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SPACE-TIME PROCESSES, PARABOLIC FUNCTIONS AND ONE-DIMENSIONAL DIFFUSIONS[1]

BY

TZE LEUNG LAI

ABSTRACT. In this paper, we study the properties of the space-time process and of parabolic functions associated with a Markov process. Making use of these properties and the asymptotic behavior of the first passage probabilities near the boundary points, we prove certain theorems concerning when $u(X(t), t)$ is a martingale, where $X(t)$ is a conservative regular one-dimensional diffusion with inaccessible boundaries. A characterization of the class of parabolic functions associated with classical diffusions is also obtained.

1. **Introduction.** Robbins and Siegmund [10] have made use of the martingale

$$\int_0^\infty \exp\left(yW(t) - \frac{t}{2}y^2\right) dF(y), \qquad t \geq 0,$$

to evaluate the probability that the standard Wiener process $W(t)$ would ever cross certain boundaries which are moving with time. In [8], by making use of martingales of the form $u(X(t), t)$, we extend the Robbins-Siegmund method to find boundary crossing probabilities for other Markov processes $X(t)$, and the question of when $u(X(t), t)$ is a martingale is also considered. As a moving boundary in space is a fixed boundary in space-time, it is natural for us to look at the space-time process. In §2, we shall examine the properties of the space-time process, taking into account the special features of the time variable. The continuity properties of parabolic functions associated with continuous strong Feller processes are also established. Making use of these results, we characterize the class of parabolic functions associated with classical nonstationary diffusions in §3. Our main interest, however, lies in martingales of the form $u(X(t), t)$ when $X(t)$ is a conservative one-dimensional regular diffusion. Theorem 5 is concerned with the situation when both boundaries are natural, while Theorem 6 deals with entrance boundaries. In §5, we give some applications of our results to find certain boundary crossing probabilities for $X(t)$.

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2. **Characteristic operators of space-time processes and parabolic functions.** Let $Y(t)$, $0 \le t < \zeta$, be a Markov process which may not be temporally homogeneous and which is defined up to a lifetime $\zeta \in [0, \infty]$. Let M denote the state space of $Y(t)$, and let $P(s, x; t, \Gamma)$ be its transition function. It is well known (see for example [4] or [6]) that the corresponding space-time process $Z(t)$ is a temporally homogeneous Markov process. The state space of $Z(t)$ is $M \times [0, \infty)$, and its transition function $\bar{P}(t, z, \Lambda)$ with $z = (x, s)$ and Λ being any measurable subset of $M \times [0, \infty)$ is determined by

$$\bar{P}(t, (x, s), \Gamma \times C) = \begin{cases} P(s, x; s+t, \Gamma) & \text{if } s + t \in C, \\ 0 & \text{otherwise,} \end{cases}$$

where Γ is any measurable subset of M and C is any Borel subset of $[0, \infty)$. Obviously $Z(t)$ is conservative iff $Y(t)$ is conservative, while $Z(t)$ is normal iff $Y(t)$ is normal. (As in [5], we say that the Markov process $Y(t)$ is normal if $\lim_{t \downarrow s} P(s, x; t, M) = 1$ for all x, s.)

Let $\bar{B}$ be the Banach space of all bounded measurable functions on $M \times [0, \infty)$. For each $u \in \bar{B}$, $\|u\|$ is defined as $\sup_{x \in M, t > 0} |u(x, t)|$ and we define

$$\begin{aligned} \bar{T}_t u(x, s) &= E[u(Z(t)) | Z(0) = (x, s)] = \int u(y, t+s) P(s, x; t+s, dy), \\ \bar{G}_\alpha u(x, s) &= \int_0^\infty e^{-\alpha t} \bar{T}_t u(x, s)\, dt. \end{aligned} \tag{1}$$

Then $\{\bar{T}_t, t \ge 0\}$ is a contraction semigroup and $\bar{G}_\alpha$, $\alpha > 0$, is the resolvent. Let $\bar{A}$ be the (strong) infinitesimal generator of the semigroup and its domain be $\mathfrak{D}(\bar{A})$. Then for $u \in \mathfrak{D}(\bar{A})$, $\bar{A}u = 0$ implies that $\bar{T}_t u = u$ and so $u(Y(t+s), t+s)$, $t \ge 0$, is a martingale with respect to $P_{s,x}$ for any s, x. (Here we follow the usual convention by defining $u(Y(t), t) = 0$ if $t \ge \zeta$.) The same conclusion is also true if we replace the strong infinitesimal generator $\bar{A}$ by the weak infinitesimal generator of the semigroup $\{\bar{T}_t, t \ge 0\}$. We now investigate to what extent the above conclusion carries over when we replace $\bar{A}$ which is global in nature by the characteristic operator $\mathfrak{U}$ (also known as Dynkin's generator) of the space-time process $Z(t)$.

Let $Z_T(t)$ denote the part of $Z(t)$ on $M \times [T, \infty)$, $T \ge 0$. $Z_T(t)$ is just the space-time process of the process $\{Y(t), T \le t < \zeta\}$. Assume that M is a locally compact Hausdorff space satisfying the second axiom of countability and that $Y(t)$ has continuous sample paths. We shall say that a real-valued function $u(x, t)$ defined on an open subset G of $M \times [T, \infty)$ is parabolic for the process $Y(t)$ on the set G if it is harmonic for the space-time process $Z_T(t)$ on G. Note that if V is an open subset of M and $T_1 > T > 0$, then $V \times [T, T_1)$ is open in $M \times [T, \infty)$, but it is not open in $M \times [0, \infty)$ and its boundary in $M \times [0, \infty)$

contains $V \times \{T\}$ which is a set of irregular boundary points for the process $Z(t)$. Similarly we define $v: G \to (-\infty, \infty]$ to be superparabolic for $Y(t)$ on G if it is superharmonic for $Z_T(t)$ on G. (We say that a function $f: D \to (-\infty, \infty]$ is superharmonic for a temporally homogeneous continuous Markov process $X(t)$ on an open subset D of the state space which is a locally compact, second countable, Hausdorff space if the function f is nearly Borel, continuous in the intrinsic topology, and given any open subset U with compact closure contained in D, f is bounded below on U and $E_x f(X(\tau_U)) \le f(x)$ for any $x \in D$, where τ_U denotes the first exit time from U. If both f and $-f$ are superharmonic for $X(t)$ on D, then we say that f is harmonic for $X(t)$ on D. For further properties of harmonic and superharmonic functions, see Chapter 12 of [5].) Let u be superparabolic for $Y(t)$ on G, and let U be any open subset of $M \times [T, \infty)$ with compact closure contained in G. It is easy to see that $u(Y((t \wedge \tau_U) + s), (t \wedge \tau_U) + s)$, $t \ge 0$, is a supermartingale with respect to $P_{s,x}$ for any $(x, s) \in G$, where τ_U is the first exit time from U by $Z(t)$.

Suppose $Y(t)$, $0 \le t < \zeta$, is a continuous Markov process on M and $Z(t)$, $t \ge 0$, is its space-time process. Then $Z(t)$ is a continuous Markov process and so its characteristic operator $\mathfrak{U}$ obeys the minimum principle (cf. [5]). In fact, it obeys a stronger form of the minimum principle: Let G be an open subset of $M \times [0, \infty)$ and $(x_0, t_0) \in G$. Let f be a real-valued function on G such that $\mathfrak{U}f(x_0, t_0)$ exists. If f attains its minimum on $G \cap (M \times [t_0, \infty))$ at (x_0, t_0) and $f(x_0, t_0) \le 0$, then $\mathfrak{U}f(x_0, t_0) \ge 0$. (Compare the classical maximal principles for parabolic and elliptic operators.)

By the definition of the characteristic operator $\mathfrak{U}$, in order that $f(x, t)$ is parabolic on G, it is necessary that $\mathfrak{U}f(x, t) = 0$ for all $(x, t) \in G$. To what extent is the condition $\mathfrak{U}f = 0$ on G also sufficient? Dynkin [5] has proved a theorem relating continuous harmonic functions on an open set G for a continuous standard process to continuous solutions of $\mathfrak{U}f = 0$ on G. One of the conditions in his theorem is that the processes stopped at the first exit times from certain open sets with compact closures contained in G are Feller processes. This condition, however, often fails for space-time processes. The following theorem is a modification of Dynkin's result, taking into consideration the special features of the time variable.

Theorem 1. *Let $(Y(t), \zeta, \mathfrak{F}_t^s, P_{s,x})$ be a normal strong Markov process with continuous sample paths, lifetime ζ and state space M which is a locally compact Hausdorff space satisfying the second axiom of countability. Let $\mathfrak{U}$ denote the characteristic operator of the corresponding space-time process $Z(t)$. Let G be an open subset of $M \times [T, \infty)$ $(T \ge 0)$. Suppose that given any open subset U of $M \times [T, \infty)$ with compact closure contained in G, there exist $t_n > \cdots > t_0 = T$ $(n \ge 1)$ and disjoint sets U_j^i $(i = 1, \cdots, n;\ j = 1, \cdots, n_i)$ such that*

(i) $\tilde{U} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} U_j^i$ *has a compact closure contained in* G, *and* $U \subset \tilde{U}$.

(ii) U_j^i *is an open subset of* $M \times [t_{i-1}, t_i)$ *for all* i, j.

(iii) $Z_i(t \wedge \tau_{ij})$ *is a Feller process for all* i, j, *where* $Z_i(t)$ *denotes the part of* $Z(t)$ *on* $M \times [t_{i-1}, \infty)$, *and* $\tau_{ij} = \inf\{t \geq 0 | Z_i(t) \notin U_j^i\}$.

Under these conditions, in order for a continuous function $f\colon G \to (-\infty, \infty)$ *to be parabolic for* $Y(t)$ *on the set* G, *it is necessary and sufficient that* $f \in \mathfrak{D}_{\mathfrak{U}}(G)$ *and* $\mathfrak{U}f(x, t) = 0$ *for all* $(x, t) \in G$ (*where* $\mathfrak{D}_{\mathfrak{U}}(G)$ *denotes the class of all functions* u *for which* $\mathfrak{U}u(x, t)$ *exists at every point* $(x, t) \in G$).

Proof. The time-homogeneous process $Z(t)$ is a continuous strong Markov normal process on $M \times [0, \infty)$ which is a locally compact, second countable, Hausdorff space. Let $f \in \mathfrak{D}_{\mathfrak{U}}(G)$ be a continuous function on G such that $\mathfrak{U}f = 0$ on G. The process $Z_i(t \wedge \tau_{ij})$ is a Feller process on $M \times [t_{i-1}, \infty)$. Obviously it is also a $\hat{C}$ process. (As in [5], we say that a temporally homogeneous Markov process $X(t)$ on a locally compact, second countable, Hausdorff space H is a $\hat{C}$ process if, given any $t \geq 0$ and $f \in \hat{C}$, the function $g(x) = E_x f(X(t))$ also belongs to $\hat{C}$, where $\hat{C}$ denotes the class of all continuous real-valued functions f on H such that, given $\epsilon > 0$, there exists a compact subset K for which $|f(x)| < \epsilon$ if $x \notin K$.) Let $\hat{A}$ denote the $\hat{C}$-infinitesimal generator and $\hat{\mathfrak{U}}$ the characteristic operator of the process $Z_i(t \wedge \tau_{ij})$. Then $\hat{A}$ is the restriction of $\hat{\mathfrak{U}}$ to the class of all functions $g \in \mathfrak{D}_{\hat{\mathfrak{U}}} \cap \hat{C}$ such that $\hat{\mathfrak{U}}g \in \hat{C}$. The space $M \times [0, \infty)$ is metrizable, and let us assume that some metric ρ is introduced into it. Let V be any open subset of $M \times [t_{i-1}, \infty)$ containing U_j^i such that the closure of V is compact and is contained in G. Let V^c denote the complement of V in $M \times [t_{i-1}, \infty)$. We define

$$\hat{f}(z) = \begin{cases} \dfrac{\rho(z, V^c)}{\rho(z, V^c) + \rho(z, U_j^i)} f(z) & \text{if } z \in V, \\[2ex] 0 & \text{if } z \in V^c. \end{cases}$$

Then $\hat{f} \in \mathfrak{D}_{\hat{\mathfrak{U}}} \cap \hat{C}$ and $\hat{\mathfrak{U}}\hat{f} = 0$. Hence $\hat{A}\hat{f} = 0$. Also $\hat{f}(z) = f(z)$ for all $z \in U_j^i$. From this it follows that $E_{s,x} f(Z_i(t \wedge \tau_{ij})) = f(x, s)$ for all $(x, s) \in U_j^i$ and $t > 0$. Clearly given $(x, s) \in U_j^i$, $E_{s,x} f(Z_i(t \wedge \tau_{ij})) = E_{s,x} f(Z(\tau_{ij}))$ for all large t. Therefore, for any $(x, s) \in U_j^i$, $E_{s,x} f(Z(\tau_{ij})) = f(x, s)$. This implies that, for all $(x, s) \in \tilde{U}$, $E_{s,x} f(Z(\tau_{\tilde{U}})) = f(x, s)$, as can be easily verified by repeated use of the strong Markov property and conditional expectations. Given an open set U with compact closure contained in G, we can find an open set U_1 containing the closure of U such that the closure of U_1 is a compact subset of G. Take $\tilde{U}_1$ containing U_1 and satisfying conditions (i), (ii), (iii). Then $E_{s,x} f(Z(\tau_{\tilde{U}_1})) = f(x, s)$ for all $(x, s) \in \tilde{U}_1$. From this, it then follows that $E_{s,x} f(Z(\tau_U)) = f(x, s)$ for all $(x, s) \in U$, and f is parabolic for $Y(t)$ on G. Q.E.D.

We say that a Markov process $(Y(t), \zeta, \mathcal{F}_t^s, P_{s,x})$ on a topological measurable space is a strong Feller process if given any bounded measurable function ϕ on the state space M, the function $F_T(x, s) = \int\phi(y)P(s, x; T, dy)$ is continuous in (x, s) for $s < T$ (cf. [4]). When $Y(t)$ is strong Feller, the space-time process may not be strong Feller. However the strong Feller property of $Y(t)$ may be exploited to investigate properties of parabolic functions on cylinder sets (i.e., sets of the form $V \times (t_1, t_2)$), and these properties can then be extended to the case of parabolic functions on an arbitrary open subset G of $M \times [T, \infty)$.

Theorem 2. *Let $(Y(t), \zeta, \mathcal{F}_t^s, P_{s,x})$ be a continuous strong Feller process on a metric space M. Let V be an open subset of M and $\sigma_V^s = \inf\{t \geq s \mid Y(t) \notin V\}$. Suppose for any $\epsilon > 0$, $T > 0$ and any open set Γ with compact closure contained in V,*

$$P(s, x; t, B_\epsilon(x)) \longrightarrow 1 \quad \textit{as } t \downarrow s \textit{ uniformly for } x \in \Gamma \textit{ and } s \in [0, T], \tag{2}$$
$$\textit{where } B_\epsilon(x) = \{y \in M \mid \rho(x, y) < \epsilon\}.$$

Then given any $T > 0$ and any bounded measurable function f on $M \times [0 \ \infty)$, the functions

$$F_T(x, s) = E_{s,x} f(Y(T), T) I_{[\sigma_V^s > T]}$$

and

$$\Psi_T(x, s) = E_{s,x} f(Y(\sigma_V^s), \sigma_V^s) I_{[\sigma_V^s \leq T]}$$

are continuous in $(x \ \ s)$ for $x \in V$ and $0 \leq s < T$ where we use I_A to denote the indicator function (or commonly called the characteristic function) of the event A.

Lemma 1. *Let $Y(t)$, V be the same as in Theorem 2. Then for any $\epsilon > 0$, $T > 0$ and any open set Γ with compact closure contained in V,*

(a) $$\limsup_{\delta \downarrow 0} \sup_{x \in \Gamma} \sup_{0 \leq s \leq T} P_{s,x}\left[\sup_{s \leq t_1 \leq t_2 \leq s+\delta} \rho(Y(t_1), Y(t_2)) > \epsilon\right] = 0,$$

(b) $$\limsup_{\delta \downarrow 0} \sup_{x \in \Gamma} \sup_{0 \leq s \leq T} P_{s,x}[\zeta \leq s + \delta] = 0,$$

(c) $$\limsup_{\delta \downarrow 0} \sup_{x \in \Gamma} \sup_{0 \leq s \leq T} P_{s,x}[\sigma_V^s \leq s + \delta] = 0.$$

Proof. The proof of (a) is similar to that of Lemma 13.2 in [5], using Lemma 6.6 in [4] in place of the estimate 3.11B in [5]. As to (b), we note that $P_{s,x}[\zeta \leq s + \delta] = 1 - P(s, x; s + \delta, M)$ and so (b) follows from (2). If $V = M$, then

(c) simply reduces to (b). If $V \neq M$, we let $\epsilon = \rho(\Gamma, M - V)$. Then for all $x \in \Gamma$,

$$P_{s,x}[\sigma_V^s \leq s + \delta] \leq P_{s,x}\left[\sup_{s \leq t_1 \leq t_2 \leq s+\delta} \rho(Y(t_1), Y(t_2)) \geq \epsilon\right] + P_{s,x}[\zeta \leq s + \delta]$$

and (c) follows from (a) and (b). Q.E.D.

Proof of Theorem 2. Take $s_0 \in [0, T)$, $x_0 \in V$. Take an open neighborhood Γ of x_0 having compact closure contained in V. Given $\epsilon > 0$, by Lemma 1(c), we can can choose $\delta \in (0, T - s_0)$ such that $P_{s,x}[\sigma_V^s \leq s + 2\delta] < \epsilon$ for all $x \in \Gamma$ and $0 \leq s \leq s_0 + \delta$. Then if $x \in \Gamma$, $s \in (s_0 - \delta, s_0 + \delta)$, we obtain using the strong Markov property

$$\begin{aligned}\Psi_T(x, s) = E_{s,x} f(Y(\sigma_V^s), \sigma_V^s) I_{[\sigma_V^s \leq s_0 + \delta]} + \int \Psi_T(y, s_0 + \delta) P(s, x; s_0 + \delta, dy) \\ - E_{s,x} \Psi_T(Y(s_0 + \delta), s_0 + \delta) I_{[\sigma_V^s \leq s_0 + \delta]}.\end{aligned}$$

Since $Y(t)$ is strong Feller, $\int \Psi_T(y, s_0 + \delta) P(s, x; s_0 + \delta, dy)$ is a continuous function in (x, s) for $s < s_0 + \delta$. From this, it is then clear that Ψ_T is continuous at (x_0, s_0). The continuity of F_T can be proved similarly. Q.E.D.

Corollary. *Let $(Y(t), \zeta, \mathfrak{F}_t^s, P_{s,x})$ be a continuous strong Feller process on a metric space such that, for any $\epsilon > 0$, $T > 0$ and any open set Γ with compact closure contained in M, (2) holds. Let $Z(t)$ be the corresponding space-time process.*

(i) *Let V be an open subset of M having compact closure. Let f: $(\partial V \times [s_1, s_2)) \cup (\bar{V} \times \{s_2\}) \to (-\infty, \infty)$ be a bounded measurable function. For $(x, s) \in [s_1, s_2) \times V$, define $u(x, s) = E_{s,x} f(Z(\tau_U))$, where τ_U is the first exit time by $Z(t)$ from $U = V \times [s_1, s_2)$. Then u is continuous on U.*

(ii) *Let G be an open subset of $M \times [T, \infty)$ and f: $G \to (-\infty, \infty)$ be parabolic for $Y(t)$ on G. Then f is continuous on G.*

3. Semigroups and parabolic functions associated with classical diffusions.

In this section, we consider the case of a diffusion process on R^d corresponding to continuous coefficients $a = a_{ij}(x, t)$, $1 \leq i, j \leq d$, $b = b_i(x, t)$, $1 \leq i \leq d$, and $c = c(x, t) \geq 0$. We assume that for any $T > 0$, the matrix a is uniformly positive definite and the coefficients a_{ij} are bounded and Hölder continuous on $R^d \times [0, T]$, while the coefficients b_i and c are bounded and satisfy a Hölder condition with respect to x on $R^d \times [0, T]$. The equation

$$\frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x, t) \frac{\partial u}{\partial x_i} - c(x, t) u + \frac{\partial u}{\partial t} = 0$$

has a unique fundamental solution $p(s, x; t, y)$ (see Theorem 0.4 of [5]). The function $p(s, x; t, y)$ defines the transition density of a normal continuous strong Markov process $Y(t)$. It can be shown that the corresponding space-time process $Z(t)$ is a $\hat{C}$ process. The process $Y(t)$ has the strong Feller property; however, $Z(t)$ fails to be a strong Feller process, as it is easy to give examples of bounded measurable functions f such that $\overline{T}_t f$ is discontinuous. Because of this, various useful properties of harmonic functions associated with strong Feller processes, which have led to a nice characterization of the class of all harmonic functions for a canonical diffusion (see Chapter 13 of [5]), cannot be applied here.

The (strong) $\hat{C}$-infinitesimal generator of the space-time process $Z(t)$ coincides with the weak $\hat{C}$-infinitesimal generator and is the restriction of the differential operator L to the class

$$\left\{u \mid u \in \hat{C}, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial u}{\partial t} \in \hat{C} \ (i, j = 1, \dots, d)\right\},$$

where L is given by

$$Lu = \frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x, t) \frac{\partial u}{\partial x_i} - c(x, t)u. \tag{3}$$

Now consider the characteristic operator $\mathfrak{U}$ of $Z(t)$. Let G be an open subset of $R^d \times [0, \infty)$. Let $f: G \to (-\infty, \infty)$ be continuous and

$$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \quad (i, j = 1, \dots, d)$$

be continuous on G, and we shall denote the set of all such functions f by $C^{2,1}(G)$. We assert that $f \in \mathfrak{D}_{\mathfrak{U}}(G)$ and $\mathfrak{U}f = Lf$ on G. To prove this, let $(x_0, s_0) \in G$ and U_0 be a compact neighborhood of (x_0, s_0). By modifying f outside U_0, we have a function $g \in C^{2,1}(R^d \times [0, \infty))$ such that $g = f$ on U_0 and g has compact support. Then g is in the domain of the $\hat{C}$-infinitesimal generator and $Lg = \mathfrak{U}g$. Since $f = g$ on U_0, $\mathfrak{U}f(x_0, s_0) = \mathfrak{U}g(x_0, s_0) = Lg(x_0, s_0)$.

Using the estimate (0.33) in [5], for every $T > 0$, there exist constants $\alpha_T > 0$, $M_T > 0$ such that

$$p(s, x; t, y) \le M_T(t - s)^{-d/2} \exp(-\alpha_T |y - x|^2/(t - s))$$

for all $x, y \in R^d$, $0 \le s < t \le T$. Therefore given $\epsilon > 0$,

$$\int_{|x-y| \ge \epsilon} p(s, x; t, y)\, dy \le M_T \int_{|z| \ge \epsilon(t-s)^{-1/2}} \exp(-\alpha_T |z|^2)\, dz \to 0 \quad \text{as } t \downarrow s;$$
$$\text{uniformly for } 0 \le s \le T - \eta.$$

On the other hand, it is easy to see that $P(s, x; s + \delta, R^d) \uparrow 1$ uniformly for (x, s) in every compact set as $\delta \downarrow 0$. Therefore we have proved that (2) indeed holds, and Theorem 2 and its corollary can be applied. Hence if f is parabolic for $Y(t)$ on an open subset G of $R^d \times [0, \infty)$, then f is continuous on G. In the case where $Y(t)$ is the d-dimensional Brownian motion, this result was established by Doob [3]. Actually Doob did not work with the space-time process $Z(t)$, but worked with the Brownian trajectory process $\tilde{Z}(t) = \phi(Z(t))$, where $\phi(x, t) = (x, -t)$. His approach made use of the explicit form of the exit distribution from a rectangular domain S by $\tilde{Z}(t)$, so that if (x, s) is an interior point of S, then $E[f(\tilde{Z}(\tau_S)) | Z(0) = (x, s)]$ can be expressed in terms of integrals involving the boundary function f, the Green's function for the heat equation in S and partial derivatives of the Green's function (cf. [6]), where τ_S denotes the first exit time by $\tilde{Z}(t)$ from the rectangular domain S.

We now assert that the class of all parabolic functions on G for the diffusion $Y(t)$ coincides with the class of all continuous solutions of $\mathfrak{U}f = 0$ on G, $\mathfrak{U}$ being the space-time characteristic operator. Let V be a bounded open subset of R^d. We say that $a \in \partial V$ (where ∂V denotes the boundary of V) is a regular point with respect to $Y(t)$ if, for all $s \geq 0$, $P_{s,a}[\sigma_V^{s+} > s] = 0$, where $\sigma_V^{s+} = \lim_{\epsilon \downarrow 0} \sigma_V^{s+\epsilon}$ ($\sigma_V^{s+\epsilon}$ being defined as in Theorem 2). Omitting the details here, it can be shown that if V is a bounded open subset of R^d such that ∂V is regular with respect to $Y(t)$ and $U = V \times [T, T_1)$, $T_1 > T \geq 0$, and if f is a bounded continuous function on $R^d \times [T, \infty)$, then, for any $t > 0$, $F_t(x, s) = E_{s,x} f(Z_T(t \wedge \tau_U))$ defines a continuous function of (x, s) in $R^d \times [T, \infty)$, where $Z_T(t)$ is the part of $Z(t)$ on $[T, \infty)$ and $\tau_U = \inf\{t \geq 0 | Z_T(t) \notin U\}$. The process $Z_T(t \wedge \tau_U)$ is therefore a Feller process on $R^d \times [T, \infty)$. As in Theorem 13.8 of [5], we can prove that $a \in \partial V$ is a regular point with respect to $Y(t)$ if ∂V is differentiable at a, i.e., there exist a neighborhood K of a and a function $\psi(x)$, differentiable at a and having a nonzero differential, such that $K \cap V = \{x \in K | \psi(x) < 0\}$. Let G be an open subset of $R^d \times [T, \infty)$. Then given any open subset U of $R^d \times [T, \infty)$ with compact closure contained in G, we can find $t_n > \cdots > t_0 \geq T$ $(n \geq 1)$ and disjoint sets U_j^i $(i = 1, \ldots, n;\ j = 1, \ldots, n_i)$ such that U_j^i is of the form $V_j^i \times [t_{i-1}, t_i)$ just considered and the closure of $\tilde{U} = \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} U_j^i$ is contained in G, while $\tilde{U}$ contains U. By Theorem 1, the set of all parabolic functions for $Y(t)$ on G coincides with the set of all continuous solutions of the equation $\mathfrak{U}f(x, t) = 0$, $(x, t) \in G$, as we have seen that any parabolic function for $Y(t)$ is continuous.

Let us further assume that, for any $T > 0$, the coefficients b_i and c also satisfy a Hölder condition with respect to t on $R^d \times [0, T]$. Then we assert that the class of all parabolic functions for $Y(t)$ on G coincides with the set of all solutions $f \in C^{2,1}(G)$ of $Lf = 0$ on G, where L is the backward parabolic operator defined by (3). To see this, we shall show that any function f which is parabolic for $Y(t)$ on G belongs to $C^{2,1}(G)$. Take any $s_2 > s_1 \geq 0$ and any open sphere V in R^d such that $\bar{V} \times [s_1, s_2] \subset G$. ($\bar{V}$ denotes the closure of V.) Let $D = V \times (s_1, s_2)$, $B = V \times \{s_1\}$, $\Sigma = (\partial V \times [s_1, s_2)) \cup (\bar{V} \times \{s_2\})$. Define ϕ: $R^d \times [0, s_2] \rightarrow R^d \times [0, s_2]$ by $\phi(x, t) = (x, s_2 - t)$, and let $\tilde{a}_{ij}(x, t) = a_{ij}(x, s_2 - t)$, $\tilde{b}_i(x, t) = b_i(x, s_2 - t)$, $\tilde{c}(x, t) = c(x, s_2 - t)$ and $\tilde{f}(x, t) = f(x, s_2 - t)$. Now $\tilde{f}$ is continuous and there exists a unique solution v of the first initial-boundary value problem:

$$\frac{1}{2}\sum_{i,j=1}^{d} \tilde{a}_{ij}(x, t)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \tilde{b}_i(x, t)\frac{\partial v}{\partial x_i} - \tilde{c}(x, t)v - \frac{\partial v}{\partial t} = 0 \quad \text{on } \phi[D] \cup \phi[B],$$

$$v = \tilde{f} \quad \text{on} \quad \phi[\Sigma].$$

Furthermore, $v \in C^{2,1}(\phi[D] \cup \phi[B])$. Let $u(x, t) = v(x, s_2 - t)$, $(x, t) \in \bar{D}$. Then u satisfies $Lu = 0$ on $D \cup B$ and $u \in C^{2,1}(D \cup B)$. Therefore $u \in \mathfrak{D}_{\mathfrak{U}}(V \times [s_1, s_2))$, $\mathfrak{U}u = Lu = 0 = \mathfrak{U}f$ on $V \times [s_1, s_2)$ and $u = f$ on Σ. This implies that $u = f$ on $\bar{V} \times [s_1, s_2]$, in accordance with the following lemma:

Lemma 2. (a) *Let U be a bounded open subset of $R^d \times [s_1, s_2)$, $0 \leq s_1 < s_2$, of the form $D \cup B_1$, where D is an open subset of $R^d \times (s_1, s_2)$, whose boundary ∂D consists of a domain B_1 lying on $t = s_1$, the closure of a domain B_2 lying on $t = s_2$, and a manifold S lying in the strip $s_1 \leq t < s_2$ such that S has the outside strong sphere property* (*see p. 69 of* [7]). *Then given any $\epsilon > 0$, there exists h: $\bar{U} \rightarrow (-\infty, \infty)$ such that $0 \leq h \leq \epsilon$, $h \in \mathfrak{D}_{\mathfrak{U}}(U)$ and $\mathfrak{U}h < 0$ on U.*

(b) *Let U be as in* (a). *Suppose f: $\bar{U} \rightarrow (-\infty, \infty)$ is continuous, $f \in \mathfrak{D}_{\mathfrak{U}}(U)$, $\mathfrak{U}f \leq 0$ on U, $f \geq 0$ on $\bar{B}_2 \cup S$. Then $f \geq 0$ on U.*

Proof. (a) Define $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}$, ϕ as before. Then there exists a unique solution g of the first initial-boundary value problem:

$$\frac{1}{2}\sum_{i,j} \tilde{a}_{ij}(x, t)\frac{\partial^2 g}{\partial x_i \partial x_j} + \sum_{i} \tilde{b}_i(x, t)\frac{\partial g}{\partial x_i} - \tilde{c}(x, t)g - \frac{\partial g}{\partial t} = -1 \quad \text{on } \phi[D] \cup \phi[B_1],$$

$$g = 0 \quad \text{on } \phi[\bar{B}_2] \cup \phi[S].$$

Furthermore, $g \in C^{2,1}(\phi[D] \cup \phi[B_1])$ and g is continuous on $\phi[\bar{D}]$. It follows from the classical maximum principle that $g \geq 0$ on $\phi[\bar{D}]$. Let $k(x, t) = g(x, s_2 - t)$, $(x, t) \in \bar{D}$. Then $k \in C^{2,1}(D \cup B_1)$, k is continuous on $\bar{D}$, $k \geq 0$ and $Lk = -1$ on $D \cup B_1 = U$. Therefore $k \in \mathfrak{D}_{\mathfrak{U}}(U)$ and $\mathfrak{U}k = Lk = -1$ on U. Given $\epsilon > 0$, we can choose a positive constant c such that $ck \leq \epsilon$ on $\bar{U}$. Let $h = ck$. Then $\mathfrak{U}h = c\mathfrak{U}k < 0$ on U.

(b) Given $\epsilon > 0$, let h be the function constructed in (a). Let $\tilde{f} = f + h$.

Then $\mathfrak{U}\tilde{f} = \mathfrak{U}f + \mathfrak{U}h < 0$ on U, $\tilde{f} \geq 0$ on $\bar{B}_2 \cup S$. By the minimum principle for $\mathfrak{U}$ (the version we referred to in §2), $\tilde{f} \geq 0$ on U. This means that $f \geq -\epsilon$ on U. Since ϵ is arbitrary, the desired conclusion follows. Q.E.D.

Summing up, we have the following theorem.

Theorem 3. *Let $Y(t)$ be the diffusion on R^d corresponding to diffusion coefficients $a_{ij}(x, t)$, $b_i(x, t)$, $i, j = 1, \cdots, d$, and $c(x, t) \geq 0$ such that, for each $T > 0$, the matrix a is uniformly positive definite and the coefficients a_{ij}, b_i and c are bounded and Hölder continuous on $R^d \times [0, T]$. If G is an open subset of $R^d \times [T, \infty)$ $(T \geq 0)$ and $u: G \to (-\infty, \infty)$, then the following statements are equivalent*:

(A) *u is continuous on G and $\mathfrak{U}u = 0$ on G,*

(B) *$u \in C^{2,1}(G)$ and $Lu = 0$ on G, where L is the differential operator defined in* (3),

(C) *u is parabolic for $Y(t)$ on G,*

(D) *u is bounded on every compact subset of G; and if U is any open subset of $R^d \times [T, \infty)$ with compact closure contained in G and τ_U is the first exit time by the corresponding space-time process $Z(t)$ from U, then $u(Y(s + (t \wedge \tau_U)), s + (t \wedge \tau_U))$, $t \geq 0$, is a martingale with respect to $P_{s,x}$ for all $(x, s) \in G$.*

4. One-dimensional regular diffusions with inaccessible boundaries.

Consider a conservative continuous strong Markov process $X(t)$ on an interval I with endpoints r_0, r_1, where $-\infty \leq r_0 < r_1 \leq \infty$, such that all the interior points of I are regular points. Let C_I denote the space of all continuous functions on I such that $\lim_{x \to r_i} f(x)$ exists (as a finite number) for $i = 0, 1$. To simplify the notation, we shall, without loss of generality, assume that $0 \in (r_0, r_1)$, $s(0) = m(0) = m(0-) = 0$, where s is the scale (which is unique up to a linear transformation) and the measure which assigns mass $m(b) - m(a)$ to the interval $(a, b]$ for $r_0 < a < b < r_1$ is the speed measure of $X(t)$. Define

$$u_1(x) = \int_0^x m(z)\, ds(z), \qquad \omega_1(x) = \int_0^x s(z)\, dm(z).$$

r_i is called an accessible boundary if $u_1(r_i)$ is finite, and is said to be inaccessible if otherwise. An accessible boundary r_i is said to be regular if $\omega_1(r_i)$ is finite, and is called an exit boundary if otherwise. An inaccessible boundary r_i is an entrance boundary if $\omega_1(r_i)$ is finite, and is a natural boundary otherwise. It is well known (see Chapter 2 of [9]) that $D_m D_s^+$, when restricted to the class F_I of all functions $f \in C_I$ such that $D_m D_s^+ f \in C_I$, is the strong C_I-infinitesimal generator for the process $X(t)$ if r_0, r_1 are inaccessible boundaries.

Suppose hereafter that r_0, r_1 are inaccessible boundaries. Let $Z(t)$ be the corresponding space-time process and $\mathfrak{U}$ the characteristic operator of $Z(t)$. Let $x_0 \in (r_0, r_1)$ and $t_0 \geq 0$. Let G be an open subset of $(r_0, r_1) \times [0, \infty)$ containing

(x_0, t_0) and $f\colon G \to (-\infty, \infty)$ such that $\partial f/\partial t$ is continuous at (x_0, t_0) and $f(x, t_0)$, $D_m D_s^+ f(x, t_0)$ are both continuous in x for all x in some neighborhood of x_0. Then we assert that $\mathfrak{U}f(x_0, t_0)$ exists and

$$\mathfrak{U}f(x_0, t_0) = D_m D_s^+ f(x_0, t_0) + \partial f(x_0, t_0)/\partial t. \tag{4}$$

To prove this, let U_ϵ be any open rectangular neighborhood of (x_0, t_0) contained in G such that

$$|\partial f(y, t)/\partial t - \partial f(x_0, t_0)/\partial t| \leq \epsilon \quad \forall (y, t) \in U_\epsilon$$

Let U be any open neighborhood of (x_0, t_0) whose closure is contained in U_ϵ, and let τ_U be the first exit time from U by $Z(t)$. Write

$$\begin{aligned} E[f(X(t_0 + \tau_U), t_0 + \tau_U) | X(t_0) = x_0] - f(x_0, t_0) \\ = E[f(X(t_0 + \tau_U), t_0 + \tau_U) - f(X(t_0 + \tau_U), t_0) | X(t_0) = x_0] \\ + E[f(X(t_0 + \tau_U), t_0) - f(x_0, t_0) | X(t_0) = x_0]. \end{aligned}$$

Since

$$f(X(t_0 + \tau_U), t_0 + \tau_U) - f(X(t_0 + \tau_U), t_0) = \tau_U (\partial f/\partial t)(X(t_0 + \tau_U), \hat{t}),$$

where $\hat{t}$ lies between t_0 and $t_0 + \tau_U$, and since $(X(t_0 + \tau_U), \hat{t}) \in U_\epsilon$, we have

$$\frac{E(|f(X(t_0 + \tau_U), t_0 + \tau_U) - f(X(t_0 + \tau_U), t_0) - \tau_U \partial f(x_0, t_0)/\partial t| \, |X(t_0) = x_0)}{E(\tau_U | X(t_0) = x_0)} \leq \epsilon.$$

Let $f_{t_0}(x) = f(x, t_0)$. We need only prove that

$$\lim_{U \downarrow (x_0, t_0)} \frac{E[f_{t_0}(X(t_0 + \tau_U)) - f_{t_0}(x_0) | X(t_0) = x_0]}{E(\tau_U | X(t_0) = x_0)} = D_m D_s^+ f_{t_0}(x_0).$$

The functions f_{t_0}, $D_m D_s^+ f_{t_0}$ are continuous on $[a, b]$, where $r_0 < a < x_0 < b < r_1$. Choose $x_1 \in (a, b)$ such that $m(x_1) = m(x_1 -)$. Define

$$\tilde{f}(x) = f_{t_0}(x) - s(x) D_s^+ f_{t_0}(x_1), \qquad x \in [a, b].$$

Then $\tilde{f}$, $D_m D_s^+ \tilde{f}$ are continuous on $[a, b]$ and $D_s^+ \tilde{f}(x_1) = 0$, $D_m D_s^+ \tilde{f}(x_0) = D_m D_s^+ f(x_0, t_0)$. Let $h = D_m D_s^+ \tilde{f}$ on $[a, b]$. Since h is continuous on $[a, b]$, we can extend h to a continuous function h_I on I such that h_I vanishes outside $[a', b']$ with $r_0 < a' < a < b < b' < r_1$. Define

$$\hat{f}(x) = \int_{x_1}^{x} \int_{x_1}^{y} h_I(z) \, dm(z) \, ds(y) + \tilde{f}(x_1), \qquad x \in I.$$

Now $\hat{f}$ is a continuous function on I, is constant outside $[a', b']$, and $D_m D_s^+ \hat{f} = h_I$ is a continuous function on I with compact support. Therefore $\hat{f} \in F_I$. Since $D_m D_s^+ \tilde{f}$ is continuous on $[a, b]$, we have, for all $x \in [a, b]$,

$$\tilde{f}(x) = \int_{x_1}^{x} \int_{x_1}^{y} D_m D_s^+ \tilde{f}(z) \, dm(z) \, ds(y) + \tilde{f}(x_1) + (s(x) - s(x_1)) D_s^+ \tilde{f}(x_1).$$

But $D_s^+\tilde{f}(x_1) = 0$ and so $\hat{f}(x) = \tilde{f}(x)$ for all $x \in [a, b]$. By the Markov property, $E[\hat{f}(X(t_0 + \tau_U)) | X(t_0) = x_0] = E_{x_0}\hat{f}(X(\zeta_U))$ and $E(\tau_U | X(t_0) = x_0) = E_{x_0}\zeta_U$, where $\zeta_U = \inf\{t \geq 0 | (X(t), t + t_0) \in U\}$. Since $\hat{f}$ is in the domain F_I of the infinitesimal generator, and ζ_U is a Markov time with $E_{x_0}\zeta_U < \infty$, it follows from Dynkin's formula that

$$E_{x_0}\hat{f}(X(\zeta_U)) - \hat{f}(x_0) = E_{x_0}\int_0^{\zeta_U} D_m D_s^+\hat{f}(X(s))\, ds.$$

Therefore we have

$$\lim_{U \downarrow (x_0, t_0)} \frac{E_{x_0}\tilde{f}(X(\zeta_U)) - \tilde{f}(x_0)}{E_{x_0}\zeta_U} = \lim_{U \downarrow (x_0, t_0)} \frac{E_{x_0}\hat{f}(X(\zeta_U)) - \hat{f}(x_0)}{E_{x_0}\zeta_U}$$
$$= \lim_{U \downarrow (x_0, t_0)} \left(E_{x_0}\int_0^{\zeta_U} D_m D_s^+\tilde{f}(X(s))\, ds\right) \Big/ E_{x_0}\zeta_U$$
$$= D_m D_s^+\tilde{f}(x_0) = D_m D_s^+ f(x_0, t_0).$$

To complete the proof of (4), it remains to show that

$$\lim_{U \downarrow (x_0, t_0)} \frac{E_{x_0}s(X(\zeta_U)) - s(x_0)}{E_{x_0}\zeta_U} = 0.$$

But this is obvious, since s is harmonic for $X(t)$.

Let $r_0 < a < b < r_1$. Define first passage times

(5) $$\xi_a = \inf\{t \geq 0 | X(t) = a\}, \quad \xi_{ab} = \xi_a \wedge \xi_b.$$

Lemma 3. *For $r_0 < a < x < b < r_1$, $u \geq 0$, define*

$$\Psi_x(u) = P_x[\xi_a = \xi_{ab} \leq u], \quad \Phi_x(u) = P_x[\xi_b = \xi_{ab} \leq u],$$
$$\rho_1(x, u) = P_x[\xi_b \leq u], \quad \rho_2(x, u) = P_x[\xi_a \leq u], \quad \rho_3(x, u) = P_x[\xi_{ab} \leq u].$$

Then for $u > 0$, $x_0 \in (a, b)$ and any continuous function g: $[0, u] \to (-\infty, \infty)$,

$$\lim_{x \to x_0} \Psi_x(u) = \Psi_{x_0}(u), \quad \lim_{x \to x_0} \int_0^u g(\lambda)\, d\Psi_x(\lambda) = \int_0^u g(\lambda)\, d\Psi_{x_0}(\lambda).$$

Similar assertions hold for $\Phi_x(u)$. Furthermore, the functions $\rho_i(x, u)$, $i = 1, 2, 3$, are jointly continuous in (x, u) for $(x, u) \in (a, b) \times (0, \infty)$. Also $\lim_{x \uparrow b} \rho_1(x, u) = \lim_{x \downarrow a} \rho_2(x, u) = 1$, *for all $u > 0$.*

Proof. We first prove that $\rho_1(x, u)$ is jointly continuous in (x, u) for $(x, u) \in (r_0, b) \times (0, \infty)$. Consider the process $\tilde{X}(t)$ obtained from $X(t)$ by curtailment of its lifetime up to ξ_b, i.e., $\tilde{X}(t)$ is the part of $X(t)$ on (r_0, b). Its transition function is determined by $D_m D_s^+$ (restricted to (r_0, b)) subject to the boundary condition $f(b) = 0$. This transition function of $\tilde{X}(t)$ has a density $\tilde{p}(t, x, y)$ with respect to speed measure and $\tilde{p}$ is continuous on $(0, \infty) \times (r_0, b) \times (r_0, b)$ (see

p. 149 of [6]). Now

$$\rho_1(x, u) = 1 - P_x[\tilde{X}(u) \in (r_0, b)] = 1 - \int_{(r_0, b)} \tilde{p}(u, x, y)\, m(dy).$$

Since $\tilde{p}$ is continuous in (x, u), it follows that ρ_1 is continuous on $(r_0, b) \times (0, \infty)$. We can prove the assertions for ρ_2, ρ_3 in a similar way.

By making use of the strong Markov property, it is easy to prove that $\lim_{x \to x_0} \Phi_x(u) = \Phi_{x_0}(u)$, $u > 0$. Obviously $\Phi_x(0) = 0$ for all $x \in (a, b)$. Therefore by the Helly-Bray Lemma, for any continuous function g on $[0, u]$, we have

$$\lim_{x \to x_0} \int_0^u g(\lambda)\, d\Phi_x(\lambda) = \int_0^u g(\lambda)\, d\Phi_{x_0}(\lambda).$$

The assertions for Ψ_x can be proved similarly.

To prove that $\lim_{x \downarrow a} \rho_2(x, u) = 1$ for all $u > 0$, we consider the matching number $e_3 = \lim_{x \downarrow a} E_x \exp(-\xi_a)$ (cf. [6]). Since the diffusion is regular, $e_3 = 1$. This implies that, for any $u > 0$, $\rho_2(x, u) = P_x[\xi_a \le u] \to 1$ as $x \downarrow a$. In a similar way, we can prove that $\lim_{x \uparrow b} P_x[\xi_b \le u] = 1$. Q.E.D.

Lemma 4. *For any compact subset C of (r_0, r_1) and any $\epsilon > 0$,*

$$\lim_{\delta \downarrow 0} \sup_{\substack{x \in (r_0, r_1) \\ t \ge 0}} P_x\left[X(t) \in C, \sup_{t \le t_1 \le t_2 \le t + \delta} |X(t_1) - X(t_2)| > \epsilon\right] = 0.$$

Proof. For any open set U whose closure is a compact set in (r_0, r_1) and any $\epsilon > 0$,

$$\lim_{t \downarrow 0} \sup_{x \in U} t^{-1} P(t, x, (r_0, r_1) - (x - \epsilon, x + \epsilon)) = 0$$

(cf. [5]). Therefore we can apply Lemma 13.2 of [5] and obtain that, for any $\epsilon > 0$ and any open set U whose closure is a compact subset of (r_0, r_1),

$$\lim_{u \downarrow 0} \sup_{x \in U} P_x\left[\sup_{0 \le t_1 \le t_2 \le u} |X(t_1) - X(t_2)| > \epsilon\right] = 0.$$

Let C be a compact subset of (r_0, r_1) and $\epsilon > 0$. Given $\eta > 0$, choose $\delta_0 > 0$ such that

$$\sup_{y \in C} P_y\left[\sup_{0 \le t_1 \le t_2 \le \delta_0} |X(t_1) - X(t_2)| > \epsilon\right] < \eta.$$

Then for $x \in (r_0, r_1)$, $t \ge 0$, $\delta < \delta_0$,

$$P_x\left[X(t) \in C, \sup_{t \le t_1 \le t_2 \le t + \delta} |X(t_1) - X(t_2)| > \epsilon\right]$$
$$= \int_{[X(t) \in C]} P_{X(t)}\left[\sup_{0 \le t_1 \le t_2 \le \delta} |X(t_1) - X(t_2)| > \epsilon\right] dP_x \le \eta. \quad \text{Q.E.D.}$$

Lemma 5. *Let $r_0 < a < b < r_1$, $T > T_1 \ge 0$ and $U = (a, b) \times [T_1, T)$. Let $Z(t)$ be the space-time process of $X(t)$, and $Z_1(t)$ be the part of $Z(t)$ on $(r_0, r_1) \times [T_1, \infty)$. Let τ_U denote the first exit time of U by $Z_1(t)$. Then the process $Z_1(t \wedge \tau_U)$, $t \ge 0$, is a Feller process on $(r_0, r_1) \times [T_1, \infty)$.*

Proof. Let f be any continuous function on $(r_0, r_1) \times [T_1, \infty)$. We shall prove that, given $t > 0$, the function

$$F_t(x, s) = E[f(Z_1(t \wedge \tau_U)) | Z_1(0) = (x, s)]$$

is a continuous function on $(r_0, r_1) \times [T_1, \infty)$. Let ξ_a, ξ_b, ξ_{ab} be the first passage times defined by (5) and write $\xi = \xi_{ab}$. In the case $t + s \leq T$, we have

$$F_t(x, s) = \int_{[\xi > t]} f(X(t), t + s)\, dP_x + \int_{[\xi = \xi_a \leq t]} f(a, \xi + s)\, dP_x + \int_{[\xi = \xi_b \leq t]} f(b, \xi + s)\, dP_x$$

$$= \pi_1(x, s) + \nu_1(x, s) + \mu_1(x, s), \quad \text{say.}$$

On the other hand, if $t + s \geq T$, then

$$F_t(x, s) = \int_{[\xi > T - s]} f(X(T - s), T)\, dP_x + \int_{[\xi = \xi_a \leq T - s]} f(a, \xi + s)\, dP_x$$
$$+ \int_{[\xi = \xi_b \leq T - s]} f(b, \xi + s)\, dP_x$$
$$= \pi_2(x, s) + \nu_2(x, s) + \mu_2(x, s), \quad \text{say.}$$

Making use of Lemmas 3 and 4, it is not hard to show that π_i, ν_i and μ_i $(i = 1, 2)$ are continuous on U. Hence F_t is continuous on U.

We now prove the continuity of F_t on the boundary of U (in the space $(r_0, r_1) \times [T_1, \infty)$). We first show that

$$\lim_{\substack{s \uparrow T \\ x \to x_1, x \in (a, b)}} F_t(x, s) = f(x_1, T) \quad \text{for any } x_1 \in [a, b]. \tag{6}$$

For s sufficiently close to T, $t + s > T$ and so

$$F_t(x, s) = \int f(X(\xi \wedge (T - s)), (\xi + s) \wedge T)\, dP_x$$
$$= f(x, T) + \int_{[|X(\xi \wedge (T - s)) - x| \leq \delta]} \{f(X(\xi \wedge (T - s)), (\xi + s) \wedge T) - f(x, T)\}\, dP_x$$
$$+ \int_{[|X(\xi \wedge (T - s)) - x| > \delta]} \{f(X(\xi \wedge (T - s)), (\xi + s) \wedge T) - f(x, T)\}\, dP_x,$$

where δ is chosen as follows. Given $\epsilon > 0$, we choose $\delta \in (0, t)$ such that $|f(y, s) - f(z, T)| < \epsilon$ if $T - \delta \leq s \leq T$ and $|y - z| \leq \delta$, $z \in [a, b]$. Then for $x \in (a, b)$ and $s \in [T - \delta, T]$,

$$\int_{[|X(\xi \wedge (T - s)) - x| \leq \delta]} |f(X(\xi \wedge (T - s)), (\xi + s) \wedge T) - f(x, T)|\, dP_x < \epsilon.$$

By Lemma 4, we can choose $\delta_1 \in (0, \delta)$ such that

$$P_x\left[\sup_{0 \leq t_1 \leq t_2 \leq \delta_1} |X(t_1) - X(t_2)| > \delta\right] < \epsilon \quad \text{for all } x \in (a, b).$$

Therefore if $x \in (a, b)$ and $s \in [T - \delta_1, T]$, then

$$\int_{[|X(\xi \wedge (T - s)) - x| > \delta]} |f(X(\xi \wedge (T - s)), (\xi + s) \wedge T) - f(x, T)|\, dP_x \leq 2\|f\|\epsilon.$$

Since $\lim_{x \to x_1} f(x, T) = f(x_1, T)$, (6) follows. Finally, making use of Lemma 3, it is easy to show that if $T_1 \leq s_0 < T$, then

$$\lim_{\substack{s \to s_0 \\ x \downarrow a}} F_t(x, s) = f(a, s_0) \quad \text{and} \quad \lim_{\substack{s \to s_0 \\ x \uparrow b}} F_t(x, s) = f(b, s_0). \qquad \text{Q.E.D.}$$

Let G be an open subset of $(r_0, r_1) \times [T, \infty)$ $(T \geq 0)$. Then given any open subset U of $(r_0, r_1) \times [T, \infty)$ with compact closure contained in G, we can find $t_n > \cdots > t_0 \geq T$ and disjoint sets U_j^i $(i = 1, \cdots, n;\ j = 1, \ldots, n_i)$ such that U_j^i is of the form $(a_j^i, b_j^i) \times [t_{i-1}, t_i)$, $r_0 < a_j^i < b_j^i < r_1$, and the closure of $\tilde{U}$ is contained in G, while $\tilde{U}$ contains U, where $\tilde{U} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} U_j^i$. Let $Z_i(t)$ be the part of the space-time process $Z(t)$ on $(r_0, r_1) \times [t_{i-1}, \infty)$ and let $\tau_{ij} = \inf\{t \geq 0 \mid Z_i(t) \notin U_j^i\}$. By Lemma 5, $Z_i(t \wedge \tau_{ij})$ is a Feller process. Hence Theorem 1 can be applied and the set of all continuous parabolic functions for $X(t)$ on G coincides with the set of all continuous solutions of the equation $\mathfrak{U} f(x, t) = 0$, $(x, t) \in G$, where $\mathfrak{U}$ is the space-time characteristic operator. In particular, if $f: G \to (-\infty, \infty)$ is continuous, $\partial f/\partial t$ is continuous and $D_m D_s^+ f$ exists and satisfies $\partial f/\partial t + D_m D_s^+ f = 0$ on G, then $f(X(t \wedge \tau), s + (t \wedge \tau))$, $t \geq 0$, is a martingale with respect to P_x for any $(x, s) \in G$, where $\tau = \inf\{t \geq 0 \mid (X(t), t + s) \notin U\}$ and U is an arbitrary open subset of $(r_0, r_1) \times [T, \infty)$ with compact closure contained in G. Furthermore, it can be shown that Theorem 2 is applicable here, and so if f is parabolic for $X(t)$ on G, then it has to be continuous on G.

Let us now consider solutions of $\partial f/\partial t + D_m D_s^+ f = 0$ obtained by the separation of variables, viz., $f(x, t) = g(x)h(t)$. Such solutions have the form $f(x, t) = e^{-\alpha t} g(x)$, where α is a real number and $D_m D_s^+ g(x) = \alpha g(x)$. As before, we assume for notational convenience that $s(0) = m(0) = m(0-) = 0$. A solution of $D_m D_s^+ g = \alpha g$ is given by

$$u(x, \alpha) = \sum_{n=0}^{\infty} \alpha^n u_n(x), \qquad x \in (r_0, r_1), \tag{7}$$

where

$$u_0(x) = 1, \quad u_{n+1}(x) = \int_0^x \int_0^y u_n(z)\, dm(z)\, ds(y).$$

The functions $u_j(x)$, $j \geq 1$, are everywhere nonnegative, increase for $x > 0$ and decrease for $x < 0$, and since $u_k(x) \leq u_j(x)(u_1(x))^{k-j}/(k-j)!$ for $j \leq k$, the series $\sum_{n=0}^{\infty} \alpha^n u_n(x)$ converges for all real α. Another solution of $D_m D_s^+ g = \alpha g$ is given by

$$v(x, \alpha) = \sum_{n=0}^{\infty} \alpha^n v_n(x), \qquad x \in (r_0, r_1), \tag{8}$$

where

$$v_0(x) = s(x), \quad v_{n+1}(x) = \int_0^x \int_0^y v_n(z)\, dm(z)\, ds(y).$$

The functions $v_n(x)$ are increasing on (r_0, r_1), negative if $x < 0$, and positive if $x > 0$. For $n = 0, 1, 2, \cdots,$

$$v_n(x) \leq \frac{s(x)}{n!}\left(\int_0^x s(y)\,dm(y)\right)^n, \qquad x > 0,$$
$$|v_n(x)| \leq \frac{|s(x)|}{n!}\left(\int_x^0 |s(y)|\,dm(y)\right)^n, \qquad x < 0.$$

Therefore the series $\sum_{n=0}^{\infty} \alpha^n v_n(x)$ converges for all real α.

Consider the solutions of

$$D_m D_s^+ g = \lambda g \qquad (\lambda > 0). \tag{9}$$

We have seen that $u(x, \lambda)$, $v(x, \lambda)$ are solutions of (9). Since $\lambda > 0$, we have $1 + \lambda u_1(x) \leq u(x, \lambda) \leq \exp(\lambda u_1(x))$, and the functions

$$\begin{aligned} u_+(x, \lambda) &= u(x, \lambda)\int_x^{r_1} \frac{ds(y)}{(u(y, \lambda))^2}, \\ u_-(x, \lambda) &= u(x, \lambda)\int_{r_0}^x \frac{ds(y)}{(u(y, \lambda))^2}, \end{aligned} \tag{10}$$

are respectively positive decreasing and positive increasing solutions of (9) (cf. [9]). As both boundaries are inaccessible, $D_s^+ u_+(r_1, \lambda) = D_s^+ u_-(r_0, \lambda) = 0$. If r_1 is not an entrance boundary, then $u_+(r_1, \lambda) = 0$, and if r_0 is not an entrance boundary, then $u_-(r_0, \lambda) = 0$. The solutions $u(x, \lambda)$, $v(x, \lambda)$, $u_+(x, \lambda)$, $u_-(x, \lambda)$ of (9) are all continuous functions in x. By our previous result, any continuous solution of $D_m D_s^+ f + \partial f/\partial t = 0$ with $\partial f/\partial t$ continuous on $(r_0, r_1) \times [0, \infty)$ is parabolic for $X(t)$. In particular, $e^{-\lambda t}u(x, \lambda)$, $e^{-\lambda t}v(x, \lambda)$, $e^{-\lambda t}u_+(x, \lambda)$ and $e^{-\lambda t}u_-(x, \lambda)$ are parabolic for $X(t)$, and so if V_n is an interval such that $\bar{V}_n \subset (r_0, r_1)$ and σ_n denotes the first exit time from V_n by $X(t)$, then $\{\exp(-\lambda(t \wedge \sigma_n))u(X(t \wedge \sigma_n), \lambda), t \geq 0\}$, etc., are martingales with respect to P_x for all $x \in (r_0, r_1)$. Now suppose that $V_n \uparrow (r_0, r_1)$ as $n \uparrow \infty$. Then $\sigma_n \uparrow \infty$ since the boundaries are inaccessible, and it is natural to ask if $\{e^{-\lambda t}u(X(t), \lambda), t \geq 0\}$, etc., are still martingales. We shall show below that if both boundaries are natural, then the answer to the above question is affirmative.

Theorem 4. *Let $X(t)$ be a conservative regular continuous strong Markov process on an interval I with endpoints r_0, r_1, where $-\infty \leq r_0 < r_1 \leq \infty$. Let f: $I \times [0, \infty) \to (-\infty, \infty)$. For $r_0 < a < b < r_1$, let ξ_a, ξ_b, ξ_{ab} be the first passage times defined by* (5). *Let $x \in (r_0, r_1)$. In order that $f(X(t), t + r)$, $t \geq 0$, is a martingale with respect to P_x for all $r \geq 0$, it is necessary that the following conditions hold for all $t \geq 0$, $r \geq 0$,*

(i) $$\lim_{a \downarrow r_0} \int_{[\xi_a < r]} f(a, t + \xi_a)\,dP_x = 0 \quad \textit{if } r_0 \textit{ is inaccessible,}$$

(ii) $$\lim_{b \uparrow r_1} \int_{[\xi_b < r]} f(b, t + \xi_b)\,dP_x = 0 \quad \textit{if } r_1 \textit{ is inaccessible,}$$

(iii) $$\lim_{\substack{a \downarrow r_0 \\ b \uparrow r_1}} \int_{[\xi_{ab} \geq r]} f(X(r), t+r)\, dP_x = \int f(y, t+r) P(r, x, dy)$$

if both r_0, r_1 are inaccessible,

(iv) $$\int_{[\xi_a < r \wedge \xi_b]} f(a, t+\xi_a)\, dP_x + \int_{[\xi_b < r \wedge \xi_a]} f(b, t+\xi_b)\, dP_x \to 0 \quad \text{as } a \downarrow r_0,\ b \uparrow r_1,$$

if both r_0, r_1 are inaccessible.

Conversely, if r_0, r_1 are both inaccessible boundaries, f is parabolic for $X(t)$, and conditions (iii), (iv) *hold for all $x \in (r_0, r_1)$ and all $t \geq 0$, $r \geq 0$, then $f(X(t), t+r)$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$ and all $r \geq 0$.*

Proof. Suppose $f(X(t), t+r)$, $t \geq 0$, is a martingale with respect to P_x for all $r \geq 0$. Let r_0 be inaccessible. For any $a \in (r_0, r_1)$, $f(X(t \wedge \xi_a), (t \wedge \xi_a) + r)$, $t \geq 0$, is also a matringale with respect to P_x for all $r \geq 0$. Therefore for any $r \geq 0$, $t \geq 0$,

$$f(x, t) = \int_{[\xi_a \geq r]} f(X(r), r+t)\, dP_x + \int_{[\xi_a < r]} f(a, \xi_a + t)\, dP_x.$$

Since r_0 is inaccessible, $P_x[\lim_{a \downarrow r_0} \xi_a = \infty] = 1$. Also $E_x f(X(r), t+r) = f(x, t)$. Therefore as $a \downarrow r_0$,

$$\int_{[\xi_a \geq r]} f(X(r), t+r)\, dP_x \to \int f(X(r), t+r)\, dP_x = f(x, t)$$

and so (i) holds. In a similar way, we prove (ii).

Now assume that both r_0, r_1 are inaccessible. For $r_0 < a < b < r_1$, $f(X(t \wedge \xi_{ab}), (t \wedge \xi_{ab}) + r)$, $t \geq 0$, is a martingale with respect to P_x for all $r \geq 0$. Therefore for any $r \geq 0$, $t \geq 0$,

$$\begin{aligned} f(x, t) = {} & \int_{[\xi_{ab} \geq r]} f(X(r), r+t)\, dP_x \\ & + \int_{[\xi_a < \xi_b,\, \xi_a < r]} f(a, \xi_a + t)\, dP_x + \int_{[\xi_b < \xi_a,\, \xi_b < r]} f(b, \xi_b + t)\, dP_x. \end{aligned} \tag{11}$$

Since r_0, r_1 are both inaccessible, $P_x[\lim_{a \downarrow r_0; b \uparrow r_1} \xi_{ab} = \infty] = 1$. Therefore

$$\lim_{\substack{a \downarrow r_0 \\ b \uparrow r_1}} \int_{[\xi_{ab} \geq r]} f(X(r), t+r)\, dP_x = \int f(X(r), t+r)\, dP_x = f(x, t)$$

and so

$$\lim_{\substack{a \downarrow r_0 \\ b \uparrow r_1}} \left\{ \int_{[\xi_a < r \wedge \xi_b]} f(a, \xi_a + t)\, dP_x + \int_{[\xi_b < r \wedge \xi_a]} f(b, \xi_b + t)\, dP_x \right\} = 0.$$

Conversely, suppose that r_0, r_1 are both inaccessible, f is parabolic for $X(t)$ and conditions (iii), (iv) both hold. Since f is parabolic, for any $r_0 < a < b$

$< r_1$, $f(X(t \wedge \xi_{ab}), (t \wedge \xi_{ab}) + r)$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$ and $r \geq 0$. Therefore for any $x \in (r_0, r_1)$, $t \geq 0$, $r \geq 0$, (11) holds, and letting $a \downarrow r_0$, $b \uparrow r_1$ in (11), we obtain by (iii) and (iv) that $f(x, t) = \int f(y, t+r)P(r, x, dy)$. Hence $f(X(t), t+r)$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$ and $s \geq 0$. Q.E.D.

Lemma 6. *Assume that r_0, r_1 are natural boundaries. If $g(x)$ is any solution of (9), then for any $t > 0$ and any $x \in (r_0, r_1)$,*

(i) $\int |g(y)| P(t, x, dy) < \infty$, *and* $\int_{[\xi_{ab} \geq t]} g(X(t))\, dP_x \to \int g(y) P(t, x, dy)$ *as* $a \downarrow r_0$ *and* $b \uparrow r_1$,

(ii) $\lim_{a \downarrow r_0} g(a) P_x[\xi_a < t] = 0$,

(iii) $\lim_{b \uparrow r_1} g(b) P_x[\xi_b < t] = 0$.

Proof. Since g is a solution of (9), there exist constants c_1, c_2 such that $g(x) = c_1 u_+(x, \lambda) + c_2 u_-(x, \lambda)$. Therefore we need only prove (i), (ii) and (iii) for u_+ and u . We first consider u . Since u > 0 on (r_0, r_1), the monotone convergence theorem implies that

$$\lim_{a \downarrow r_0; b \uparrow r_1} \int_{[\xi_{ab} \geq t]} u_-(X(t), \lambda)\, dP_x = \int u_-(X(t), \lambda)\, dP_x.$$

Now $u_-(X(t \wedge \xi_{ab}), \lambda) \exp(-\lambda(t \wedge \xi_{ab}))$, $t \geq 0$, is a martingale with respect to P_x and so

$$u_-(x, \lambda) \geq e^{-\lambda t} \int_{[\xi_{ab} \geq t]} u_-(X(t), \lambda)\, dP_x.$$

Therefore $\int u_-(y, \lambda) P(t, x, dy) < \infty$ and (i) is proved. Since $u_-(r_0, \lambda) = 0$, (ii) is trivial. To prove (iii), we fix $\alpha > 0$ and note that, given any $t > 0$,

$$\begin{aligned} u_-(b, \lambda) P_x[\xi_b < t] &\leq e^{(\alpha+\lambda)t} u_-(b, \lambda) \int_0^\infty e^{-(\alpha+\lambda)r}\, dP_x[\xi_b < r] \\ &= e^{(\alpha+\lambda)t} u_-(b, \lambda) u_-(x, \alpha+\lambda)/u_-(b, \alpha+\lambda), \quad \text{if } b > x. \end{aligned}$$

Therefore we need only prove that $\lim_{b \uparrow r_1} u_-(b, \lambda)/u_-(b, \alpha+\lambda) = 0$. To prove this, it suffices to show that $\lim_{b \uparrow r_1} u(b, \lambda)/u(b, \alpha+\lambda) = 0$. Given any $\delta > 0$, we can choose n_0 such that $\lambda^n/(\lambda+\alpha)^n < \delta$ for all $n \geq n_0$. Then

$$\frac{u(b, \lambda)}{u(b, \alpha+\lambda)} \leq \frac{\sum_{n=0}^{n_0} \lambda^n u_n(b)}{\sum_{n=n_0+1}^{\infty} (\alpha+\lambda)^n u_n(b)} + \delta \to \delta \quad \text{as } b \uparrow r_1.$$

The last relation above holds by Lemma 7 below. Since δ is arbitrary, we indeed have $\lim_{b \uparrow r_1} u(b, \lambda)/u(b, \alpha+\lambda) = 0$. In a similar way, we can prove that (i), (ii), (iii) hold for u_+. Q.E.D.

Lemma 7. *Assume that r_1 is a natural boundary.*

(i) $\int_0^{r_1} u_n(x)\, dm(x) = \infty$ *for all* $n \geq 1$.

(ii) *For* $n = 0, 1, 2, \cdots$, $\lim_{b \uparrow r_1} u_{n+1}(b) = \infty$, $\lim_{b \uparrow r_1} u_{n+1}(b)/u_n(b) = \infty$.

(iii) *For* $n = 0, 1, 2, \cdots$ *and* $\lambda > 0$, $\lim_{b \uparrow r_1} u_n(b)/u(b, \lambda) = 0$.
Similar statements hold for r_0 *if it is natural.*

Proof. To prove (i), let $n \geq 1$ and $r_0 < 0 < b < r_1$. Then since u_{n-1} is continuous and positive on $(0, r_1)$ and m is strictly increasing, it follows that $\int_0^b u_{n-1}(z)\,dm(z) > 0$. Letting $C = \int_0^b u_{n-1}(z)\,dm(z)$, we have

$$\int_0^{r_1} u_n(x)\,dm(x) \geq \int_b^{r_1}\int_b^x\int_0^y u_{n-1}(z)\,dm(z)\,ds(y)\,dm(x)$$
$$\geq C\int_b^{r_1}(s(x) - s(b))\,dm(x) = \infty.$$

We shall prove (ii) by induction. Obviously (ii) is true for $n = 0$ and it is easy to show that (ii) also holds for $n = 1$. Now assume that (ii) holds for $n = k - 1$ $(k \geq 2)$. By induction assumption, $\lim_{b \uparrow r_1} u_k(b)/u_{k-1}(b) = \infty$. Hence given $\Delta > 0$, we can choose $b_0 \in (0, r_1)$ such that $u_k(b)/u_{k-1}(b) > \Delta$ if $b \in [b_0, r_1)$. Then for $b \in (b_0, r_1)$,

$$\frac{u_{k+1}(b)}{u_k(b)} \geq \frac{\Delta\int_{b_0}^b\int_{b_0}^y u_{k-1}(z)dm(z)\,ds(y)}{\int_0^b\int_0^y u_{k-1}(z)dm(z)\,ds(y)}$$

$$(12) \qquad = \Delta\left[\frac{\int_0^{b_0}\int_0^y u_{k-1}(z)\,dm(z)\,ds(y)}{\int_{b_0}^b\int_{b_0}^y u_{k-1}(z)dm(z)ds(y)} + \frac{\int_{b_0}^b\int_0^{b_0} u_{k-1}(z)\,dm(z)\,ds(y)}{\int_{b_0}^b\int_{b_0}^y u_{k-1}(z)\,dm(z)\,ds(y)} + 1\right]^{-1}$$

$$= \Delta[G_1(b) + G_2(b) + 1]^{-1}, \quad \text{say}$$

It is easy to see that $\int_{b_0}^{r_1}\int_{b_0}^y u_{k-1}(z)\,dm(z)\,ds(y) = \infty$, and therefore we have $\lim_{b \uparrow r_1} G_1(b) = 0$. If $s(r_1) < \infty$, then

$$\int_{b_0}^{r_1}\int_0^{b_0} u_{k-1}(z)\,dm(z)\,ds(y) < \infty$$

and so $\lim_{b \uparrow r_1} G_2(b) = 0$. Suppose $s(r_1) = \infty$. By (i), given any $\delta > 0$, we can choose $b_1 \in (b_0, r_1)$ such that

$$\int_{b_0}^{b_1} u_{k-1}(x)\,dm(x) > \delta^{-1}.$$

Then for $b \in (b_1, r_1)$,

$$G_2(b) \leq \delta u_{k-1}(b_0)m(b_0)(s(b) - s(b_0))/(s(b) - s(b_1))$$
$$\longrightarrow \delta u_{k-1}(b_0)m(b_0) \quad \text{as } b \uparrow r_1.$$

Since δ is aribtrary, $\lim_{b \uparrow r_1} G_2(b) = 0$. Therefore (12) implies that $\liminf_{b \uparrow r_1} u_{k+1}(b)/u_k(b) \geq \Delta$. But Δ is arbitrary, and so $\lim_{b \uparrow r_1} u_{k+1}(b)/u_k(b) = \infty$. Hence we have proved (ii), and (iii) is an obvious consequence of (ii). Q.E.D.

In a similar way, we can prove the following lemma.

Lemma 8. *Assume that* r_1 *is a natural boundary.*

(i) $\int_0^{r_1} v_n(x)\, dm(x) = \infty$ *for all* $n \geq 0$.

(ii) *For* $n = 0, 1, 2, \cdots$, $\lim_{b\uparrow r_1} v_{n+1}(b) = \infty$, $\lim_{b\uparrow r_1} v_{n+1}(b)/v_n(b) = \infty$.

(iii) *For* $n = 0, 1, 2, \cdots$ *and* $\lambda > 0$, $\lim_{b\uparrow r_1} v_n(b)/v(b, \lambda) = 0$.

Similar statements hold for r_0 *if it is natural.*

Theorem 5. *Let* $X(t)$ *be a conservative regular continuous strong Markov process on* (r_0, r_1) *such that the boundaries* r_0, r_1 *are natural. Let* $s(x)$ *be the scale and* m *the speed measure of* $X(t)$.

(i) *If* $f\colon (r_0, r_1) \times [0, \infty) \to (-\infty, \infty)$ *is continuous,* $\partial f/\partial t$ *is also continuous and* $D_m D_s^+ f$ *exists and satisfies* $\partial f/\partial t + D_m D_s^+ f = 0$ *on* $(r_0, r_1) \times [0, \infty)$, *then* f *is parabolic for* $X(t)$. *If moreover, for any* $T > 0$, *there exist* $\alpha_T > 0$ *and a function* $g_T\colon (r_0, r_1) \to (-\infty, \infty)$ *satisfying* $D_m D_s^+ g_T = \alpha_T g_T$ *such that* $\max_{0\leq t\leq T} |f(x, t)| = O(|g_T(x)|)$ *as* $x \uparrow r_1$ *and as* $x \downarrow r_0$, *then* $f(X(t), t + r)$, $t \geq 0$, *is a martingale with respect to* P_x *for all* $x \in (r_0, r_1)$ *and all* $r \geq 0$.

(ii) *For any real number* α,

$D_m D_s^+ g(x) = \alpha g(x)$ *for all* $x \in (r_0, r_1)$

$\Rightarrow e^{-\alpha t} g(X(t))$, $t \geq 0$, *is a martingale with respect to* P_x *for all* $x \in (r_0, r_1)$.

$D_m D_s^+ g(x) = \alpha$ *for all* $x \in (r_0, r_1)$

$\Rightarrow g(X(t)) - \alpha t$, $t \geq 0$, *is a martingale with respect to* P_x *for all* $x \in (r_0, r_1)$.

(iii) *If* $f(x, t) = t^n \phi_0(x) + t^{n-1}\phi_1(x) + \cdots + \phi_n(x)$ *satisfies* $D_m D_s^+ f + \partial f/\partial t = 0$, *then* $f(X(t), t + r)$, $t \geq 0$, *is a martingale with respect to* P_x *for all* $x \in (r_0, r_1)$ *and all* $r \geq 0$.

Proof. We may without loss of generality assume that $0 \in (r_0, r_1)$ and $s(0) = m(0) = m(0-) = 0$.

To prove (i), we can use Lemma 6 and the assumption that $\max_{0\leq t\leq T} |f(x, t)| = O(|g_T(x)|)$ to show that the function f satisfies conditions (iii) and (iv) of Theorem 4. Hence by that theorem, $f(X(t), t + r)$, $t \geq 0$, is a martingale with respect to P_x for $x \in (r_0, r_1)$ and $r \geq 0$.

To prove (ii), first consider solutions of $D_m D_s^+ g = \alpha$. For all $x \in (r_0, r_1)$, we have

$$g(x) = \alpha u_1(x) + g(0) + v_0(x) D_s^+ g(0).$$

Set $f(x, t) = g(x) - \alpha t$. Then $D_m D_s^+ f + \partial f/\partial t = 0$, and f is continuous. Take $\lambda > 0$. By Lemmas 7 and 8, there exist constants K_1, K_2 such that $u_1(x) \leq K_1 u(x, \lambda)$ and $|v_0(x)| \leq K_2 |v(x, \lambda)|$ for all $x \in (r_0, r_1)$. The desired conclusion then follows from (i).

Now consider solutions $D_m D_s^+ g = \alpha g$. If $\alpha > 0$, then the desired conclusion follows from (i). If $\alpha = 0$, then this is a special case of $D_m D_s^+ g = \alpha$ already considered. Let $\alpha < 0$. Then $u(x, \alpha)$ and $v(x, \alpha)$ are two linearly independent solutions since $D_s^+u(0, \alpha) = 0$, $D_s^+v(0, \alpha) = 1$ and $u(0, \alpha) = 1$. Clearly $|u(x, \alpha)| \le u(x, -\alpha)$, $|v(x, \alpha)| \le |v(x, -\alpha)|$. Since $-\alpha > 0$ and $u(x, -\alpha)$, $v(x, -\alpha)$ are solutions of $D_m D_s^+ g = (-\alpha)g$, the desired conclusion follows from (i).

We now prove (iii). Consider solutions of the equation $D_m D_s^+ f + \partial f/\partial t = 0$ of the form $f(x, t) = t^n\phi_0(x) + t^{n-1}\phi_1(x) + \cdots + \phi_n(x)$. We have for all $t \ge 0$, $x \in (r_0, r_1)$,

$$t^n D_m D_s^+\phi_0(x) + t^{n-1}(n\phi_0(x) + D_m D_s^+\phi_1(x)) + \cdots + (\phi_{n-1}(x) + D_m D_s^+\phi_n(x)) = 0.$$

Therefore $D_m D_s^+\phi_0(x) = 0$, and in general for $1 \le i \le n$, $D_m D_s^+\phi_i(x) = -(n-i+1)\phi_{i-1}(x)$. Hence there exist arbitrary constants $c_0, d_0, c_1, d_1, \cdots, c_n, d_n$ such that

$$\phi_0(x) = c_0 + d_0 s(x) \tag{13a}$$

and for $1 \le i \le n$,

$$\phi_i(x) = -(n-i+1)\left[\int_0^x \int_0^y \phi_{i-1}(z)\, dm(z)\, ds(y) + c_i + d_i s(x)\right]. \tag{13b}$$

It is easy to see by induction that ϕ_i is a linear combination of $u_0, \cdots, u_i$, $v_0, \cdots, v_i$. Therefore by Lemmas 7 and 8, given $\lambda > 0$, there exist constants K_1, K_2 such that, for $i = 0, \cdots, n$, $|\phi_i(x)| \le K_1 u(x, \lambda) + K_2|v(x, \lambda)|$. Hence the desired conclusion follows from (i). Q.E.D.

From (iii) of Theorem 5, it follows that $t^n\phi_0(X(t)) + t^{n-1}\phi_1(X(t)) + \cdots + \phi_n(X(t))$, $t \ge 0$, is a martingale, where $\phi_0, \cdots, \phi_n$ are given by (13a, b). We shall call such martingales time-polynomial martingales, since they are polynomials of the time variable. By first setting $c_0 = (-1)^n/n!$, $c_1 = \cdots = c_n = d_0 = \cdots = d_n = 0$, and then setting $d_0 = (-1)^n/n!$, $d_1 = \cdots = d_n = c_0 = \cdots = c_n = 0$, one obtains the two kinds of time-polynomial martingales of degree n discussed by Arbib [1] in the case of a regular diffusion with generator $D_m D^+$ and natural boundaries at $\pm\infty$. Arbib's proof depends on the direct evaluation of $E_0[\phi_i(X(t))]$ which in turn makes use of the behavior of the positive decreasing and increasing solutions of $D_m D^+ g = \alpha g$ at the boundaries $+\infty$ and $-\infty$ respectively. Arbib [1] has obtained a characterization of the diffusion with generator $D_m D^+$ and natural boundaries $\pm\infty$ in terms of time-polynomial martingales. In our present case, in order that a stochastic process $X(t)$, $t \ge 0$, taking values in (r_0, r_1) be a regular diffusion with generator $D_m D_s^+$ and natural boundaries at r_0, r_1, it is both necessary and sufficient that almost all sample paths of $X(t)$ are continuous, and

$$\{s(X(t)),\ t \ge 0\}, \quad \left\{\int_0^{X(t)} m(0, y]\, ds(y) - t,\ t \ge 0\right\}$$

are both martingales.

Under certain assumptions on the diffusion coefficients $\sigma(x) > 0$ and $\mu(x)$, a standard Ito process $X(t)$ on the real line can be considered as a regular diffusion with generator $D_m D_s^+$ and natural boundaries at $\pm\infty$, where

$$s(x) = \int_0^x e^{-B(y)}dy, \quad m(x) = \int_0^x \frac{2e^{B(y)}}{\sigma^2(y)}dy, \quad B(x) = \int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy.$$

In such cases, by (ii) of Theorem 5, for any real number α, if

$$\frac{\sigma^2(x)}{2}\frac{d^2g}{dx^2} + \mu(x)\frac{dg}{dx} = \alpha g,$$

then $e^{-\alpha t}g(X(t))$, $t \geq 0$, is a martingale, while if

$$\frac{\sigma^2(x)}{2}\frac{d^2g}{dx^2} + \mu(x)\frac{dg}{dx} = \alpha,$$

then $g(X(t)) - \alpha t$, $t \geq 0$, is a martingale (cf. Doob [2]).

Let us turn to the case where r_0, r_1 are both inaccessible, but not both natural. Let $X(t)$ be the Bessel process of order $m \geq 2$, i.e., $X(t)$ has transition density

$$p(t, x, y) = t^{-1}(xy)^{1-m/2}y^{m-1}I_{m/2-1}(xy/t)\exp(-(x^2+y^2)/2t), \qquad t > 0,\ x, y > 0.$$

($I_r(z)$ denotes the modified Bessel function of the first kind.) Here 0 is an entrance boundary and ∞ is a natural boundary. In [8], it is shown that for the increasing solution

$$g_1(x) = x^{1-m/2}I_{m/2-1}(\sqrt{2\alpha}\,x)$$

of

$$\frac{1}{2}\left(\frac{d^2g}{dx^2} + \frac{m-1}{x}\frac{dg}{dx}\right) = \alpha g \qquad (\alpha > 0),$$

$e^{-\alpha t}g_1(X(t))$, $t \geq 0$, is a martingale, whereas the decreasing solution $g_2(x) = x^{1-m/2}K_{m/2-1}(\sqrt{2\alpha}\,x)$ ($K_r(z)$ being the modified Bessel function of the second kind) fails to give a martingale. In the case $\alpha < 0$, the second order differential equation above has two linearly independent solutions, one of which is $\tilde{g}_1(x) = x^{1-m/2}J_{m/2-1}(\sqrt{2|\alpha|}\,x)$, where $J_r(z)$ is the Bessel function of the first kind. The other linearly independent solution $\tilde{g}_2(x)$ is $x^{1-m/2}J_{1-m/2}(\sqrt{2|\alpha|}\,x)$ if $m \neq 2, 4, 6, \cdots$, and is $x^{1-m/2}Y_{m/2-1}(\sqrt{2|\alpha|}\,x)$ if $m = 2, 4, 6, \cdots$, where $Y_r(z)$ is the Bessel function of the second kind. It is easy to see that $e^{-\alpha t}\tilde{g}_1(X(t))$, $t \geq 0$, is a martingale, while $e^{-\alpha t}\tilde{g}_2(X(t))$, $t \geq 0$, is not a martingale. For $\alpha = 0$, two linearly independent solutions of the above second order differential equation are $g_1^* = 1$ and

$$g_2^*(x) = \begin{cases} -\, x^{2-m} & (m > 2) \\ \log x & (m = 2). \end{cases}$$

Clearly $g_1^*(X(t))$, $t \geq 0$, is a martingale. However $g_2^*(X(t))$, $t \geq 0$, is not a martingale. The function g_2^* is the scale of $X(t)$. Such a phenomenon is actually common to all one-dimensional diffusions with an entrance boundary, as we shall show below.

Let $X(t)$ be a conservative regular strong Markov process on (r_0, r_1) such that r_0 is an entrance boundary. Again for notational convenience, assume that $0 \in (r_0, r_1)$, $s(0) = m(0) = m(0-) = 0$, and we define $u(x, \alpha)$ as in (7), $v(x, \alpha)$ as in (8), $u_+(x, \lambda)$, $u_-(x, \lambda)$ for $\lambda > 0$ as in (10). Since r_0 is an entrance boundary, $m(r_0)$ is finite, but $s(r_0) = -\infty$. Also $\int_{r_0}^0 |s(z)|\, dm(z) < \infty$.

Lemma 9. *Assume that r_0 is an entrance boundary, $0 \in (r_0, r_1)$, $s(0) = m(0) = m(0-) = 0$. For $x \in (r_0, 0)$, define $\psi_1(x) = -\, m(x)$, and for $n \geq 1$,*

$$\psi_{n+1}(x) = \int_x^0 \int_y^0 \psi_n(z)\, ds(z)\, dm(y).$$

Also define

$$\omega_1(x) = -\int_x^0 s(z)\, dm(z)$$

and for $n \geq 1$,

$$\omega_{n+1}(x) = \int_x^0 \int_y^0 \omega_n(z)\, ds(z)\, dm(y).$$

Then for $n \geq 1$, $x \in (r_0, 0)$,

$$0 < \psi_n(x) \leq ((\omega_1(x))^{n-1}/(n-1)!)\psi_1(x), \tag{14}$$

$$0 < \omega_n(x) \leq (\omega_1(x))^n/n!. \tag{15}$$

Furthermore, $\lim_{x \downarrow r_0} \psi_n(x)$, $\lim_{x \downarrow r_0} \omega_n(x)$ exist and are finite positive numbers. Define $\psi_n(r_0) = \lim_{x \downarrow r_0} \psi_n(x)$, $\omega_n(r_0) = \lim_{x \downarrow r_0} \omega_n(x)$, $n = 1, 2, \cdots$. Then the series $\sum_{n=1}^\infty \alpha^n \psi_n(r_0)$, $\sum_{n=1}^\infty \alpha^n \omega_n(r_0)$ both converge for any real α.

Proof. The inequality (14) clearly holds for $n = 1$. Assume that it holds for n. Then for $x \in (r_0, 0)$,

$$\begin{aligned} \psi_{n+1}(x) &\leq \int_x^0 \int_y^0 (\omega_1(z))^{n-1} |m(z)|\, ds(z)\, dm(y)/(n-1)! \\ &\leq \frac{|m(x)|}{(n-1)!} \int_x^0 (\omega_1(y))^{n-1} \int_y^0 ds(z)\, dm(y) = (\omega_1(x))^n \psi_1(x)/n!. \end{aligned}$$

Similarly we can prove (15). Clearly the functions $\psi_n(x)$, $\omega_n(x)$ are decreasing for $x < 0$, and so $\lim_{x \downarrow r_0} \psi_n(x)$, $\lim_{x \downarrow r_0} \omega_n(x)$ exist and are positive. Since (14) holds for all $x \in (r_0, 0)$, it follows that

$$\psi_n(r_0) \le (\omega_1(r_0))^{n-1}\psi_1(r_0)/(n-1)! < \infty.$$

Therefore $\sum_{n=1}^{\infty} \alpha^n \psi_n(r_0)$ converges for all real α. The assertions for $\omega_n(r_0)$ are proved similarly.

Lemma 10. *With the same assumptions as in Lemma 9, and $u_n(x)$, $v_n(x)$ being defined by (7), (8), we have the following equalities for all $x \in (r_0, 0)$ and $n \ge 1$:*

$$u_n(x) = \int_x^0 \psi_n(y)\, ds(y), \qquad v_n(x) = -\int_x^0 \omega_n(y)\, ds(y).$$

Therefore $u_n(x) \le \psi_n(r_0)|s(x)|$ and $|v_n(x)| \le \omega_n(r_0)|s(x)|$. As $x \downarrow r_0$, $u_n(x) \sim \psi_n(r_0)|s(x)|$ and $v_n(x) \sim \omega_n(r_0)s(x)$.

Proof. It is easy to see by induction that, for $n \ge 1$, $u_n(x) = \int_x^0 \psi_n(y)\, ds(y)$. From this, it follows that $u_n(x) \le \psi_n(r_0)|s(x)|$. On the other hand, given $\epsilon > 0$, we can choose $a_0 \in (r_0, 0)$ such that, if $r_0 < x \le a_0$, then $\psi_n(x) > \psi_n(r_0) - \epsilon$. For $x \in (r_0, a_0)$,

$$u_n(x) \ge \int_x^{a_0} \psi_n(y)\, ds(y) \ge (\psi_n(r_0) - \epsilon)(s(a_0) - s(x)).$$

Therefore $\liminf_{x\downarrow r_0} u_n(x)/|s(x)| \ge \psi_n(r_0) - \epsilon$. Since ϵ is arbitrary, $u_n(x) \sim \psi_n(r_0)|s(x)|$ as $x \downarrow r_0$. The assertions for $v_n(x)$ can be proved similarly. Q.E.D.

Lemma 11. *With the same assumptions as in Lemma 9, for any real number α, the series $\sum_{n=1}^{\infty} \alpha^n \psi_n(r_0)$, $\sum_{n=1}^{\infty} \alpha^n \omega_n(r_0)$ are convergent, and so we can define*

$$H(\alpha) = \sum_{n=1}^{\infty} \alpha^n \psi_n(r_0), \qquad K(\alpha) = 1 + \sum_{n=1}^{\infty} \alpha^n \omega_n(r_0). \tag{16}$$

Then

$$\lim_{x\downarrow r_0} u(x, \alpha)/|s(x)| = H(\alpha), \qquad \lim_{x\downarrow r_0} v(x, \alpha)/s(x) = K(\alpha).$$

Proof. Since $\sum_{n=1}^{\infty} |\alpha|^n \psi_n(r_0)$ converges by Lemma 9, given $\epsilon > 0$, we can choose n_0 such that $\sum_{n=n_0+1}^{\infty} |\alpha|^n \psi_n(r_0) < \epsilon$. By Lemma 10, for any $x \in (r_0, 0)$, $u_n(x) \le \psi_n(r_0)|s(x)|$, and so

$$\left| \sum_{n=n_0+1}^{\infty} \alpha^n u_n(x) \right| < \epsilon |s(x)|. \tag{17}$$

Moreover, Lemma 10 implies

$$\lim_{x\downarrow r_0} (1 + \alpha u_1(x) + \cdots + \alpha^{n_0} u_{n_0}(x))/|s(x)| = \sum_{n=1}^{n_0} \alpha^n \psi_n(r_0). \tag{18}$$

Since $H(\alpha) - \epsilon \le \sum_{n=1}^{n_0} \alpha^n \psi_n(r_0) \le H(\alpha) + \epsilon$, it follows from (17) and (18) that

$$H(\alpha) + 2\epsilon \ge \limsup_{x \downarrow r_0} \frac{u(x, \alpha)}{|s(x)|} \ge \liminf_{x \downarrow r_0} \frac{u(x, \alpha)}{|s(x)|} \ge H(\alpha) - 2\epsilon.$$

But ϵ is arbitrary, and so $\lim_{x \downarrow r_0} u(x, \alpha)/|s(x)| = H(\alpha)$. The proof for $v(x, \alpha)$ is similar. Q.E.D.

Lemma 12. *With the same assumptions as in Lemma* 9, *if* $\lambda > 0$, *then* $H(\lambda) > 0$ *and as* $a \downarrow r_0$,

$$|s(a)| \sim u_+(a, \lambda) \Big/ \left(H(\lambda) \int_{r_0}^{r_1} \frac{ds(y)}{(u(y, \lambda))^2} \right).$$

Proof. Since $\psi_n(r_0) > 0$ for all n, it is obvious that if $\lambda > 0$, then $H(\lambda) > 0$. By Lemma 11, $u(a, \lambda) \sim H(\lambda)|s(a)|$ as $a \downarrow r_0$. Since $\int_{r_0}^{r_1} ds(y)/(u(y, \lambda))^2 < \infty$, the desired conclusion follows. Q.E.D.

Lemma 13. *With the same assumptions as in Lemma* 9,

(i) *If* $f(a) = o(s(a))$ *as* $a \downarrow r_0$, *then given any* $x \in (r_0, r_1)$ *and* $t > 0$,

$$\lim_{a \downarrow r_0} f(a) P_x[\xi_a < t] = 0.$$

(ii) *Given any* $x \in (r_0, r_1)$, *there exists* $t > 0$ *such that*

$$\limsup_{a \downarrow r_0} |s(a)| P_x[\xi_a < t] > 0.$$

Proof. (i) We fix $\lambda > 0$ and note that, given any $t > 0$, $x \in (r_0, r_1)$,

$$|f(a)| P_x[\xi_a < t] \le e^{\lambda t} |f(a)| \int_0^\infty e^{-\lambda \tau} dP_x[\xi_a < \tau]$$

$$= e^{\lambda t} |f(a)| u_+(x, \lambda)/u_+(a, \lambda) \qquad \text{if } a < x.$$

Since $f(a) = o(s(a))$ and $s(a) = O(u_+(a, \lambda))$, (i) follows immediately.

(ii) Given $x \in (r_0, r_1)$, we take any $\lambda > 0$ and define

$$F_a(t) = |s(a)| \int_0^t e^{-\lambda \tau} dP_x[\xi_a < \tau], \qquad t \ge 0.$$

F_a is a continuous increasing function on $[0, \infty)$ with $F_a(0) = 0$. By Lemma 12, there exist a constant $c > 0$ and $a_0 \in (r_0, x \wedge 0)$ such that $|s(a)| \le c u_+(a, \lambda)/u_+(x, \lambda)$ for all $a \in (r_0, a_0)$. Now if $a \in (r_0, a_0)$,

$$\int_0^\infty dF_a = |s(a)| \frac{u_+(x, \lambda)}{u_+(a, \lambda)} \le c$$

and so F_a defines a measure on $[0, \infty)$ with total mass $\le c$. By the Helly-Bray Lemma,

$$\lim_{a\downarrow r_0} F_a(t) = 0 \quad \forall t \in [0, \infty) \quad \Rightarrow \lim_{a\downarrow r_0} \int_0^\infty e^{-\alpha t} dF_a(t) = 0 \quad \forall \alpha > 0.$$

Take any $\alpha > 0$. For $a \in (r_0, a_0)$,

$$\int_0^\infty e^{-\alpha t} dF_a(t) = |s(a)| \frac{u_+(x, \alpha+\lambda)}{u_+(a, \alpha+\lambda)}$$

$$\to u_+(x, \alpha+\lambda) \Big/ \left(H(\alpha+\lambda) \int_{r_0}^{r_1} \frac{ds(y)}{(u(y, \alpha+\lambda))^2} \right) \quad \text{as } a \downarrow r_0.$$

The last relation follows from Lemma 12. Since $F_a(0) = 0$ for all a, we can therefore find $t_0 \in (0, \infty)$ such that $F_a(t_0)$ does not converge to 0 as $a \downarrow r_0$, i.e., $\limsup_{a\downarrow r_0} F_a(t_0) > 0$. But $F_a(t_0) \leq |s(a)| P_x[\xi_a < t_0]$ and so $\limsup_{a\downarrow r_0} |s(a)| P_x[\xi_a < t_0] > 0$. Q.E.D.

Theorem 6. *Let $X(t)$ be a conservative regular continuous strong Markov process on an interval I whose endpoints are r_0, r_1, where $-\infty \leq r_0 < r_1 \leq \infty$. Let $s(x)$ be the scale and m the speed measure of $X(t)$.*

(i) *Suppose r_0, r_1 are inaccessible. If $f: (r_0, r_1) \times [0, \infty) \to (-\infty, \infty)$ is continuous, $\partial f/\partial t$ is also continuous and $D_m D_s^+ f$ exists and satisfies $\partial f/\partial t + D_m D_s^+ f = 0$ on $(r_0, r_1) \times [0, \infty)$, then f is parabolic for $X(t)$. Suppose furthermore that for $i = 0, 1$ and any $T > 0$,*

(a) $\max_{0\leq t\leq T} |f(x, t)| = o(s(x))$ *as $x \to r_i$, if r_i is an entrance boundary and*

(b) *there exists $\alpha_T > 0$ and a function $g_T: (r_0, r_1) \to (-\infty, \infty)$ satisfying $D_m D_s^+ g_T = \alpha_T g_T$ such that* $\max_{0\leq t\leq T} |f(x, t)| = O(|g_T(x)|)$ *as $x \to r_i$, if r_i is a natural boundary.*

Then $f(X(t), t+r)$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$ and all $r \geq 0$.

(ii) *If r_0 is an entrance boundary and g is a positive decreasing solution of $D_m D_s^+ g = \lambda g$ with $\lambda > 0$, then* $\lim_{x\downarrow r_0} g(x)/|s(x)|$ *exists and is positive, and $\{s(X(t)), t \geq 0\}$, $\{e^{-\lambda t} g(X(t)), t \geq 0\}$ are not martingales with respect to P_x for any $x \in (r_0, r_1)$. A similar assertion (replacing 'positive drecreasing solution' in the above by 'positive increasing solution') holds if r_1 is an entrance boundary.*

(iii) *Suppose r_0 is an entrance boundary and r_1 is a natural boundary. If h is a positive increasing solution of $D_m D_s^+ h = \lambda h$ with $\lambda > 0$, then $e^{-\lambda t} h(X(t))$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$. Let $\alpha < 0$. For notational convenience, assume $0 \in (r_0, r_1)$, $s(0) = m(0) = m(0-) = 0$, and define $H(\alpha)$, $K(\alpha)$ by* (16). *If $H(\alpha)$ and $K(\alpha)$ are not both zero, then the vector space of all solutions g of $D_m D_s^+ g = \alpha g$ such that $e^{-\alpha t} g(X(t))$, $t \geq 0$, is a martingale with respect to P_x for any $x \in (r_0, r_1)$ has dimension* 1. *If $H(\alpha)$ and $K(\alpha)$ are both equal to zero, then, for any solution g of $D_m D_s^+ g = \alpha g$, $e^{-\alpha t} g(X(t))$, $t \geq 0$, is a* *martingale with respect to* P_x *for any* $x \in (r_0, r_1)$.

Proof. We may without loss of generality assume throughout that $0 \in (r_0, r_1)$, $s(0) = m(0) = m(0-) = 0$.

(i) Suppose r_0, r_1 are both inaccessible. If $f\colon (r_0, r_1) \times [0, \infty) \to (-\infty, \infty)$ is continuous, $\partial f/\partial t$ is also continuous and $D_m D_s^+ f$ exists and satisfies $\partial f/\partial t + D_m D_s^+ f = 0$ on $(r_0, r_1) \times [0, \infty)$, then we have already proved that f is parabolic for $X(t)$. Let $r \geq 0$, $t \geq 0$. If r_0 is an entrance boundary, then it follows from condition (a) and Lemma 13(i) that, for $x \in (r_0, r_1)$, $r_0 < a < b < r_1$,

$$\int_{[\xi_a < r \wedge \xi_b]} |f(a, t + \xi_a)|\, dP_x \leq \Big(\max_{0 \leq \sigma \leq t+r} |f(a, \sigma)|\Big) P_x[\xi_a < r] \to 0 \quad \text{as } a \downarrow r_0.$$

If r_0 is a natural boundary, then it follows from condition (b) and Lemma 6 that

$$\int_{[\xi_a < r \wedge \xi_b]} |f(a, t + \xi_a)|\, dP_x \leq C|g_T(a)| \cdot P_x[\xi_a < r] \to 0 \quad \text{as } a \downarrow r_0.$$

Similarly we can prove that, under conditions (a) and (b),

$$\lim_{b \uparrow r_1} \int_{[\xi_b \leq r \wedge \xi_a]} |f(b, t + \xi_b)|\, dP_x = 0.$$

Take any $t \geq 0$, $r \geq 0$ and $\lambda > 0$. By Lemma 11, $s(x) = O(u(x, \lambda))$ as $x \to r_i$ if r_i is an entrance boundary. Therefore conditions (a) and (b) imply that there exist K such that $|f(x, t + r)| \leq Ku(x, \lambda)$ for all $x \in (r_0, r_1)$. Now $u(x, \lambda) \geq e^{-\lambda r} \int u(y, \lambda) P(r, x, dy)$. Therefore $\int |f(y, t + r)| P(r, x, dy) < \infty$, and by the dominated convergence theorem,

$$\lim_{a \downarrow r_0,\, b \uparrow r_1} \int_{[\xi_{ab} \geq r]} f(X(r), t + r)\, dP_x = \int f(y, t + r) P(r, x, dy).$$

Hence by Theorem 4, $f(X(t), t + r)$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$ and $r \geq 0$.

(ii) Since g is a positive decreasing solution of $D_m D_s^+ g = \lambda g$, $g(x) = c_1 u_+(x, \lambda) + c_2 u_-(x, \lambda)$ with $c_1 > 0$. (Note that c_2 must vanish if r_1 is also an inaccessible boundary.) Therefore by Lemma 12,

$$\lim_{x \downarrow r_0} g(x)/|s(x)| = \lim_{x \downarrow r_0} c_1 u_+(x, \lambda)/|s(x)|$$

indeed exists and is positive.

Take any $x \in (r_0, r_1)$. By Lemma 13, there exists $t_0 > 0$ such that $\limsup_{a \downarrow r_0} |s(a)| P_x[\xi_a < t_0] > 0$. Therefore we also have $\limsup_{a \downarrow r_0} g(a) P_x[\xi_a < t_0] > 0$. Apply Theorem 4 with $f(z, t)$ equal to $s(z)$ or $e^{-\lambda t} g(z)$. Then condition (i) of that theorem is violated and therefore $\{s(X(t)), t \geq 0\}$ and $\{e^{-\lambda t} g(X(t)), t \geq 0\}$ cannot be martingales with respect to P_x.

(iii) If h is a positive increasing solution of $D_m D_s^+ h = \lambda h$ with $\lambda > 0$, then

$f(x, t) = e^{-\lambda t}h(x)$ satisfies conditions (a), (b) of (i), and so $e^{-\lambda t}h(X(t))$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$.

Take $\alpha < 0$. Suppose $H(\alpha) \neq 0$, By Lemma 11, $u(a, \alpha) \sim H(\alpha)|s(a)|$ as $a \downarrow r_0$. Therefore given $x \in (r_0, r_1)$, there exists $t_0 > 0$ such that

$$\limsup_{a \downarrow r_0} |u(a, \alpha)| P_x[\xi_a < t_0] > 0,$$

and $e^{-\alpha t}u(X(t), \alpha)$, $t \geq 0$, is not a martingale with respect to P_x for any $x \in (r_0, r_1)$.

Assume for definiteness that $H(\alpha) > 0$. (The case $H(\alpha) < 0$ is treated similarly.) Then $u(a, \alpha) \sim H(\alpha)|s(a)|$ as $a \downarrow r_0$. Therefore we can choose $c > 0$ and $x_0 < 0$ such that $u(y, \alpha) \geq c|s(y)|$ on $(r_0, x_0]$. We can then define $u_{(\alpha)}: (r_0, x_0) \to (-\infty, \infty)$ by

$$u_{(\alpha)}(x) = u(x, \alpha) \int_{r_0}^{x} \frac{ds(y)}{(u(y, \alpha))^2}.$$

It can be proved that $D_m D_s^+ u_{(\alpha)}(x) = \alpha u_{(\alpha)}(x)$, $x \in (r_0, x_0)$ (cf. pp. 27–28 of [9]). Since $u(x, \alpha)$, $v(x, \alpha)$ are two linearly independent solutions of $D_m D_s^+ g = \alpha g$ on (r_0, x_0), there exist unique constants c_1, c_2 such that

$$u_{(\alpha)}(x) = c_1 u(x, \alpha) + c_2 v(x, \alpha), \qquad x \in (r_0, x_0).$$

Let $w(x) = c_1 u(x, \alpha) + c_2 v(x, \alpha)$, $x \in (r_0, r_1)$. Then $D_m D_s^+ w = \alpha w$ on (r_0, r_1). It is easy to see that

$$\lim_{x \downarrow r_0} w(x) = \lim_{x \downarrow r_0} u_{(\alpha)}(x) = 1/H(\alpha).$$

From this, it is clear that $u(x, \alpha)$ and $w(x)$ are linearly independent. Also the function $e^{-\alpha t}w(x)$ satisfies conditions (a), (b) of (i) and so $e^{-\alpha t}w(X(t))$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$.

Let g be any solution of $D_m D_s^+ g = \alpha g$ such that $e^{-\alpha t}g(X(t))$, $t \geq 0$, is a martingale with respect to P_x for any $x \in (r_0, r_1)$. Then $g(x) = K_1 u(x, \alpha) + K_2 w(x)$. If $K_1 \neq 0$, $\lim_{x \downarrow r_0} g(x)/|s(x)| = K_1 H(\alpha) \neq 0$, and so $e^{-\alpha t}g(X(t))$, $t \geq 0$, cannot be a martingale with respect to P_x for any $x \in (r_0, r_1)$, contradicting the assumption on g. Therefore $K_1 = 0$ and g is a scalar multiple of w.

The case $K(\alpha) \neq 0$ can be treated in an analogous way.

Now suppose that $H(\alpha) = 0$ and $K(\alpha) = 0$. Then by Lemma 11, $u(a, \alpha) = o(s(a))$ as $a \downarrow r_0$ and $v(a, \alpha) = o(s(a))$ as $a \downarrow r_0$. Therefore for any solution g of $D_m D_s^+ g = \alpha g$, $g(a) = o(s(a))$ as $a \downarrow r_0$. Hence the function $f(x, t) = e^{-\lambda t}g(x)$ satisfies conditions (a), (b) of (i) and so $e^{-\lambda t}g(X(t))$, $t \geq 0$, is a martingale with respect to P_x for all $x \in (r_0, r_1)$. Q.E.D.

5. **Application to boundary crossing probabilities.** Let $(Y(t), \zeta, \mathfrak{F}_t^s, P_{s,x})$ be a continuous normal strong Markov process on a locally compact, second count-

able, Hausdorff space M. Let $Z(t)$ be the corresponding space-time process. For any open subset G of $M \times [0, \infty)$, set

$$\psi(x, s) = P_{s,x}[(Y(t), t) \notin G \text{ for some } s \le t < \zeta].$$

The function ψ is parabolic for $Y(t)$ on G. In particular, let us consider the case where $Y(t)$ is the diffusion in Theorem 3. Then by that theorem, ψ belongs to $C^{2,1}(G)$ and satisfies the partial differential equation $L\psi = 0$ on G, where L is the backward parabolic operator defined in (3).

Now suppose that $X(t)$ is a regular conservative continuous strong Markov process on (r_0, r_1) with speed measure m and scale function s such that the boundaries r_0, r_1 are both inaccessible $(-\infty \le r_0 < r_1 \le \infty)$. For $\lambda > 0$, let $f_\lambda^{(0)}$ be a positive decreasing solution and let $f_\lambda^{(1)}$ be a positive increasing solution of $D_m D_s^+ f = \lambda f$. Let F_i $(i = 0, 1)$ be a nontrivial measure on $(0, \infty)$ such that $\phi_i(x, \tau) < \infty$ for all $x \in (r_0, r_1)$, where we define

$$\phi_i(x, t) = \int_0^\infty e^{-\lambda t} f_\lambda^{(i)}(x)\, dF_i(\lambda).$$

If r_0, r_1 are both natural boundaries, then since $\lim_{x\uparrow r_1} f_\lambda^{(1)}(x) = \infty$ and $\lim_{x\downarrow r_0} f_\lambda^{(1)}(x) = 0$, therefore given $\epsilon > 0$, we have a unique solution $x = B_1(t, \epsilon)$, with $r_0 < x < r_1$, of the equation $\phi_1(x, t) = \epsilon$ for any $t \ge \tau$. By our results in §4, $\phi_1(X(t), t)$, $t \ge \tau$, is a martingale with respect to P_x for any $x \in (r_0, r_1)$. If $\phi_1(X(t), t)$ converges to 0 in probability as $t \to \infty$ on the event $[\sup_{t\ge\tau} \phi_1(X(t), t) < \epsilon]$, then it follows from [10] that

$$\begin{aligned} &P_x[X(t) \ge B_1(t, \epsilon) \text{ for some } t \ge \tau] \\ (19)\quad &= P_x[X(\tau) \ge B_1(\tau, \epsilon)] + \frac{1}{\epsilon}\int_0^\infty \int_{[X(\tau) < B_1(\tau,\epsilon)]} e^{-\lambda\tau} f_\lambda^{(1)}(X(\tau))\, dP_x\, dF_1(\lambda). \end{aligned}$$

Likewise given any $\epsilon > 0$, there exists a unique solution $x = B_0(t, \epsilon)$ of the equation $\phi_0(x, t) = \epsilon$ for $t \ge \tau$. Also $\phi_0(X(t), t)$, $t \ge \tau$, is a martingale with respect to P_x for any $x \in (r_0, r_1)$. If $\phi_0(X(t), t)$ converges to 0 in probability on $[\sup_{t\ge\tau} \phi_0(X(t), t) < \epsilon]$, then again it follows from [10] that

$$\begin{aligned} &P_x[X(t) \le B_0(t, \epsilon) \text{ for some } t \ge \tau] \\ (20)\quad &= P_x[X(\tau) \le B_0(\tau, \epsilon)] + \frac{1}{\epsilon}\int_0^\infty \int_{[X(\tau) > B_0(\tau,\epsilon)]} e^{-\lambda\tau} f_\lambda^{(0)}(X(\tau))\, dP_x\, dF_0(\lambda). \end{aligned}$$

When the boundaries are both inaccessible, but not both natural, we still have $\lim_{x\to r_i} f_\lambda^{(i)}(x) = \infty$, $i = 0, 1$. But $\lim_{x\to r_0} f_\lambda^{(1)}(x)$ may not be zero if r_0 is not natural. However, given any $\epsilon > \phi_1(r_0, \tau)$, we still have a unique solution $x = B_1(t, \epsilon)$ of the equation $\phi_1(x, t) = \epsilon$ for any $t \ge \tau$. If r_1 is not natural, then $e^{-\lambda t} f_\lambda^{(1)}(X(t))$, $t \ge 0$, fails to be a martingale. However, the function $\phi_1(x, t)$ is parabolic for $X(t)$. Let $\epsilon > \phi_1(r_0, \tau)$. Take any $\epsilon' > \epsilon$ and define $\sigma = \inf\{t \ge \tau \mid \phi_1(X(t), t) \ge \epsilon'\}$. Then $\phi_1(X(t \wedge \sigma), t \wedge \sigma)$, $t \ge \tau$, is a martingale with respect to P_x for any $x \in (r_0, r_1)$. Therefore if $\phi_1(X(t), t)$ converges to 0 in probability as $t \longrightarrow \infty$ on $[\sup_{t \geq \tau} \phi_1(X(t), t) < \epsilon]$, then it follows from [10] that

$$
\begin{aligned}
&P_x[X(t) \geq B_1(t, \epsilon) \text{ for some } t \geq \tau] \\
(21) \qquad &= P_x[\phi_1(X(t \wedge \sigma), t \wedge \sigma) \geq \epsilon \text{ for some } t \geq \tau] \\
&= P_x[X(\tau) \geq B_1(\tau, \epsilon)] + \frac{1}{\epsilon}\int_0^\infty \int_{[X(\tau)<B_1(\tau,\epsilon)]} e^{-\lambda\tau} f_\lambda^{(1)}(X(\tau))\, dP_x\, dF_1(\lambda).
\end{aligned}
$$

In the case where $m(r_1) < \infty$ and $m(r_0) > -\infty$, the transition probabilities $P(t, x, \cdot)$ converge weakly to a probability distribution on (r_0, r_1) whose distribution function $G(y)$ is $(m(y) - m(r_0))(m(r_1) - m(r_0))^{-1}$ as $t \longrightarrow \infty$ (cf. [9]). It is obvious in this case that, for $i = 0, 1$, the martingale $e^{-\lambda t} f_\lambda^{(i)}(X(t))$ converges to zero in probability (and therefore almost surely by the martingale convergence theorem) as $t \longrightarrow \infty$ for any initial state x. Suppose F_i is any finite measure with bounded support on $(0, \infty)$. Then it easily follows that $\phi_i(X(t), t)$ converges to zero almost surely as $t \longrightarrow \infty$ for any initial state x, and so (19), (20) and (21) can be applied to give the probability that $X(t)$ would ever cross certain moving boundaries.

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DEPARTMENT OF MATHEMATICAL STATISTICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027