lambda_m(a_{1,m} \otimes \cdots \otimes a_{n,m})$ of x,

$$\hat{x}(\phi_E) = \sum_m \lambda_m \hat{a}_{1,m}(\phi_{A_1}) \cdots \hat{a}_{n,m}(\phi_{A_n}).$$

Let $\mathbf{h}(J)$ denote the hull [17, p. 78] of J in $\mathfrak{M}_E$, i.e.

$$\mathbf{h}(J) = \{\phi_E \in \mathfrak{M}_E \mid \hat{j}(\phi_E) = 0 \ \ \forall j \in J\}.$$

It is known that the structure space of the quotient of a commutative Banach algebra by a closed ideal is identifiable with the hull of the ideal (cf. [17, Theorem 3.1.17]), and therefore there is a homeomorphism ω of $\mathfrak{M}_D$ onto $\mathbf{h}(J)$, a closed subset of $\mathfrak{M}_E$, such that if $\phi_D \in \mathfrak{M}_D$ and $\omega(\phi_D) = \phi_E \in \mathbf{h}(J)$, then for all $z \in D$ and representatives x/J of z $(x \in E)$, $\hat{z}(\phi_D) = \hat{x}(\phi_E)$.

Let $\tau\colon \mathfrak{M}_D \to \eta(\mathbf{h}(J)) \subseteq \mathfrak{M}_{A_1} \times \cdots \times \mathfrak{M}_{A_n}$ be the homeomorphism and composite map of the mappings

$$\begin{array}{ccccc} \mathfrak{M}_D & \xrightarrow{\omega} & \mathbf{h}(J) & \xrightarrow{\eta|_{\mathbf{h}(J)}} & \eta(\mathbf{h}(J)) \\ & & \cap| & & \cap| \\ & & \mathfrak{M}_E & & \mathfrak{M}_{A_1} \times \cdots \times \mathfrak{M}_{A_n} \end{array}$$

Since $\mathbf{h}(J)$ is closed in $\mathfrak{M}_E$ and η is a homeomorphism, $\tau(\mathfrak{M}_D) = \eta(\mathbf{h}(J))$ is a closed subset of $\mathfrak{M}_{A_1} \times \cdots \times \mathfrak{M}_{A_n}$. Furthermore, if $\phi_D \in \mathfrak{M}_D$ and $\tau(\phi_D) = \eta(\omega(\phi_D)) = (\phi_{A_1}, \cdots, \phi_{A_n})$, then for every $z \in D$ and representative $\sum_m \lambda_m (a_{1,m} \otimes_C \cdots \otimes_C a_{n,m})$ of z, we have with $x = \sum_m \lambda_m (a_{1,m} \otimes \cdots \otimes a_{n,m}) \in E$, $z = x/J$ and

$$\hat{z}(\phi_D) = \hat{x}(\omega(\phi_D)) = \sum_m \lambda_m \hat{a}_{1,m}(\phi_{A_1}) \cdots \hat{a}_{n,m}(\phi_{A_m}).$$

We now show $\tau(\mathfrak{M}_D) = (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta^0)$. First, observe that $\tau(\mathfrak{M}_D)$ consists of precisely those n-tuples $(\phi_{A_1}, \cdots, \phi_{A_n})$ such that

$$(3.1) \qquad \left(\prod_{k=1; k \neq i}^{n} \hat{a}_k(\phi_{A_k}) \right) [c \cdot a_i]\hat{}(\phi_{A_i}) = \left(\prod_{k=1; k \neq j}^{n} \hat{a}_k(\phi_{A_k}) \right) [c \cdot a_j]\hat{}(\phi_{A_j})$$

for all $1 \leq i, j \leq n$, $a_k \in A_k$ $(k = 1, \cdots, n)$, and $c \in C$. Indeed, suppose $(\phi_{A_1}, \cdots, \phi_{A_n}) \in \mathfrak{M}_{A_1} \times \cdots \times \mathfrak{M}_{A_n}$ is an n-tuple satisfying the above equality; then for all i, j, a_k, and c,

$$[a_1 \otimes \cdots \otimes c \cdot a_i \otimes \cdots \otimes a_n - a_1 \otimes \cdots \otimes c \cdot a_j \otimes \cdots \otimes a_n]\hat{}(\eta^{-1}(\phi_{A_1}, \cdots, \phi_{A_n})) = 0.$$

Since J is the closed linear span of the elements $a_1 \otimes \cdots \otimes c \cdot a_i \otimes \cdots \otimes a_n - a_1 \otimes \cdots \otimes c \cdot a_j \otimes \cdots \otimes a_n$, $1 \leq i, j \leq n$, $a_k \in A_k$ $(k = 1, \cdots, n)$, $c \in C$, we have $\hat{j}(\eta^{-1}(\phi_{A_1}, \cdots, \phi_{A_n})) = 0$, $j \in J$. Thus, $\eta^{-1}(\phi_{A_1}, \cdots, \phi_{A_n}) \in \mathbf{h}(J)$ and $(\phi_{A_1}, \cdots, \phi_{A_n}) \in \eta(\mathbf{h}(J)) = \tau(\mathfrak{M}_D)$. Conversely, if $(\phi_{A_1}, \cdots, \phi_{A_n}) \in \tau(\mathfrak{M}_D)$, it is easily seen that the above steps are all reversible and hence $(\phi_{A_1}, \cdots, \phi_{A_n})$ is an n-tuple satisfying equality (3.1).

Using the adjoint maps $\mu_k\colon \mathfrak{M}_{A_k} \to \mathfrak{M}_C^0$, equality (3.1) can be written as

$$(3.2) \qquad \prod_{k=1}^{n} \hat{a}_k(\phi_{A_k}) \hat{c}(\mu_i(\phi_{A_i})) = \prod_{k=1}^{n} \hat{a}_k(\phi_{A_k}) \hat{c}(\mu_j(\phi_{A_j}))$$

for all $1 \leq i, j \leq n$, $a_k \in A$ $(k = 1, \cdots, n)$, and $c \in C$. Choosing a_k so that $\hat{a}_k(\phi_{A_k}) \neq 0$, $k = 1, \cdots, n$, and then canceling the product $\prod_{k=1}^{n} \hat{a}_k(\phi_{A_k})$, we have

$$\hat{c}(\mu_i(\phi_{A_i})) = \hat{c}(\mu_j(\phi_{A_j})), \qquad c \in C,\ 1 \le i,\ j \le n, \tag{3.3}$$

i.e., $\mu_i(\phi_{A_i}) = \mu_j(\phi_{A_j})$, $1 \le i, j \le n$. Thus, if $(\phi_{A_1}, \cdots, \phi_{A_n})$ is in $\tau(\mathfrak{M}_D)$, then $(\phi_{A_1}, \cdots, \phi_{A_n}) \in (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n^0)$. Conversely, if $(\phi_{A_1}, \cdots, \phi_{A_n})$ is in $(\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n^0)$, then equalities (3.3), (3.2), and (3.1) each hold in succession for all $1 \le i, j \le n$, $a_k \in A_k$ $(k = 1, \cdots, n)$, and $c \in C$. Thus, $(\phi_{A_1}, \cdots, \phi_{A_n}) \in \tau(\mathfrak{M}_D)$. Therefore $\tau(\mathfrak{M}_D) = (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n^0)$ and the proof is complete.

Remark 3.4. When $n = 3$, $A_1 = C$, $A_2 = A$, and $A_3 = B$, we can recover the characterization of the structure space of $\mathfrak{D} = \mathfrak{F}_C(A, B)/\mathfrak{I}_C$ obtained by Gelbaum in [4]. Namely, let μ and ν denote the adjoint maps associated with the action of C on A and B, respectively. The adjoint map of the natural action of C on C is simply the identity map $1: \mathfrak{M}_C \to \mathfrak{M}_C \subseteq \mathfrak{M}_C^0$. By Theorem 3.3 the structure space of $C \otimes_C A \otimes_C B$ is naturally identifiable with

$$\begin{aligned}(1 \times \mu \times \nu)^{-1}(\Delta_3^0) &= \{(\phi_C, \phi_A, \phi_B) |\ \mu(\phi_A) = \nu(\phi_B) = \phi_C\} \\ &= \{(\phi_C, \phi_A, \phi_B) |\ \mu(\phi_A) = \nu(\phi_B) \in \mathfrak{M}_C\},\end{aligned}$$

where $\Delta_3^0 = \operatorname{diag}(\mathfrak{M}_C^0 \times \mathfrak{M}_C^0 \times \mathfrak{M}_C^0)$. It is easily seen that the natural projection of $\mathfrak{M}_C \times \mathfrak{M}_A \times \mathfrak{M}_B$ onto $\mathfrak{M}_A \times \mathfrak{M}_B$, when restricted to the closed subset $(1 \times \mu \times \nu)^{-1}(\Delta_3^0)$, is a homeomorphism onto the subset $(\mu \times \nu)^{-1}(\Delta_2)$ in $\mathfrak{M}_A \times \mathfrak{M}_B$ where $\Delta_2 = \operatorname{diag}(\mathfrak{M}_C \times \mathfrak{M}_C)$. Since $\mathfrak{D}$ is isometrically C-isomorphic to $C \otimes_C A \otimes_C B$ (Theorem 2.1), there is a natural homeomorphism τ' of $\mathfrak{M}_{\mathfrak{D}}$ onto the subset $(\mu \times \nu)^{-1}(\Delta_2)$ in $\mathfrak{M}_A \times \mathfrak{M}_B$ and the characterization by Gelbaum [4, §2, Theorem 1] of $\mathfrak{M}_{\mathfrak{D}}$ is recovered.

4. Algebra actions between group algebras.

The remainder of this paper is devoted to analyzing the algebra actions of one group algebra on another and using the ensuing analysis to characterize tensor products of group algebras. The main objective is to show to what extent the tensor product of group algebras is the group algebra of the tensor product of groups, and thereby bring to completion a study initiated by Gelbaum [4], and continued by Natzitz [14], Gelbaum [5], and Lardy [13].

If G is a locally compact Abelian (LCA) group, we will let $L^1(G)$ and $M(G)$ denote the group algebra and measure algebra on G, respectively. $\hat{G}$ will denote the dual group of G, and we make the (usual) identification of $\hat{G}$ with the structure space of $L^1(G)$, and $\hat{a}$, $\hat{\mu}$ denote the Fourier and Fourier-Stieltjes transforms of $a \in L^1(G)$, $\mu \in M(G)$:

$$\hat{a}(\alpha) = \int_G a(x)\overline{\alpha(x)}dx, \qquad \hat{\mu}(\alpha) = \int_G \overline{\alpha(x)}d\mu(x),$$

for $\alpha \in \hat{G}$.

In this section we describe the algebra actions of $L^1(K)$ on $L^1(G)$ for LCA groups K and G. In [12] we have discussed this problem in detail. Briefly, attention is centered on showing that any algebra action of $L^1(K)$ on $L^1(G)$ is naturally induced by a continuous homomorphism between the underlying groups. Since only certain special results are needed from [12], we find it convenient to state and briefly prove those facts needed in this paper. We begin with a crucial lemma concerning algebra actions on a group algebra. We remark that this is a special case of Theorem 2.2 of [12] since the algebras we consider here are commutative.

Lemma 4.1. *Let C be a commutative Banach algebra and let G be a LCA group. If $L^1(G)$ is a C-algebra and $\mu: \hat{G} \to \hat{K}^0$ is the adjoint map, then there is a unique, necessarily continuous homomorphism $\Psi: C \to M(G)$ such that*

(i) $c \cdot a = \Psi(c) * a$ *for all* $(c, a) \in C \times L^1(G)$;

(ii) $\|\Psi\|$ *is the norm of the action of C on $L^1(G)$;*

(iii) $[\Psi(c)]\hat{} = \hat{c} \circ \mu$, $c \in C$, *i.e., μ is the adjoint map of Ψ.*

Proof. Since μ is the adjoint map of the action of C on $L^1(G)$, $[c \cdot a]\hat{} = \hat{c} \circ \mu\, \hat{a}$ for each $c \in C$, $a \in L^1(G)$. Thus, for each fixed $c \in C$, $\hat{c} \circ \mu$ is a multiplier of $L^1(G)\hat{}$ and hence by Helson and Edwards (cf. [20, Theorem 3.8.1]) $\hat{c} \circ \mu$ is the Fourier-Stieltjes transform of some measure $\Psi(c)$ in $M(G)$. Thus, $c \cdot a = \Psi(c) * a$ and $[\Psi(c)]\hat{} = \hat{c} \circ \mu$ for all $c \in C$ and $a \in L^1(G)$. Ψ is evidently a continuous homomorphism of C into $M(G)$. Since for all $c \in C$, $a \in L^1(G)$, $\|c \cdot a\|_1 = \|\Psi(c) * a\|_1 \le \|\Psi\|\, \|c\|_C \|a\|_1$, the norm χ of the action of C on $L^1(G)$ does not exceed $\|\Psi\|$. On the other hand, since in general the operator $a \to \mu * a$ on $L^1(G)$, $\mu \in M(G)$, has norm, $\|\mu\|$ (cf. [20, Corollary 3.8.2]), and since for all $c \in C$, $a \in L^1(G)$, $\|\Psi(c) * a\|_1 = \|c \cdot a\|_1 \le \chi \|c\|_C \|a\|_1$, we have $\|\Psi(c)\| \le \chi \|c\|_C$, and hence $\|\Psi\| \le \chi$. Thus, $\|\Psi\| = \chi$ and the proof is complete.

When C is the group algebra of a LCA group, Cohen's theory [1] of homomorphisms of group algebras applies; specifically the adjoint map is characterized. We paraphrase Cohen's result in the context of algebra actions (with the specific references from [20, Theorem 4.1.3 and §4.6.3]).

Corollary 4.2 (Cohen). *If $L^1(G)$ is an $L^1(K)$-algebra for LCA groups K and G with $\mu: \hat{G} \to \hat{K}^0$ the adjoint map and $\chi \ge 0$ the norm of the action, then*

(i) *$X = \mu^{-1}(\hat{K})$ belongs to the open coset ring in $\hat{G}$ and μ is a piecewise affine map of X into $\hat{K}$;*

(ii) *if $0 < \chi \le 1$, then $X = \mu^{-1}(\hat{K})$ is an open coset in $\hat{G}$ and μ is an affine map of X into $\hat{K}$.*

Conversely, if X is the open coset ring [resp., an open coset] in $\hat{G}$ and μ is a piecewise affine [resp., affine] map of X into $\hat{K}$, then μ is the adjoint map of a [resp., isometric] Banach $L^1(K)$-algebra action on $L^1(G)$.

Just as the multiplicative structure in a group algebra is related to the group structure in the underlying space, the algebra actions of one group algebra on another are induced by an interaction between the underlying groups. We realize this connection in the succeeding discussion.

Definition 4.3. By a LCA *K-group* we mean a LCA group G together with a compact subgroup g in G and a continuous map $(k, x) \to k \cdot x$ of $K \times G \to G/g$ such that

(i) $(kk') \cdot x = k \cdot (k' \cdot x)$,

(ii) $k \cdot (xx') = (k \cdot x)x'$ $(= x(k \cdot x'))$,

for all $k, k' \in K$, $x, x' \in G$. We call G/g the *essential part of* G and denote it by G_e (and as will be evident later, G_e is naturally a K-group under an action induced by the action of K on G). If $g = \{0\}$, and hence $G = G_e$, we call G an *essential K-group.*

Lemma 4.4. *If G is a LCA K-group, there is a unique continuous homomorphism θ_e: $K \to G_e$ such that $k \cdot x = \theta_e(k)(x/g)$ for all $k \in K$, $x \in G$. Conversely, each continuous homomorphism θ: $K \to G/g$ for compact subgroup $g \subseteq G$ makes G into a K-group under the action $k \cdot x = \theta(k)(x/g)$, $k \in K$, $x \in G$.*

Proof. If G is a K-group, set $\theta_e(k) = k \cdot 0_G$ where 0_G is the identity of G. The remaining details are left to the reader.

Once we have introduced some canonical maps to be used in the sequel, the connection between K-group actions and $L^1(K)$-algebra actions will solidify. Let g be a compact subgroup of a LCA group G. Given a Haar measure on G and with the normalized Haar measure on g, choose a Haar measure on G/g so that the Haar measures on G, g, and G/g are canonically related [16, p. 59]. Let T_g denote the canonical norm decreasing homomorphism of $L^1(G)$ onto $L^1(G/g)$ given by

$$T_g a(x/g) = \int_g a(x + p)\,dp \qquad (x/g \in G/g),$$

for all $a \in L^1(G)$ [16, p. 69]. Let π_g denote the canonical homomorphism of G onto G/g. Since g is compact, π_g induces an isometric isomorphism π_g^* of $M(G/g)$ into $M(G)$ such that

$$\langle f, \pi_g^*(\nu)\rangle = \langle T_g(f), \nu\rangle$$

for all $\nu \in M(G/g)$ and $f \in C_0(G)$, the space of continuous functions vanishing at infinity on G. Now, if $\nu = b \in L^1(G/g) \subseteq M(G/g)$, then $\pi_g^*(b) = b \circ \pi_g$ in $L^1(G)$. Also note that T_g is a left inverse for π_g^* (when restricted to $L^1(G/g)$), i.e., $T_g \circ \pi_g^* =$ identity map on $L^1(G/g)$. We will denote the kernel of T_g by I_g.

Definition 4.5. Suppose G is a LCA K-group and $\gamma \in \hat{K}$. Then γ and the action of K on G induce a natural action of $L^1(K)$ on $L^1(G)$ as follows. Let $\theta_e: K \longrightarrow G_e = G/g$ be the continuous homomorphism inducing the action of K on G. If $c \in L^1(K)$, $a \in L^1(G)$, then the induced action of c on a is defined by

$$c \cdot_{\gamma,K} a(x) \equiv \int_K \gamma(k) c(k) T_g a[x/g - \theta_e(k)]\, dk \qquad (x \in G).$$

If $\gamma \equiv 1$, then we will simply write $c \cdot_K a$ for $c \cdot_{\gamma,K} a$.

That this is indeed an $L^1(K)$-algebra action on $L^1(G)$ is formally stated in

Lemma 4.6. *$L^1(G)$ is a nondegenerate isometric Banach $L^1(K)$-algebra under the action $c \cdot_{\gamma,K} a$ defined in (4.5) and*

$$c \cdot_{\gamma,K} a = \pi_g^*[\gamma c *_{\theta_e} T_g(a)]$$

for all $c \in L^1(K)$, $a \in L^1(G)$.

Proof. Since $*_{\theta_e}$ is an isometric Banach $L^1(K)$-algebra action on $L^1(G/g)$, and since clearly $c \cdot_{\gamma,K} a = \pi_g^*[\gamma c *_{\theta_e} T_g(a)]$, it is immediate that this is an isometric $L^1(K)$-algebra action.

Now, if G is an essential LCA K-group and if $\theta: K \longrightarrow G$ is the continuous homomorphism inducing the action of K on G, then

$$c \cdot_{\gamma,K} a = \gamma c *_\theta a \equiv c *_{\gamma,\theta} a$$

for all $c \in L^1(K)$, $a \in L^1(G)$; thus, the $L^1(K)$-action induced on $L^1(G)$ when G is an essential K-group is simply θ-convolution.

Remark 4.7. If G is a LCA K-group, then the induced Banach $L^1(K)$-module action on $L^1(G)$ can also be extended to $L^p(G)$ for $1 \le p \le \infty$. Indeed, let g denote the compact subgroup of G and $\theta: K \longrightarrow G/g$ the continuous homomorphism inducing the action of K on G. It can be easily checked that

$$T_g a(x/g) = \int_g a(x+y)\, dy \qquad (x/g \in G/g)$$

defines a norm decreasing linear transformation of $L^p(G)$ onto $L^p(G/g)$ for $1 \le p \le \infty$ (of course the compactness of g is very crucial here). Furthermore, it is also easily checked that if ψ is a continuous homomorphism of K into a LCA group $\mathfrak{G}$, then $L^p(\mathfrak{G})$ $(1 \le p \le \infty)$ becomes an isometric Banach $L^1(K)$-module under the action:

$$c *_\psi a(x) = \int_K c(k)\, a(x - \psi(k))\, dk \qquad (x \in \mathfrak{G}),$$

for $c \in L^1(K)$, $a \in L^p(\mathfrak{G})$. Consequently, the action $c \cdot_{\gamma,K} a$ defined in Definition 4.5 makes $L^p(G)$ into an isometric Banach $L^1(K)$-module for all p, $1 \le p \le \infty$.

Theorem 4.8 (**Corollary 3.2** [12]). *Let K and G be LCA groups. $L^1(G)$ is a* *nondegenerate isometric Banach* $L^1(K)$*-algebra under an action* $c \cdot a$, $c \in L^1(K)$, $a \in L^1(G)$, *if and only if there is a* K*-group action on* G *and characters* $\alpha \in \hat{G}$, $\gamma \in \hat{K}$, *such that* $c \cdot a = \alpha(c \cdot_{\gamma,K} (\overline{\alpha} a))$, $c \in L^1(K)$, $a \in L^1(G)$, *where* $\cdot_{\gamma,K}$ *denotes the natural* $L^1(K)$*-algebra action on* $L^1(G)$ *induced by* γ *and the action of* K *on* G. *Finally, the* K*-group action on* G *is uniquely determined.*

Proof. We include a proof for the sake of completeness. If $L^1(G)$ is a nondegenerate isometric Banach $L^1(K)$-algebra with adjoint map $\mu: \hat{G} \longrightarrow \hat{K}^0$, then, by Corollary 4.2, $X = \mu^{-1}(\hat{K})$ is in an open coset in $\hat{G}$ and μ is an affine map of X into $\hat{K}$. Let $\alpha \in X$ and set $\gamma = \overline{\mu(\alpha)} = -\mu(\alpha) \in \hat{K}$. Let g denote the compact subgroup in G and annihilator of the open subgroup $Q = X - \alpha$ in $\hat{G}$. Using the identification $(G/g)\hat{} = g^{\perp} = Q$, let $\theta: K \longrightarrow G/g$ denote the continuous dual homomorphism of the continuous homomorphism

$$X - \alpha = Q \ni \alpha' \mapsto \mu(\alpha + \alpha') - \mu(\alpha) \in \hat{K}$$

(and note that this homomorphism is independent of the choice of $\alpha \in X$). Now, $\theta: K \longrightarrow G/g$ induces a natural K-group action on G and therefore induces a natural $L^1(K)$-algebra action on $L^1(G)$. We need only show $c \cdot a = \alpha(\gamma c \cdot_K \overline{\alpha} a)$ for all $c \in L^1(K)$, $a \in L^1(G)$. Using the facts that for $a \in L^1(G)$, $b \in L^1(G/g)$, $(T_g a)\hat{} = \hat{a}|_{g^\perp}$ on $g^{\perp} = (G/g)\hat{}$ and

$$\begin{aligned} (\pi_g^* b)\hat{}(\alpha) &= \hat{b}(\alpha) \quad \text{if } \alpha \in g^{\perp} = (G/g)\hat{}, \\ &= 0 \qquad \text{if } \alpha \in \hat{G}\backslash g^{\perp}, \end{aligned}$$

it can be easily shown that the Fourier transforms on $\hat{G}$ of $c \cdot a$ and $\alpha(\gamma c \cdot_K \overline{\alpha} a)$ are identical and hence $c \cdot a = \alpha(\gamma c \cdot_K \overline{\alpha} a)$ for all $c \in L^1(K)$, $a \in L^1(G)$.

The next lemma is crucial to Theorem 6.5 and is most easily presented now. The reader should check Definition 6.9 for the definition of a projection in a Banach C-algebra before reading the next lemma.

Lemma 4.9. *If* $L^1(G)$ *is an isometric Banach* $L^1(K)$*-algebra, there is a unique closed ideal* J *in* $L^1(G)$ *complementing* $L^1(G)_e$ *so that* $L^1(G) = J \oplus L^1(G)_e$, *and the projection of* $L^1(G)$ *onto* $L^1(G)_e$ *is of norm one. Furthermore,* J *and* $L^1(G)_e$ *satisfy*

(i) *If* $L^1(G)_e \neq \{0\}$, *there is a* K*-group action on* G *such that if* $\alpha \in \mu^{-1}(\hat{K})$, *then* $J = \alpha I_g$ *where* g *is the compact subgroup in* G *such that* $G_e = G/g$, *and* $\alpha\pi_g^*: L^1(G_e) \cong L^1(G)_e$ *is an isometric* $L^1(K)$*-algebra isomorphism where the algebra action of* $L^1(K)$ *on* $L^1(G_e)$ *is that induced by the action of* K *on* G_e *and* $\gamma = -\mu(\alpha) \in \hat{K}$.

(ii) *If* $L^1(G)_e \neq \{0\}$, *then in the notation of* (i) *the direct sum decomposition* *of an* $a \in L^1(G)$ *is given by* $a = [a - \alpha\pi_g^* T_g(\overline{\alpha}a)] + \alpha\pi_g^* T_g(\overline{\alpha}a)$ *where* π_g^* *and* T_g *are the canonical maps.*

Proof. If $L^1(K) \cdot L^1(G) = \{0\}$, then of course $J = L^1(G)$ is the unique ideal complementing $L^1(G)_e = \{0\}$. Now, if $L^1(G)$ is a nondegenerate isometric Banach $L^1(K)$-algebra, then G is a K-group, and if $\alpha \in \mu^{-1}(\hat{K})$, $\gamma = \mu(\alpha) \in \hat{K}$, then

$$c \cdot a = \alpha(c \cdot_{\gamma,K} \overline{\alpha}a) = \alpha\pi_g^*(c *_{\gamma,\theta_e} T_g(\overline{\alpha}a))$$

for all $c \in L^1(K)$, $a \in L^1(G)$. An easy computation shows that the map $a \longrightarrow \alpha\pi_g^* T_g(\overline{\alpha}a)$ is a projection of norm one on $L^1(G)$ with range, $\alpha\pi_g^*(L^1(G_e))$, and kernel, $J = \alpha \operatorname{Ker} T_g$. Now, since α, $\alpha' \in \mu^{-1}(\hat{K})$ imply $\alpha'\overline{\alpha}$ is identically one on g, $\alpha' I_g \subset \alpha I_g$; interchanging the roles of α and α', we have $\alpha I_g \subset \alpha' I_g$, and hence $\alpha I_g = \alpha' I_g$ for all α, $\alpha' \in \mu^{-1}(K)$. Now, general algebra implies that

$$L^1(G) = J \oplus \alpha\pi_g^*(L^1(G_e)).$$

Since $\alpha\pi_g^*$ is an isometric isomorphism of $L^1(G_e)$ into $L^1(G)$, we need only show it is $L^1(K)$-homogeneous and has range $L^1(G)_e$.

Now, if $c \in L^1(K)$, $b \in L^1(G_e)$, then

$$\alpha\pi_g^*(c *_{\gamma,\theta_e} b) = \alpha\pi_g^*[c *_{\gamma,\theta_e} T_g(\overline{\alpha}\alpha\pi_g^*(b))] = \alpha(c \cdot_{\gamma,K} \overline{\alpha}\alpha\pi_g^*(b)) = c \cdot \alpha\pi_g^*(b),$$

and hence $\alpha\pi_g^*$ is $L^1(K)$-homogeneous where $L^1(G_e)$ is the $L^1(K)$-algebra under the action $*_{\gamma,\theta_e}$. Now, since $L^1(G_e) = L^1(K) \cdot L^1(G)$ and $c \cdot a = \alpha\pi_g^*(b)$ for $b = c *_{\gamma,\theta_e} T_g(\overline{\alpha}a) \in L^1(G_e)$ for all c, a, it follows that $L^1(G)_e \subseteq \alpha\pi_g^*(L^1(G_e))$. To show the reverse inclusion, it suffices (by Proposition 1.3 and the fact that $\alpha\pi_g^*$ is an isometric $L^1(K)$-isomorphism) to show $L^1(G_e)$ is an essential $L^1(K)$-algebra.

Let $b \in L^1(G_e)$ and suppose $\epsilon > 0$ is given. Choose a neighborhood V of the identity in $G_e = G/g$ such that $\|L_s b - b\| < \epsilon$ if $s \in V$ where L_s denotes the left translation (by $-s$) operator. Since θ_e is continuous, there is a neighborhood U of the identity in K such that $\theta_e(U) \subseteq V$. Let c_U be a nonnegative continuous function on K with compact support in U and $\|c_U\|_1 = 1$. Setting $c = \overline{\gamma}c_U$, we have

$$\begin{aligned}
\|c *_{\gamma,\theta_e} b - b\|_1 &= \|c_U *_{\theta_e} b - b\|_1 \\
&\leq \int_{G_e} \int_K |L_{\theta_e(k)} b(s) - b(s)|\, c_U(k)\, dk\, ds \\
&= \int_U \|L_{\theta_e(k)} b - b\|_1 c_U(k)\, dk < \epsilon.
\end{aligned}$$

Thus, $L^1(K) *_{\gamma,\theta_e} L^1(G_e)$ is dense in $L^1(G_e)$ and $L^1(G_e)$ is an essential $L^1(K)$-algebra.

Finally, J is the unique closed ideal in $L^1(G)$ complementing $L^1(G)_e$. Indeed, if $L^1(G) = J' \oplus L^1(G)_e$ for a closed ideal J' in $L^1(G)$, then from

$$L^1(G) = J' + L^1(G)_e, \qquad J' \cap L^1(G)_e = \{0\},$$

it follows by taking the hull of each side that

$$\phi = \mathrm{h}(J') \cap \mathrm{h}(L^1(G)_e), \qquad \hat{G} = \mathrm{h}(J') \cup \mathrm{h}(L^1(G)).$$

Thus, $\mathrm{h}(J') = \hat{G}\backslash \mathrm{h}(L^1(G)_e)$. But

$$\mathrm{h}(L^1(G)_e) = \mathrm{h}(\alpha\pi_g^*(L^1(G_e))) = \mathrm{h}(\pi_g^*(L^1(G_e))) - \alpha = (\hat{G}\backslash g^\perp) - \alpha = \hat{G}\backslash\mu^{-1}(\hat{K}).$$

Since $\mu^{-1}(\hat{K})$ is an open coset in $\hat{G}$, it is a set of spectral synthesis [20, Theorem 7.5.2] and therefore there is a unique ideal whose hull is $\mu^{-1}(\hat{K})$. Thus, $J = J'$ and the proof is complete.

5. **Semisimplicity of tensor products of group algebras.** Using the characterization (Corollary 4.2) of the adjoint maps of algebra actions of a group algebra on a group algebra (as provided by Cohen's theory), significant additional information is gained about the structure space of $L^1(K)$-tensor products of group algebras. An important consequence is the strong semisimplicity of these tensor algebras in *all* instances of $L^1(K)$-algebra actions on the group algebras.

If $\mathfrak{G}$ is an Abelian group, $\mathfrak{R}(\mathfrak{G})$ will denote the *coset ring* of $\mathfrak{G}$, the smallest Boolean algebra of subsets of $\mathfrak{G}$ containing the cosets of all subgroups of $\mathfrak{G}$.

Theorem 5.1. *Suppose $L^1(G_1), \cdots, L^1(G_n)$ are $L^1(K)$-algebras for LCA groups $G_1, \cdots, G_n$, and K. If $\tau(\mathfrak{M}_D)$ is the identification of the structure space of*

$$D = L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n)$$

in $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$, then $\tau(\mathfrak{M}_D)$ is a closed subset in the coset ring $\mathfrak{R}(\hat{G}_1 \oplus \cdots \oplus \hat{G}_n)$ of $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$.

Proof. Let $\mu_i: \hat{G}_i \to \hat{K}^0$ denote the adjoint map of the action of $L^1(K)$ on $L^1(G_i)$ and let $X_i = \mu_i^{-1}(\hat{K}) \subseteq \hat{G}$ for each $i = 1, \cdots, n$. By Corollary 4.2(i), X_i is in the open coset ring in $\hat{G}_i$, and μ_i is a piecewise affine map of X_i into $\hat{K}$ for each $i = 1, \cdots, n$. Consequently, there are determined the following [20, p. 781] for each $i = 1, \cdots, n$:

(i) pairwise disjoint sets $S_{i,1}, \cdots, S_{i,m_i}$ belonging to the open coset ring in $\hat{G}_i$,

(ii) open cosets $C_{i,j}$ in $\hat{G}_i$, such that $S_{i,j} \subseteq C_{i,j}$, $j = 1, \cdots, m_i$,

(iii) a continuous affine map $\mu_{i,j}$ of $C_{i,j}$ into $\hat{K}$, $j = 1, \cdots, m_i$,

such that μ_i is the map of $X_i = S_{i,1} \cup \cdots \cup S_{i,m_i}$ into $\hat{K}$ which coincides on $S_{i,j}$ with $\mu_{i,j}$, $j = 1, \cdots, m_i$.

With $\Delta_n = \operatorname{diag}(\hat{K} \times \cdots \times \hat{K})$, we have

$$(\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n)$$

$$= \bigcup_{1 \le j_i \le m_i;\, 1 \le i \le n} (S_{1,j_1} \times \cdots \times S_{n,j_n}) \cap (\mu_{1,j_1} \times \cdots \times \mu_{n,j_n})^{-1}(\Delta_n).$$

Since $\mu_{1,j_1} \times \cdots \times \mu_{n,j_n}$ is a continuous affine map of $C_{1,j_1} \times \cdots \times C_{n,j_n}$ into $\hat{K}$ and Δ_n is a closed subgroup in $\hat{K} \times \cdots \times \hat{K}$, the set $(\mu_{1,j_1} \times \cdots \times \mu_{n,j_n})^{-1}(\Delta_n)$ is a closed coset in $C_{1,j_1} \times \cdots \times C_{1,j_n}$, and, therefore a coset in $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$. It is easily seen that

$$\Re(\hat{G}_1) \times \cdots \times \Re(\hat{G}_n) \subseteq \Re(\hat{G}_1 \oplus \cdots \oplus \hat{G}_n),$$

and therefore $S_{1,j_1} \times \cdots \times S_{n,j_n}$ is in $\Re(\hat{G}_1 \oplus \cdots \oplus \hat{G}_n)$. From the above expression for $(\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n)$, we conclude

$$(\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n) \in \Re(\hat{G}_1 \oplus \cdots \oplus \hat{G}_n).$$

Finally, since (by Theorem 3.3)

$$\tau(\mathfrak{M}_D) = (X_1^c \times \cdots \times X_n^c) \cup (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n),$$

and since

$$X_1^c \times \cdots \times X_n^c \in \Re(\hat{G}_1) \times \cdots \times \Re(\hat{G}_n) \subseteq \Re(\hat{G}_1 \oplus \cdots \oplus \hat{G}_n),$$

we obtain that $\tau(\mathfrak{M}_D)$ is a closed subset in the coset ring of $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$.

Definition 5.2. A nonzero commutative Banach algebra is *strongly semisimple* if the zero ideal is the intersection of all maximal regular ideals in the algebra. (Note that the existence of at least one maximal regular ideal is demanded.) We define the zero Banach algebra to be strongly semisimple.

Theorem 5.2. *The commutative Banach algebras*

$$L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n), \quad (L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n))_e,$$

are strongly semisimple in all instances of $L^1(K)$-algebra actions on $L^1(G_1)$, $\ldots$, $L^1(G_n)$ for LCA groups $G_1, \cdots, G_n$, and K.

Proof. It is a consequence of the work of Grothendieck [7] and Johnson [11] (cf. Gelbaum [2, Remark 3, p. 304]) that there is an isometric isomorphism

$$\Psi\colon\ L^1(G_1) \otimes_\gamma \cdots \otimes_\gamma L^1(G_n) \cong L^1(G_1 \oplus \cdots \oplus G_n)$$

which carries $a_1 \otimes \cdots \otimes a_n$ into $a_1(x_1) \cdots a_n(x_n)$ where $G_1 \oplus \cdots \oplus G_n$ is given the product Haar measure $dx_1 \otimes \cdots \otimes dx_n$. Moreover, if the structure spaces of the tensor algebra $L^1(G_1) \otimes_\gamma \cdots \otimes_\gamma L^1(G_n)$ and the group algebra $L^1(G_1 \oplus \cdots \oplus G_n)$ are identified with $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$ (the reason for the first following from [2] and the second from $(G_1 \oplus \cdots \oplus G_n)^\wedge \cong (\hat{G}_1 \oplus \cdots \oplus \hat{G}_n)$), the induced adjoint map $\hat{\Psi}$ between the structure spaces of the respective algebras is simply the identity map on $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$.

Let J be the closed ideal in $L^1(G_1) \otimes_\gamma \cdots \otimes_\gamma L^1(G_n)$ such that

$$D = L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n) = L^1(G) \otimes_\gamma \cdots \otimes_\gamma L^1(G_n)/J.$$

Let $J^1 \equiv \Psi(J)$, a closed ideal in $L^1(G_1 \oplus \cdots \oplus G_n)$. With J^1 as defined, Ψ induces an isometric isomorphism

$$L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n) \cong L^1(G_1 \oplus \cdots \oplus G_n)/J^1$$

which carries $a_1 \otimes \cdots \otimes a_n$ to $a_1(x_1) \cdots a_n(x_n)/J^1$. Moreover, by investigating the proof of Theorem 3.3 and using the fact that $\hat{\Psi}$ is the identity map on $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$ it is easily seen that

$$\mathrm{h}(J^1) = \mathrm{hull}(J^1) = (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n^0) \subseteq \hat{G}_1 \oplus \cdots \oplus \hat{G}_n,$$

where $\Delta_n^0 = \mathrm{diag}(\hat{K}^0 \times \cdots \times \hat{K}^0)$, i.e., $\mathrm{h}(J^1) = \tau(\mathfrak{M}_D)$.

Now, Gilbert [6, Theorem 3.9] (also Schreiber [22, Theorem 2.6 and Remark 2.8]) has shown that closed subsets in the coset ring of the dual group of a LCA group are sets of spectral synthesis. Consequently, by Theorem 5.1, $\mathrm{h}(J^1) = \tau(\mathfrak{M}_D)$ is a set of spectral synthesis in $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$. But this implies the strong semisimplicity of $L^1(G_1 \oplus \cdots \oplus G_n)/J^1$, or equivalently, that of D. Indeed, first note that by Wiener's theorem if J^1 is proper, then $\tau(\mathfrak{M}_D) = \mathrm{h}(J^1) \neq \varnothing$; thus if $D \neq \{0\}$, then $\mathfrak{M}_D \neq \varnothing$. Now, it is an easily proven general fact that if A is a commutative Banach algebra and I is a closed proper ideal in A with $\mathrm{h}(I) \neq \varnothing$, then the strong semisimplicity of A/I is equivalent to the spectral synthesis of the reducing ideal I, i.e., the spectral synthesis of $\mathrm{h}(I)$ in $\mathfrak{M}_A$.

Finally, the essential part of D, D_e, is strongly semisimple since D_e is isomorphic to the strongly semisimple tensor algebra

$$L^1(K) \otimes_{L^1(K)} L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n)$$

by Corollary 1.13.

Remark 5.3. In the work of Gelbaum [4], [5], and Natzitz [14] the strong semisimplicity of the tensor algebra was the crucial ingredient in their characterizations of tensor products of group algebras. Corollary 5.2 generalizes their work and in fact the important point here is that the tensor product of group algebras over a group algebra is always strongly semisimple in *all instances* of algebra actions. On the other hand, as noted in [13] it is possible for the C-tensor product of Banach C-algebra to be strongly semisimple and yet the factors themselves may not be strongly semisimple.

Corollary 5.4. *Suppose $L^1(G_1), \cdots, L^1(G_n)$ are $L^1(K)$-algebras and μ_i: $\hat{G}_i \to \hat{K}^0$, $i = 1, \cdots, n$, are the respective adjoint maps. Let D be the $L^1(K)$-tensor product of $L^1(G_1), \cdots, L^1(G_n)$. Then*

(i) $D = \{0\}$ *if and only if some* $L^1(G_i)$ *is an essential* $L^1(K)$*-algebra and* $S \equiv \mu_1(\hat{G}_1) \cap \cdots \cap \mu_n(\hat{G}_n) \cap \hat{K} = \varnothing$.

(ii) $D_e = \{0\}$ *if and only if* $S = \varnothing$.

(iii) $D = D_e$ *if and only if some* $L^1(G_i)$ *is an essential* $L^1(K)$*-algebra.*

Proof. In the notation of the proof of Theorem 5.1,

$$\tau(\mathfrak{M}_D) = X_1^c \times \cdots \times X_n^c \cup (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n).$$

From the proof of Theorem 5.2, $D = \{0\}$ if and only if $\tau(\mathfrak{M}_D) = \varnothing$, and this happens iff $X_i^c = \varnothing$ for some i and $(\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n) = \varnothing$. But $X_i^c = \mathbf{h}(L^1(G_i)_e)$, and since $L^1(G_i)_e$ is a closed ideal in $L^1(G_i)$, by Wiener's theorem $X_i^c = \varnothing$ if and only if $L^1(G_i)$ is an essential $L^1(K)$-algebra. Part (iii) follows from the fact, $\tau(\text{hull}\,(D_e)) = X_1^c \times \cdots \times X_n^c$.

6. **Tensor products of group algebras. Tensor products of K-groups.** In this subsection we briefly discuss the K-tensor product of LCA K-groups. We caution the reader that the K-tensor product of groups is not related in any fashion to the tensor product of groups considered as modules over the integers; on the other hand it is more closely related to direct sums of groups with amalgamation. As usual, K will denote an arbitrary but fixed LCA group.

Definition 6.1. Let $G_1, \cdots, G_n$ be LCA K-groups. A mapping ψ of $G_1 \times \cdots \times G_n$ into a locally compact group $\mathfrak{H}$ is *K-balanced* if for each $k \in K$ and $(x_1, \cdots, x_n) \in G_1 \times \cdots \times G_n$, and all $i, j = 1, \cdots, n$,

$$\psi(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) = \psi(x_1, \cdots, x_{j-1}, z_j, x_{j+1}, \cdots, x_n)$$

whenever $y_i \in k \cdot x_i$ and $z_j \in k \cdot x_j$ (recall that $k \cdot x_l$ is a coset of a (compact) subgroup in G_l).

Definition 6.2. Let $G_1, \cdots, G_n$ be LCA K-groups. A K-tensor product for $G_1, \cdots, G_n$ is a pair $(\mathfrak{G}; \zeta)$ consisting of a LCA K-group $\mathfrak{G}$ and a continuous K-balanced homomorphism ζ of $G_1 \oplus \cdots \oplus G_n$ into $\mathfrak{G}$ satisfying the following universal property. If η is any continuous K-balanced homomorphism of $G_1 \oplus \cdots \oplus G_n$ into a locally compact group $\mathfrak{H}$, there is a unique continuous homomorphism ξ: $\mathfrak{G} \to \mathfrak{H}$ such that $\xi \circ \zeta = \eta$.

We now show how to construct the K-tensor product of LCA K-groups G_1,

$\cdots, G_n$. Let Q denote the closed subgroup generated in the LCA group $G_1 \oplus \cdots \oplus G_n$ by the set P of all elements $(x_1, \cdots, x_n)$ such that for some $i, j \in \{1, \cdots, n\}$, $i \neq j$, and some $k \in K$, $x_i \in k \cdot 0_i$, $x_j \in (-k) \cdot 0_j$ and $x_l \in$ identity coset in $(G_l)_e$, $l \neq i, j$. We define

$$G_1 \otimes_K \cdots \otimes_K G_n = G_1 \oplus \cdots \oplus G_n/Q,$$

a LCA group and quotient group of $G_1 \oplus \cdots \oplus G_n$. π_Q is a continuous K-balanced homomorphism of $G_1 \oplus \cdots \oplus G_n$ onto $G_1 \otimes_K \cdots \otimes_K G_n$. We will write $x_1 \otimes \cdots \otimes x_n$ for the (generic) element $\pi_Q(x_1, \cdots, x_n)$ in $G_1 \otimes_K \cdots \otimes_K G_n$.

Theorem 6.3. *$(G_1 \otimes_K \cdots \otimes_K G_n; \pi_Q)$ is a K-tensor product for the LCA K-groups $G_1, \cdots, G_n$.*

Proof. If η is a continuous K-balanced homomorphism of $G_1 \oplus \cdots \oplus G_n$ into a locally compact group $\mathfrak{H}$, it can be shown that P is contained in the kernel of η and hence $Q \subseteq \ker(\eta)$. Then η factors through the quotient $G_1 \otimes_K \cdots \otimes_K G_n = G_1 \oplus \cdots \oplus G_n/Q$ to yield a unique continuous homomorphism ξ of $G_1 \otimes_K \cdots \otimes_K G_n$ into $\mathfrak{H}$ such that $\xi \circ \pi_Q = \eta$. Thus, $(G_1 \otimes_K \cdots \otimes_K G_n; \pi_Q)$ is a K-tensor product of $G_1, \cdots, G_n$. The K-group action on $G_1 \otimes_K \cdots \otimes_K G_n$ is to be defined later.

Lemma 6.4. *If $G_1, \cdots, G_n$ are LCA K-groups and $(G_1)_e, \cdots, (G_n)_e$ are their essential parts, there is a topological isomorphism*

$$G_1 \otimes_K \cdots \otimes_K G_n \cong (G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$$

which carries $x_1 \otimes \cdots \otimes x_n$ to $\pi_1(x_1) \otimes \cdots \otimes \pi_n(x_n)$ where $\pi_i: G_i \to (G_i)_e$ is the canonical projection.

Proof. Let Q_e denote the closed subgroup of $(G_1)_e \oplus \cdots \oplus (G_n)_e$ so that $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$ is the quotient group $(G_1)_e \oplus \cdots \oplus (G_n)_e/Q_e$. Since $\pi_{Q_e} \circ (\pi_1 \oplus \cdots \oplus \pi_n)$ is a continuous K-balanced homomorphism of $G_1 \oplus \cdots \oplus G_n$ onto $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$, there is a unique continuous homomorphism ι of $G_1 \otimes_K \cdots \otimes_K G_n$ onto $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$. Since $\pi_{Q_e} \circ (\pi_1 \oplus \cdots \oplus \pi_n)$ is an open map, ι is an open map. Finally, ι is an isomorphism since it is immediate from the definitions of Q and Q_e that $\pi_1 \oplus \cdots \oplus \pi_n(Q) = Q_e$. The proof is complete.

Now, $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$ is naturally a K-group under the action

$$k \cdot (y_1 \otimes \cdots \otimes y_n) = (k \cdot y_1) \otimes y_2 \otimes \cdots \otimes y_n$$

for $k \in K$ and $y_i \in (G_i)_e$, $i = 1, \cdots, n$. Moreover, by the construction of $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$,

$$k \cdot (y_1 \otimes \cdots \otimes y_n) = y_1 \otimes \cdots \otimes y_{i-1} \otimes k \cdot y_i \otimes y_{i+1} \otimes \cdots \otimes y_n$$

for all $i = 1, \cdots, n$. We now give $G_1 \otimes_K \cdots \otimes_K G_n$ the K-group structure induced by the K-group structure on $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e$ and the isomorphism in Lemma 6.4.

We are now prepared to state the main theorem in this section. The result here constitutes an extension of the work of Gelbaum [4], [5], Grothendieck [7], and Natzitz [14].

Theorem 6.5. *Let $G_1, \cdots, G_n$ and K be locally compact Abelian groups. If $L^1(G_i)$ is an isometric Banach $L^1(K)$-algebra, $i = 1, \cdots, n$, let D denote the commutative Banach tensor algebra*

$$L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n).$$

Let D_e denote the closed $L^1(K)$-ideal and essential part of D. Then there is a unique closed ideal N in D complementing D_e so that $D = N \oplus D_e$, and furthermore, N and D_e satisfy the following.

(i) *If $D_e \neq \{0\}$, there are K-group actions on each G_i, $i = 1, \cdots, n$, and $\gamma \in \hat{K}$, such that*

$$(L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n))_e \cong L^1(G_1 \otimes_K \cdots \otimes_K G_n)$$

where the isomorphism is isometric and algebraic, and it is $L^1(K)$-homogeneous when the $L^1(K)$-algebra action on $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$ is the natural action induced by the action of K on $G_1 \otimes_K \cdots \otimes_K G_n$ and $\gamma \in \hat{K}$.

(ii) *If $N \neq \{0\}$, there are closed ideals $I_i \subseteq L^1(G_i)$ and characters $\alpha_i \in G_i$, and a natural bicontinuous algebra isomorphism $I_1 \otimes_\gamma \cdots \otimes_\gamma I_n \cong N$ which carries $a_1 \otimes \cdots \otimes a_n$ to $(\alpha_1 a_1) \otimes \cdots \otimes (\alpha_n a_n)$ such that $I_i = I_{g_i} = \operatorname{Ker} T_{g_i}$ for a nontrivial compact subgroup $g_i \subseteq G_i$ if $L^1(G_i)$ is a nondegenerate $L^1(K)$-algebra, and $I_i = L^1(G_i)$ otherwise.*

The proof of Theorem 6.5 will follow a sequence of lemmas, the first of which is evidently a special case of Theorem 6.5; the remaining lemmas are general facts about tensor products of Banach algebras.

Theorem 6.6. *Suppose $G_1, \cdots, G_n$ are essential LCA K-groups, and suppose $\gamma \in \hat{K}$. If we regard $L^1(G_1), \cdots, L^1(G_n)$ and $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$ as the $L^1(K)$-algebras under the natural actions induced by that of K on the underlying groups and $\gamma \in \hat{K}$, then there is an isometric $L^1(K)$-isomorphism*

$$L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n) \cong L^1(G_1 \otimes_K \cdots \otimes_K G_n)$$

which carries $a_1 \otimes \cdots \otimes a_n$ *to* $T_Q[a_1(x_1) \cdots a_n(x_n)]$ *where* Q *is the reducing subgroup in* $G_1 \oplus \cdots \oplus G_n$ *yielding as quotient* $G_1 \otimes_K \cdots \otimes_K G_n$.

Proof. It has been observed in the proof of Theorem 5.2 that there is a closed ideal J^1 in $L^1(G_1 \oplus \cdots \oplus G_n)$ and an isometric algebra isomorphism

$$D = L^1(G_1) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n) \cong L^1(G_1 \oplus \cdots \oplus G_n)/J^1$$

which carries $a_1 \otimes \cdots \otimes a_n$ to $a_1(x_1) \cdots a_n(x_n)/J^1$. The crux of the proof is to show that J^1 is the kernel, I_Q, of the canonical map T_Q of $L^1(G_1 \oplus \cdots \oplus G_n)$ onto $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$; for indeed, it is well known [16, p. 69] that T_Q induces an isometric algebra isomorphism of $L^1(G_1 \oplus \cdots \oplus G_n)/I_Q$ onto $L^1(G_1 \oplus \cdots \oplus G_n/Q)$. Of course, the $L^1(K)$-homogeneity of the isomorphism will remain to be shown.

Now, recall (in the proof of Theorem 5.2) that

$$\mathrm{hull}(J^1) = (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n^0) \subseteq \hat{G}_1 \oplus \cdots \oplus \hat{G}_n,$$

where $\mu_i\colon \hat{G}_i \longrightarrow \hat{K}^0$ is the adjoint map of the action of $L^1(K)$ on $L^1(G_i)$ and $\Delta_n^0 = \mathrm{diag}\,(\hat{K}^0 \times \cdots \times \hat{K}^0)$. Since $\mathrm{h}(J^1)$ is a set of spectral synthesis in $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$ (as noted in the proof of Theorem 5.2), to show $J^1 = I_Q$ it suffices to prove $\mathrm{h}(J^1) = \mathrm{h}(I_Q)$. To this end we show that this is indeed the case.

Let $\theta_i\colon K \longrightarrow G_i$ denote the continuous homomorphism inducing the (essential) action of K on G_i. Now, for each $c \in L^1(K)$, $a \in L^1(G_i)$, and $\alpha \in \hat{G}_i$,

$$\begin{aligned}[c \cdot_{\gamma,\theta_i} a]\hat{\ }(\alpha) &= \int_{G_i} \int_K \gamma(k)\, c(k)\, a(x - \theta_i(k))\, \overline{(x, \alpha)}\, dk\, dx \\ &= \int_K \left(\int_{G_i} a(x) \overline{(x + \theta_i(k), \alpha)}\, dx \right) c(k) \overline{(k, -\gamma)}\, dk \\ &= \int_K c(k) \overline{(k, \hat{\theta}_i(\alpha) - \gamma)}\, dk\, \hat{a}(\alpha) = \hat{c}(\hat{\theta}_i(\alpha) - \gamma)\, \hat{a}(\alpha),\end{aligned}$$

where $\hat{\theta}_i\colon \hat{G}_i \longrightarrow \hat{K}$ denotes the dual map of θ_i. We conclude that $\mu_i(\alpha) = \hat{\theta}_i(\alpha) - \gamma$, $\alpha \in \hat{G}_i$, for each $i = 1, \cdots, n$. Consequently,

$$\begin{aligned}\mathrm{h}(J^1) &= \{(\alpha_1, \cdots, \alpha_n) \in \hat{G}_1 \oplus \cdots \oplus \hat{G}_n \mid \hat{\theta}_i(\alpha_i) - \gamma = \hat{\theta}_j(\alpha_j) - \gamma\} \\ &= (\hat{\theta}_i \times \cdots \times \hat{\theta}_n)^{-1}(\Delta_n) \equiv S\end{aligned}$$

where $\Delta_n = \mathrm{diag}\,(\hat{K} \times \cdots \times \hat{K})$. On the other hand, it is well known (e.g., [16, Chapter 4, §4.3]) that the Fourier transform of $T_Q f$, $f \in L^1(G_1 \oplus \cdots \oplus G_n)$, is the restriction of $\hat{f}$ to $Q^\perp$, the annihilator of Q, and therefore $\mathrm{h}(I_Q) = Q^\perp$. The proof that $\mathrm{h}(J^1) = \mathrm{h}(I_Q)$ is complete once we have shown

Lemma 6.7. $Q^{\perp} = (\hat{\theta}_1 \times \cdots \times \hat{\theta}_n)^{-1}(\text{diag } \hat{K} \times \cdots \times \hat{K})$.

Proof. Recall that Q is the closed subgroup generated in $\hat{G}_1 \oplus \cdots \oplus \hat{G}_n$ by the set P of all elements $(x_1, \cdots, x_n)$ such that for some $i, j, i \neq j$, and some $k \in K$, $x_i = \theta_i(k)$, $y_j = -\theta_j(k)$ and x_l is the identity of G_l, $l \neq i, j$. If $(\alpha_1, \cdots, \alpha_n) \in Q^{\perp}$, then for each $k \in K$ and $i, j = 1, \cdots, n$ (by an astute choice of $(x_1, \cdots, x_n) \in P$),

$$1 = (\alpha_1 \oplus \cdots \oplus \alpha_n)(x_1, \cdots, x_n) = \alpha_i(\theta_i(k))\alpha_j(-\theta_j(k)) = \hat{\theta}_i(\alpha_i)(k)\hat{\theta}_j(\alpha_j)(k).$$

Thus, $\hat{\theta}_i(\alpha_i) = \hat{\theta}_j(\alpha_j)$ for each $i, j = 1, \cdots, n$, i.e., $(\alpha_1, \cdots, \alpha_n) \in S$, and hence $Q^{\perp} \subseteq S$. Conversely, if $(\alpha_1, \cdots, \alpha_n) \in S$, then $\hat{\theta}_i(\alpha_i) = \hat{\theta}_j(\alpha_j)$ for each $i, j = 1, \cdots, n$. Reversing the steps above we see that $\alpha_1 \oplus \cdots \oplus \alpha_n$ is identically 1 on P. Since $\alpha_1 \oplus \cdots \oplus \alpha_n$ is a continuous homomorphism and P generates Q, $\alpha_1 \oplus \cdots \oplus \alpha_n$ is identically one on Q. Thus, $(\alpha_1, \cdots, \alpha_n) \in Q^{\perp}$, and $S \subseteq Q^{\perp}$. The lemma is proved.

Returning to the proof of Theorem 6.6, we must show that the isomorphism Λ of D onto $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$ is $L^1(K)$-homogeneous. Suppose $a_1(x_1), \cdots, a_n(x_n)$, and $c(k)$ are continuous functions with compact supports on $G_1, \cdots, G_n$, and K, respectively. For all $(x_1, \cdots, x_n)/Q$ in $G_1 \otimes_K \cdots \otimes_K G_n$,

$$\begin{aligned}
&\Lambda(c \cdot (a_1 \otimes \cdots \otimes a_n))((x_1, \cdots, x_n)/Q) \\
&\quad = \Lambda(c *_{\gamma,\theta_1} a_1 \otimes a_2 \otimes \cdots \otimes a_n)((x_1, \cdots, x_n)/Q) \\
&\quad = \int_Q \int_K \gamma(k) c(k) a_1(p_1 + x - \theta_1(k)) a_2(p_2 + x_2) \cdots a_n(p_n + x_n)\, dk\, dq
\end{aligned}$$

(where $q = (p_1, \cdots, p_n)$ denotes a generic element of Q)

$$\begin{aligned}
&\quad = \int_K \gamma(k) c(k) T_Q(a_1(\cdot) \cdots a_n(\cdot))((x_1 - \theta_1(k), x_2, \cdots, x_n)/Q)\, dk \\
&\quad = (c \cdot_{\gamma,K} \Lambda(a_1 \otimes \cdots \otimes a_n))((x_1, \cdots, x_n)/Q).
\end{aligned}$$

Thus, Λ commutes with the action of $L^1(K)$ on elementary tensors in which the factors are continuous functions with compact support. Since the linear span of such elementary tensors is dense in D, the module actions are continuous and Λ is linear and continuous, it follows the Λ is $L^1(K)$-homogeneous on D into $L^1(G_1 \otimes_K \cdots \otimes_K G_n)$.

Remark 6.8. Although it is not needed in the proof of Theorem 6.5, we remark that the conclusion of Theorem 6.6 remains true (verbatim) if we merely assume that some one of the LCA K-groups $G_1, \cdots, G_n$ is essential.

Definition 6.9. If A is a commutative Banach C-algebra, a map $P: A \to A$ is a *projection on* A if P is a continuous C-algebra homomorphism satisfying $P^2 = P$. The *range of* P, denoted $\text{Rng}(P)$, is the set $\{a \in A \mid P(a) = a\}$.

Remark 6.10. It is clear that $\operatorname{Rng}(P)$ and $\operatorname{Ker}(P)$ are closed C-ideals in A and $A = \operatorname{Ker}(P) \oplus \operatorname{Rng}(P)$, where $a = [a - P(a)] + Pa$ is the direct sum decomposition of $a \in A$.

Lemma 6.11. *If P_i is a projection on the commutative Banach C-algebra A_i, $i = 1, \cdots, n$, then the tensor product of $P_1, \cdots, P_n$, $P_1 \otimes \cdots \otimes P_n$, is a projection on $A_1 \otimes_C \cdots \otimes_C A_n$ and there obtains the natural bicontinuous C-algebra isomorphism*

$$\operatorname{Rng}(P_1) \otimes_C \cdots \otimes_C \operatorname{Rng}(P_n) \cong \operatorname{Rng}(P_1 \otimes \cdots \otimes P_n).$$

If each P_i is of norm one, then the isomorphism is an isometry.

Proof. The proof is routine and omitted.

Lemma 6.12. *Suppose C has a bounded approximate identity. If A_i is a commutative Banach C-algebra and if $(A_i)_e$ is the range of a projection P_i in A for each $i = 1, \cdots, n$, then there obtains a natural bicontinuous C-algebra isomorphism*

$$(A_1)_e \otimes_C \cdots \otimes_C (A_n)_e \simeq (A_1 \otimes_C \cdots \otimes_C A_n)_e,$$

and the isomorphism is an isometry if each P_i is of norm one. Furthermore, there is a natural bicontinuous isomorphism

$$\operatorname{Ker} P_1 \otimes_\gamma \cdots \otimes_\gamma \operatorname{Ker} P_n \cong \operatorname{Ker}(P_1 \otimes \cdots \otimes P_n).$$

Proof. By Lemma 6.11 there obtains the natural bicontinuous C-algebra isomorphism of $(A_1)_e \otimes_C \cdots \otimes_C (A_n)_e$ onto the closed ideal $\operatorname{Rng}(P_1 \otimes \cdots \otimes P_n) \equiv I$. Therefore, we need only show $I = D_e$ where $D = A_1 \otimes_C \cdots \otimes_C A_n$. Now, since I is an essential Banach C-subalgebra of D, $I \subseteq D_e$ by Proposition 1.3. On the other hand if $z \in D_e$, then by Hewitt's factorization theorem, $z = c \cdot z'$ for $c \in C$, $z' \in D_e$. Let

$$z' = \sum_{m=1}^{\infty} a_{1,m} \otimes \cdots \otimes a_{n,m}, \qquad \sum_{n=1}^{\infty} \|a_{1,m}\| \cdots \|a_{n,m}\| < \infty.$$

Since in general $c \cdot a_i = c \cdot P_i a_i$, we have

$$z = c \cdot z' = \sum_{m=1}^{\infty} c \cdot (a_{1,m} \otimes \cdots \otimes a_{n,m}) = \sum_{m=1}^{\infty} (c \cdot a_{1,m}) \otimes \cdots \otimes a_{n,m}$$

$$= \sum_{m=1}^{\infty} P_1 a_{1,m} \otimes (c \cdot a_{2,m}) \otimes \cdots \otimes a_{n,m}$$

$$= \cdots = \sum_{m=1}^{\infty} c \cdot (P_1 a_{1,m} \otimes \cdots \otimes P_n a_{n,m})$$

$$= c \cdot (P_1 \otimes \cdots \otimes P_n)(z') = P_1 \otimes \cdots \otimes P_n(z).$$

Thus, $D_e \subseteq I$, and hence $I = D_e$, and the first isomorphism is proven.

To prove the second isomorphism in Lemma 6.12, note that, since $C \cdot \operatorname{Ker}(P_i) = \{0\}$ for each i,

$$\operatorname{Ker}(P_1) \otimes_C \cdots \otimes_C \operatorname{Ker}(P_n) = \operatorname{Ker}(P_1) \otimes_\gamma \cdots \otimes_\gamma \operatorname{Ker}(P_2).$$

Now, since $1_{A_i} - P_i$ is a projection on A_i and $\operatorname{Rng}(1_{A_i} - P_i) = \operatorname{Ker}(P_i)$, there is by Lemma 6.11 a bicontinuous isomorphism

$$\operatorname{Ker}(P_i) \otimes_\gamma \cdots \otimes_\gamma \operatorname{Ker}(P_n) \cong \operatorname{Rng}((1_{A_1} - P_1) \otimes \cdots \otimes (1_{A_n} - P_n)).$$

To complete the proof it suffices to show

$$1_D - (P_1 \otimes \cdots \otimes P_n) = (1_{A_1} - P_1) \otimes \cdots \otimes (1_{A_n} - P_n).$$

First observe that

$$(1_{A_1} - P_1) \otimes \cdots \otimes (1_{A_n} - P_n) = \sum_{\chi \in \{0,1\}^{\mathbf{N}_n}} (-1)^{\sigma(\chi)} P_1^{\chi(1)} \otimes \cdots \otimes P_n^{\chi(n)},$$

where $\{0, 1\}^{\mathbf{N}_n}$ is the set of all 0, 1-valued functions on $\mathbf{N}_n = \{1, 2, \cdots, n\}$, where

$$\sigma(\chi) = \sum_{i=1}^{n} \chi(i) \qquad (\chi \in \{0, 1\}^{\mathbf{N}_n}),$$

and where $P_i^0 \equiv 1_{A_i}$. Now, we assert that if $\chi \neq 0$, then $P_1^{\chi(1)} \otimes \cdots \otimes P_n^{\chi(n)} = P_1 \otimes \cdots \otimes P_n$. Indeed, suppose $\chi(i_0) \neq 0$. If $a_i \in A_i$, let $P_{i_0}(a_{i_0}) = c \cdot P_{i_0}(a'_{i_0})$ by Hewitt's factorization theorem. Using the general fact that $c \cdot a_i = c \cdot P_i(a_i)$ and

$$(P_1^{\chi(1)} \otimes \cdots \otimes P_n^{\chi(n)})(a_1 \otimes \cdots \otimes a_n)$$

$$= P_1^{\chi(1)}(a_1) \otimes \cdots \otimes c \cdot P_{i_0}(a'_{i_0}) \otimes \cdots \otimes P_n^{\chi(n)}(a_n),$$

one obtains by distributing c through the tensor that

$$P_1^{\chi(1)} \otimes \cdots \otimes P_n^{\chi(n)}(a_1 \otimes \cdots \otimes a_n) = P_1 \otimes \cdots \otimes P_n(a_1 \otimes \cdots \otimes a_n).$$

Since $P_1^{\chi(1)} \otimes \cdots \otimes P_n^{\chi(n)}$ and $P_1 \otimes \cdots \otimes P_n$ agree on the elementary tensors, they are equal. It follows that

$$(1_{A_1} - P_1) \otimes \cdots \otimes (1_{A_n} - P_n) = 1_D + \sum_{\chi \neq 0} (-1)^{\sigma(\chi)} P_1 \otimes \cdots \otimes P_n.$$

Since $\Sigma_{\chi \neq 0}(-1)^{\sigma(\chi)} = -1$, the proof is completed.

Remark 6.13. We wish to elaborate slightly on the above lemma. It equivalently states that if there is a closed ideal N_i in A_i such that $A_i = N_i \oplus (A_i)_e$ for each $i = 1, \cdots, n$, then there is a closed ideal N in D such that $D = N \oplus D_e$ and moreover the natural bicontinuous C-algebra isomorphisms obtain

$$N_1 \otimes_\gamma \cdots \otimes_\gamma N_n \cong N, \qquad (A_1)_e \otimes_C \cdots \otimes_C (A_n)_e \cong D_e.$$

Proof of Theorem 6.5. By Lemma 4.9 there is for each $i = 1, \cdots, n$, a closed ideal J_i in $L^1(G_i)$ complementing $L^1(G_i)_e$. Therefore, by Lemma 6.12 and Remark 6.13 there is a closed ideal N in D such that $D = N \oplus D_e$ and the natural isometric C-algebra isomorphism

$$L^1(G_1)_e \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n)_e \cong D_e$$

and bicontinuous algebra isomorphism $J_1 \otimes_\gamma \cdots \otimes_\gamma J_n \cong N$ obtain. Let μ_i: $\hat{G}_i \to \hat{K}^0$ denote the adjoint map.

If $D_e \neq \{0\}$, then by Corollary 5.4(ii), $S = (\mu_1 \times \cdots \times \mu_n)^{-1}(\Delta_n) \neq \emptyset$, where $\Delta_n = \operatorname{diag}(\hat{K} \times \cdots \times \hat{K})$. Let $(\alpha_1, \cdots, \alpha_n) \in S$ and set $\gamma = -\mu_1(\alpha_1) = \cdots = -\mu_n(\alpha_n) \in \hat{K}$. Now, by Lemma 4.9(ii) there is a compact subgroup g_i in G_i and a K-group action on G_i with $G_e = G_i/g_i$ such that $\alpha_i \pi^*_{g_i}$ is an isometric $L^1(K)$-algebra isomorphism of $L^1((G_i)_e)$ onto $(L^1(G_i))_e$ where $L^1((G_i)_e)$ is the $L^1(K)$-algebra whose action is induced by the essential K-group action on $(G_i)_e$ and $\gamma \in \hat{K}$ (and where the Haar measures are appropriately related). It follows that $\alpha_1 \pi^*_{g_1} \otimes \cdots \otimes \alpha_n \pi^*_{g_n}$ defines an isometric $L^1(K)$-algebra isomorphism

$$L^1((G_1)_e) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1((G_n)_e) \cong L^1(G_1)_e \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1(G_n)_e.$$

Now, let ι denote the topological isomorphism of $G_1 \otimes_K \cdots \otimes_K G_n \equiv \mathfrak{G}$ onto $(G_1)_e \otimes_K \cdots \otimes_K (G_n)_e \equiv \mathfrak{H}$ defined in Lemma 6.4. Given a Haar measure ds on $\mathfrak{G}$ choose a Haar measure dt on $\mathfrak{H}$ such that

$$\int_{\mathfrak{H}} f(t)\, dt = \int_{\mathfrak{G}} f(\iota(s))\, ds$$

for all $f \in C_{00}(\mathfrak{H})$; then the map $a \to a \circ \iota$ defines an isometric isomorphism ι^* of $L^1(\mathfrak{H})$ onto $L^1(\mathfrak{G})$ and it is $L^1(K)$-homogeneous when $L^1(\mathfrak{H})$ and $L^1(\mathfrak{G})$ are the $L^1(K)$-algebras whose actions are induced by the actions of K on $\mathfrak{H}$ and $\mathfrak{G}$, and $\gamma \in \hat{K}$, respectively. With the product Haar measure on $(G_1)_e \oplus \cdots \oplus (G_n)_e$ choose a Haar measure on Q_e (the reducing subgroup in $(G_1)_e \oplus \cdots \oplus (G_n)_e$ yielding as quotient $\mathfrak{H}$) so that the Haar measures of $(G_1)_e \oplus \cdots \oplus (G_n)_e$, Q_e, and $\mathfrak{H}$ are canonically related.

By Lemma 6.6 there is an isometric $L^1(K)$-algebra isomorphism

$$L^1((G_1)_e) \otimes_{L^1(K)} \cdots \otimes_{L^1(K)} L^1((G_n)_e) \cong L^1(\mathfrak{H}).$$

Together with the isomorphism of $L^1(\mathfrak{H})$ onto $L^1(\mathfrak{G})$, we yield an isometric $L^1(K)$-isomorphism of D_e onto $L^1(\mathfrak{G})$ and hence (i) in Theorem 6.5 is proven.

Remark 6.14. As is apparent to the reader, considerable care was taken in the choice of Haar measures. The purpose was of course to assure that the isomorphism T of D_e onto $L^1(\mathfrak{G})$ is in fact an *isometry*. In fact, by carefully retracing the intermediary isomorphisms one can show that the value of T on the elementary tensors in D_e is given as follows. If $a_i \in L^1(G_i)_e$, $i = 1, \cdots, n$, then

$$T(a_1 \otimes \cdots \otimes a_n)(s) = T_{Q_e}[T_{g_1}(\overline{\alpha}_1 a_1) \cdots T_{g_n}(\overline{\alpha}_n a_n)](\iota(s)) \qquad (s \in \mathfrak{G}),$$

where $T_{g_1}(\overline{\alpha}_1 a_1) \cdots T_{g_n}(\overline{\alpha}_n a_n)$ denotes the element f in $L^1((G_1)_e \oplus \cdots \oplus (G_n)_e)$ given by

$$f(y_1, \cdots, y_n) = T_{g_1}(\overline{\alpha}_1 a_1)(y_1) \cdots T_{g_n}(\overline{\alpha}_n a_n)(y_n)$$

for $(y_1, \cdots, y_n) \in (G_1)_e \oplus \cdots \oplus (G_n)_e$. On the other hand, if Q is the reducing subgroup in $G_1 \oplus \cdots \oplus G_n$ yielding the quotient $\mathfrak{G}$, and if the Haar measure on Q is chosen so that the Haar measures on $G_1 \oplus \cdots \oplus G_n$, Q, and $\mathfrak{G}$ are canonically related, then it can be shown that the isometric $L^1(K)$-algebra isomorphism T of D_e onto $L^1(\mathfrak{G})$ is that which carries $a_1 \otimes \cdots \otimes a_n$, $a_i \in L^1(G_i)_e$, to $T_Q[(\overline{\alpha}_1 a_1) \cdots (\overline{\alpha}_n a_n)]$.

Returning to the proof of Theorem 6.5, suppose $N \neq \{0\}$. Since $D \neq D_e$, each $L^1(G_i)$ is by Corollary 6.5 a nonessential $L^1(K)$-algebra, and hence by Lemma 4.9 if $L^1(G_i)$ is a nondegenerate $L^1(K)$-algebra, J_i equals $\alpha_i I_{g_i}$ for some compact subgroup $g_i \subseteq G_i$ and $\alpha_i \in \hat{G}_i$, and $J_i = L^1(G_i)$ otherwise (we set $\alpha_i = 1$ in this case). Let $I_i = \overline{\alpha}_i J_i$, $i = 1, \cdots, n$. Since multiplication by α_i is an isometric isomorphism of I_i onto J_i, there is a bicontinuous isomorphism $I_1 \otimes_\gamma \cdots \otimes_\gamma I_n \cong N$ which carries $a_1 \otimes \cdots \otimes a_n$ to $(\alpha_1 a_1) \otimes \cdots \otimes (\alpha_n a_n)$. The proof is completed.

7. **Generalized multipliers of L^1 into L^∞.** If A and B are (left and right) Banach C-modules for a Banach algebra C, then as shown by Rieffel [18, Corollary 3.21] there is an isometric isomorphism

$$(A \otimes_C B)^* \cong \mathrm{Hom}_C(A, B^*)$$

such that if $\phi \to T_\phi$ then

$$\phi(a \otimes b) = \langle b, T_\phi(a) \rangle \qquad (a \in A,\ b \in B),$$

when B^* is considered to be the left Banach C-module under the adjoint action induced by that of C on B. If $A = L^1(G)$, $B = L^1(H)$, and $C = L^1(K)$ for LCA groups G, H, and K, and if the module actions are isometric algebra actions, then Theorem 6.5 permits an analysis of

$$\operatorname{Hom}_{L^1(K)}(L^1(G), L^\infty(H)) \equiv M.$$

When $G = H = K$ and the module action is convolution, we obtain the special and well-known case of characterizing the multipliers of $L^1(G)$ into $L^\infty(G)$. Hence we may consider the characterization of M as a generalized multiplier problem.

Since $L^1(G) \otimes_{L^1(K)} L^1(H)$ is described in general as direct sum of its essential part and another closed ideal (Theorem 6.5), we note a few simple facts about Banach spaces expressed as internal direct sums.

If B is a Banach space and $V \subseteq B$, the *annihilator*, $V^\perp$, of V is the closed subspace in the Banach dual B^* given by

$$V^\perp = \{x^* \in B^* \mid \langle x, x^* \rangle = 0 \ \forall x \in V\}.$$

Lemma 7.1. *Let B be a Banach space, and, let V and W be closed subspaces in B such that $B = V \oplus W$. Then $B^* = V^\perp \oplus W^\perp$, and if $P_V: B \to V$ and $P_W: B \to W$ denote the natural mappings of B onto V and W, respectively, then $P_V^*: V^* \to B^*$ and $P_W^*: W^* \to B^*$ effect the bicontinuous isomorphisms*

$$V^* \cong W^\perp, \qquad W^* \cong V^\perp.$$

If P_V [resp. P_W] is of norm one, then the isomorphism $V^* \cong W^\perp$ [resp. $W^* \cong V^\perp$] is an isometry.

If in general S is a closed subgroup of a LCA group G, then I_S will denote ideal in $L^1(G)$ and kernel of the canonical map $T_S: L^1(G) \to L^1(G/S)$,

$$T_S(a)(x/S) = \int_S a(x+s)\,ds, \qquad a \in L^1(G),\ x \in G.$$

Lemma 7.2. *Let G and H be LCA groups and let $g \subseteq G$ and $h \subseteq H$ be compact subgroups. The following obtain.*

(i) *There is a bicontinuous isomorphism*

$$I_g \otimes_\gamma I_h \cong I_{g\oplus\{0\}} \cap I_{\{0\}\oplus h} \subseteq L^1(G \oplus H)$$

which carries $a \otimes b$ to $a(x)b(y)$, where $g \oplus \{0\}$ and $\{0\} \oplus h$ are considered as compact subgroups in $G \oplus H$.

(ii) *There is a bicontinuous isomorphism*

$$I_g \otimes_\gamma L^1(H) \cong I_{g\oplus\{0\}} \subseteq L^1(G \oplus H)$$

which carries $a \otimes b$ to $a(x)b(y)$, and similarly $L^1(G) \otimes_\gamma I_h \cong I_{\{0\}\oplus h}$.

Proof. We prove only (i), the proof of (ii) will be evident. Since g [resp. h] is compact, there is projection P_g in $L^1(G)$ onto I_g; similarly a projection P_h in $L^1(H)$ onto I_h. By Lemma 6.11 there is a natural bicontinuous isomorphism

$$I_g \otimes_\gamma I_h \cong \operatorname{Rng}(P_g \otimes P_h) \subseteq L^1(G) \otimes_\gamma L^1(H).$$

In view of the natural isomorphism, $L^1(G) \otimes_\gamma L^1(H) \cong L^1(G \oplus H)$, there is a closed ideal I in $L^1(G \oplus H)$ and a natural bicontinuous isomorphism $I_g \otimes_\gamma I_h \cong I$ which carries $a \otimes b$ to $a(x)b(y)$ and furthermore taking hulls we have

$$\begin{aligned} \mathrm{h}(I) &= \mathrm{h}(I_g) \times \hat{H} \cup \hat{G} \times \mathrm{h}(I_h) = g^\perp \times \hat{H} \cup \hat{G} \times h^\perp \\ &= (g \oplus \{0\})^\perp \cup (\{0\} \oplus h)^\perp = \mathrm{h}(I_{g\oplus\{0\}}) \cup \mathrm{h}(I_{\{0\}\oplus h}) = \mathrm{h}(I_{g\oplus\{0\}} \cap I_{\{0\}\oplus h}). \end{aligned}$$

Since $(g^\perp \times \hat{H}) \cup (\hat{G} \times h^\perp)$ is an open and closed subset in $\hat{G} \oplus \hat{H}$, it is a set of spectral synthesis and therefore $I = I_{g\oplus\{0\}} \cap I_{\{0\}\oplus h}$.

Lemma 7.3. *Let G and H be LCA groups and let $g \subseteq G$ and $h \subseteq H$ be compact subgroups. For $\phi \in L^\infty(G \oplus H)$ consider the following two conditions*:

(I) $\int_g \phi(x + s, y)\,ds = 0$ *locally a.e.* (x, y) *in* $G \oplus H$,

(II) $\int_h \phi(x, y + t)\,dt = 0$ *locally a.e.* (x, y) *in* $G \oplus H$.

Consider the subspaces of $L^\infty(G \oplus H)$ defined by

$$I^\infty_{g\oplus h} \equiv \{\phi \in L^\infty(G \oplus H) \mid \phi \text{ satisfies (I) and (II)}\},$$

$$I^\infty_{g\oplus H} \equiv \{\phi \in L^\infty(G \oplus H) \mid \phi \text{ satisfies (I)}\},$$

$$I^\infty_{G\oplus h} \equiv \{\phi \in L^\infty(G \oplus H) \mid \phi \text{ satisfies (II)}\}.$$

Then the following bicontinuous isomorphisms obtain:

$$(I_g \otimes_\gamma I_h)^* \cong I^\infty_{g\oplus h}, \quad (I_g \otimes_\gamma L^1(H))^* \cong I^\infty_{g\oplus H}, \quad (L^1(G) \otimes_\gamma I_h)^* \cong I^\infty_{G\oplus h}.$$

Furthermore, if $\phi \leftrightarrow \phi^\infty$ in any of the above isomorphisms then

$$\phi(a \otimes b) = \int_{G\oplus H} a(x)\, b(y)\, \overline{\phi^\infty(x, y)}\, dx \otimes dy$$

for all $a \otimes b$ in the respective tensor algebras.

Proof. Let I denote the closed ideal $I_{g\oplus\{0\}} \cap I_{\{0\}\oplus h}$ in $L^1(G \oplus H)$ that is naturally isomorphic to $I_g \otimes_\gamma I_h$. It can be easily shown that I is complemented in $L^1(G \oplus H)$ by the closed ideal

$$J \equiv \pi^*_{g\oplus\{0\}}(L^1(G/g \oplus H)) + \pi^*_{\{0\}\oplus h}(L^1(G \oplus H/h)) \equiv J_g + J_h$$

where $\pi^*_{g\oplus\{0\}}$ and $\pi^*_{\{0\}\oplus h}$ denote the canonical maps (indeed, using the isomorphism $L^1(G) \otimes_\gamma L^1(H) \cong L^1(G \oplus H)$ and Lemma 6.11, we find that J is the kernel of the projection $P_g \otimes P_h$ in $L^1(G \oplus H)$ onto I). We have $L^1(G \oplus H) = I \oplus J$.

By Lemma 7.1 there is a bicontinuous isomorphism $I^* \cong J^\perp$ and hence there is a bicontinuous isomorphism

$$(I_g \otimes_\gamma I_h)^* \cong J^\perp \subseteq L^\infty(G \oplus H).$$

Furthermore, if $\phi \in (I_g \otimes_\gamma I_h)^*$ corresponds to $\phi^\infty \in J^\perp$, they are related by the formula

$$\phi(a \otimes b) = \int_{G\oplus H} a(x)\, b(y)\, \overline{\phi^\infty(x, y)}\, dx \otimes dy$$

for all $a \in I_g$, $b \in I_h$.

In order to characterize $J^\perp \subseteq L^\infty(G \oplus H)$ we need to identify the ideals $J_g^\perp$ and $J_h^\perp$. The next lemma accomplishes this.

Lemma 7.4. *If* $\mathbf{G}$ *is a LCA group and* $\mathbf{g}$ *is a compact subgroup, then*

$$\pi_{\mathbf{g}}^*(L^1(\mathbf{G}/\mathbf{g}))^\perp = \left\{ \phi \in L^\infty(\mathbf{G}) \,\middle|\, \int_{\mathbf{g}} \phi(z + r)\, dr = 0 \ l.a.e. \ z \ in \ \mathbf{G} \right\}.$$

Proof. Let $\phi \in \pi_{\mathbf{g}}^*(L^1(\mathbf{G}/\mathbf{g}))^\perp$. Then for all $a \in L^1(\mathbf{G})$

$$0 = \langle \pi_{\mathbf{g}}^* T_{\mathbf{g}}(a), \phi \rangle = \int_{\mathbf{G}} \int_{\mathbf{g}} a(z + r)\, dr\, \overline{\phi(z)}\, dz = \int_{\mathbf{g}} \int_{\mathbf{G}} a(z + r)\, \overline{\phi(z)}\, dz\, dr$$

$$= \int_{\mathbf{g}} \int_{\mathbf{G}} a(z)\, \overline{\phi(z - r)}\, dz\, dr = \int_{\mathbf{G}} a(z) \overline{\left(\int_{\mathbf{g}} \phi(z + r)\, dr \right)}\, dz,$$

i.e., $\Phi(z) \equiv \int_{\mathbf{g}} \phi(z + r)\, dr = 0$ l.a.e. z in $\mathbf{G}$. Since the steps are reversible the lemma is proved.

Now, returning to the proof of Lemma 7.3, since $J = J_g + J_h$, we have $J^\perp = J_g^\perp \cap J_h^\perp$. Therefore, by Lemma 7.4, $J^\perp$ consists of precisely those $\phi^\infty \in L^\infty(G \oplus H)$ such that, for l.a.e. (x, y) in $G \oplus H$,

$$0 = \int_{g\oplus\{0\}} \phi^\infty(x + s, y + q)\, d(s, q) = \int_g \phi^\infty(x + s, y)\, ds$$

and

$$0 = \int_{\{0\}\oplus h} \phi^\infty(x + p, y + t)\, d(p, t) = \int_h \phi^\infty(x, y + t)\, dt.$$

The proof of the remaining two isomorphisms in Lemma 7.3 can be proved in a similar fashion; however direct application of Lemma 7.4 cannot be used to show that the annihilator of the complement of the closed ideal in $L^1(G \oplus H)$ corresponding to say $I_g \oplus_\gamma L^1(H)$ is $I^\infty_{g\oplus H}$; but a technique similar to that used in the proof of Lemma 7.4 will suffice. The details are left to the reader.

If G is a LCA K-group, $\gamma \in \hat{K}$, and $\alpha \in \hat{G}$, then, as noted in Remark 4.7, $L^p(G)$ $(1 \le p \le \infty)$ becomes a (left) Banach $L^1(K)$-module under the action

$$c \cdot a = \alpha[c \cdot_{\gamma, K} \bar{\alpha} a], \qquad c \in L^1(K),\ a \in L^p(G).$$

We investigate the adjoint action of $L^1(K)$ on the dual of $L^p(G)$. If $1 \le p \le \infty$, let p' denote the conjugate exponent defined by $1/p + 1/p' = 1$, and we define the dual pairing between $L^p(G)$ and $L^{p'}(G)$ by

$$\langle a, b\rangle = \int_G a(x)\overline{b(x)}\,dx, \qquad a \in L^p(G),\ b \in L^{p'}(G).$$

A computation will show that $\langle c \cdot a, b\rangle = \langle a, \overline{c^* \cdot' b}\rangle$ for all $c \in L^1(K)$, $a \in L^p(G)$, and $b \in L^{p'}(G)$ where c^* is defined by $c^*(x) = \overline{c(-x)}$, and where $c \cdot' b = \alpha[c \cdot_{-\gamma,K} \overline{\alpha} b]$, $c \in L^1(K)$, $b \in L^{p'}(G)$. (It is of interest to compare the above relationship with the special case when $K = G$ and the module action is simply convolution.) Thus the adjoint action of an element c of $L^1(K)$ on $L^{p'}(G)$ consists of the natural $L^1(K)$-action by c^* that is induced by the action of K on G, $-\gamma \in \hat{K}$, and $\alpha \in \hat{G}$.

In the next theorem if $L^1(G)$ is an (isometric) Banach $L^1(K)$-algebra and if J is the closed ideal in $L^1(G)$ complementing $L^1(G)_e$, so that $L^1(G) = J \oplus L^1(G)_e$, then we will denote the direct summands of an $a \in L^1(G)$ by $a_n \in J$ and $a_e \in L^1(G)_e$, so that $a = a_n + a_e$.

Theorem 7.5. *If $L^1(G)$ and $L^1(H)$ are commutative Banach $L^1(K)$-algebras and $L^\infty(H)$ is considered as an $L^1(K)$-module under the adjoint action, then there are closed subspaces $\mathfrak{N}$ and $\mathfrak{E}$ of $\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H))$ such that*

$$\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H)) = \mathfrak{N} \oplus \mathfrak{E}$$

and there are the respective bicontinuous and isometric isomorphisms $\mathfrak{N} \cong N^*$, $\mathfrak{E} \cong D_e^*$, where

$$D = L^1(G) \otimes_{L^1(K)} L^1(H) = N \oplus D_e.$$

Furthermore,

(i) *If $\mathfrak{E} \neq \{0\}$, there is an isometric $L^1(K)$-module isomorphism $\mathfrak{E} \cong L^\infty(G \otimes_K H)$ such that if $\mathfrak{E} \ni T_e \leftrightarrow \phi \in L^\infty(G \otimes_K H)$ then*

$$\langle b_e, T_e a_e\rangle = \langle \Gamma(a_e \otimes b_e), \phi\rangle = \int_{G\otimes_K H} T_Q[(\overline{\alpha} a_e)(\overline{\beta} b_e)](z)\overline{\phi(z)}\,dz,$$

where $\Gamma\colon D_e \cong L^1(G \otimes_K H)$ is the isomorphism of Theorem 6.5(i).

(ii) *If $\mathfrak{N} \neq \{0\}$, there is a closed subspace I^∞ in $L^\infty(G \oplus H)$ and a bicontinuous isomorphism $\mathfrak{N} \cong I^\infty$ such that if $\mathfrak{N} \ni T_n \leftrightarrow \psi \in I^\infty$ then*

$$\langle b_n, T_n a_n\rangle = \langle (\overline{\alpha} a_n)(\overline{\beta} b_n), \psi\rangle = \int_{G\oplus H} (\overline{\alpha} a_n)(x)(\overline{\beta} b_n)(y)\,\psi(x, y)\,dx \otimes dy$$

(for fixed $\alpha \in \hat{G}$, $\beta \in \hat{H}$) where $I^\infty = I^\infty_{g\oplus h}$, $I^\infty_{g\oplus\{0\}}$, $I^\infty_{\{0\}\oplus h}$, or $L^\infty(G \oplus H)$, for compact subgroups $g \subseteq G$, $h \subseteq H$, depending on whether neither, one, or both are degenerate $L^1(K)$-algebras.

Proof. As noted in the introduction to this section there is an isometric isomorphism

$$\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H)) \cong (L^1(G) \otimes_{L^1(K)} L^1(H))^* \tag{7.1}$$

such that if $T \in \mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H))$ corresponds to $\phi \in (L^1(G) \otimes_{L^1(K)} L^1(H))^*$, then

$$\phi(a \otimes b) = \langle b, T(a)\rangle$$

for all $a \in L^1(G)$, $b \in L^1(H)$. By Theorem 6.5 there is a closed $L^1(K)$-ideal N in $L^1(G) \otimes_{L^1(K)} L^1(H) \equiv D$ such that $D = N \oplus D_e$. By Lemma 7.1, $D^* = D_e^\perp \oplus N^\perp$, and furthermore the adjoints P_e^* and P_n^* of the natural mappings $P_n: D \longrightarrow N$ and $P_e: D \longrightarrow D_e$ furnish the respective bicontinuous and isometric isomorphisms $N^* \cong D_e^\perp$, $D_e^* \cong N^\perp$.

Let $\mathfrak{N}$ and $\mathfrak{E}$ denote the closed subspaces in $\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H))$ that are the images of $D_e^\perp$ and $N^\perp$ in D^* under the isomorphism in (7.1). Then

$$\mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H)) = \mathfrak{N} \oplus \mathfrak{E}$$

and there obtain the respective bicontinuous and isometric isomorphisms Φ: $\mathfrak{N} \cong N^*$, Ψ: $\mathfrak{E} \cong D_e^*$; furthermore, if $T \in \mathrm{Hom}_{L^1(K)}(L^1(G), L^\infty(H))$, $T = T_n \oplus T_e$, $T_n \in \mathfrak{N}$, $T_e \in \mathfrak{E}$, and if $\Phi(T_n) = \phi_n$, $\Psi(T_e) = \phi_e$, then for all

$$a = a_n \oplus a_e \in L^1(G) = I_n \oplus L^1(G)_e, \qquad b = b_n \oplus b_e \in L^1(H) = J_n \oplus L^1(H)_e,$$

we have

$$\langle b, T_n(a)\rangle = \langle b_n, T(a_n)\rangle = \phi_n(a_n \otimes b_n), \qquad \langle b, T_e(a)\rangle = \langle b_e, T(a_e)\rangle = \phi_e(a_e \otimes b_e).$$

The remainder of the proof now follows from Theorem 6.5.

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