TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 176, February 1973

PURELY INSEPARABLE, MODULAR EXTENSIONS OF UNBOUNDED EXPONENT

BY

LINDA ALMGREN KIME

ABSTRACT. Let K be a purely inseparable extension of a field k of characteristic $p \neq 0$. Sweedler has shown in [2, p. 403] that if K over k is of finite exponent, then K is modular over k if and only if K can be written as the tensor product of simple extensions of k. This paper grew out of an attempt to find an analogue to this theorem if K is of unbounded exponent over k. The definition of a simple extension is extended to include extensions of the form $k[x, x^{1/p}, x^{1/p^2}, \ldots][x^{1/p^\infty}]$. If K is the tensor product of simple extensions, then K is modular. The converse, however, is not true, as several counterexamples in §4 illustrate. Even if we restrict $[k : k^p] < \infty$, the converse is still shown to be false.

Given K over k modular, we construct a field $\bigcap_{i=1}^{\infty} kK^{p^i} \otimes M$ $(= Q)$ that always imbeds in K where M is the tensor product of simple extensions in the old sense. In general $K \neq Q$. For K to be the tensor product of simple extensions, we need $K = Q$, and $\bigcap_{i=1}^{\infty} kK^{p^i} = k(\bigcap_{i=1}^{\infty} K^{p^i})$. If for some finite N, $kK^{p^N} = kK^{p^{N+1}}$, then we have (by Theorem 11) that $K = Q$. This finiteness condition guarantees that M is of finite exponent. Should $\bigcap_{i=1}^{\infty} kK^{p^i} = k$, then we would have the condition of Sweedler's original theorem.

The counterexamples in §4 will hopefully be useful to others interested in unbounded exponent extensions. Of more general interest are two side theorems on modularity. These state that any purely inseparable field extension has a unique minimal modular closure, and that the intersection of modular extensions is again modular.

1. **Modular extensions.** All fields under consideration will be purely inseparable extensions of a common ground field k. They are to be viewed as contained in a common algebraically closed field (e.g. the algebraic closure of k). If two fields are said to be linearly disjoint, we mean that they are linearly disjoint over their intersection. A field extension K of k is said to be *modular* if K^{p^i} and k are linearly disjoint for all positive i. K is of exponent n over k if n is the minimal positive number such that $K^{p^n} \subset k$. If no such n exists, then K is of unbounded or infinite exponent over k. Unless otherwise indicated, the intersection and unions throughout are taken from i (or n) $= 1$ to ∞.

Received by the editors February 14, 1972.

AMS (MOS) subject classifications (1970). Primary 12F15.

Key words and phrases. Purely inseparable, modular, modular closure, infinite or unbounded exponent, tensor product of simple extensions.

Lemma 1. *Let $\{M_\alpha\}$ be a set of fields such that, for any $M_{\alpha_1}, M_{\alpha_2} \in \{M_\alpha\}$, there exists an $M_{\alpha_3} \in \{M_\alpha\}$ such that both M_{α_1} and M_{α_2} lie in M_{α_3}. If each M_α and Q are linearly disjoint, then $\bigcup_\alpha M_\alpha$ and Q are linearly disjoint.*

Proof. Given any set $\{x_1, \cdots, x_n\} \subset \bigcup_\alpha M_\alpha$, there exists an α_0 such that $\{x_1, \cdots, x_n\} \subset M_{\alpha_0} \in \{M_\alpha\}$. If $\{x_1, \cdots, x_n\}$ is linearly dependent over Q, then by the linear disjointness of M_{α_0} and Q, $\{x_1, \cdots, x_n\}$ is linearly dependent over $M_{\alpha_0} \cap Q$. Hence $\{x_1, \cdots, x_n\}$ is linear dependent over $\bigcup_\alpha M_\alpha \cap Q$ ($\supset M_{\alpha_0} \cap Q$). Q.E.D.

Lemma 2. *Let $\{M_\alpha\}$ be as in Lemma 1. If each M_α is modular over k, then $\bigcup_\alpha M_\alpha$ is modular over k.*

Proof. We must show that $(\bigcup_\alpha M_\alpha)^{p^n}$ $(= \bigcup_\alpha M_\alpha^{p^n})$ and k are linearly disjoint for all $n \geq 1$. Given any $\{x_1, \cdots, x_s\} \subset \bigcup_\alpha M_\alpha^{p^n}$ there exists an α_0 such that $\{x_1, \cdots, x_s\} \subset M_{\alpha_0}^{p^n}$. If $\{x_1, \cdots, x_s\}$ is linearly independent over $k \cap (\bigcup_\alpha M_\alpha^{p^n})$, it is linearly independent over $k \cap M_{\alpha_0}^{p^n}$. The modularity of M_{α_0} over k guarantees that $\{x_1, \cdots, x_s\}$ is linearly independent over k. Q.E.D.

Proposition 3. *K is modular over $k \Leftrightarrow k^{1/p^n} \cap K$ is modular over k for all positive n.*

Proof. Assume K modular over k. We want to show that $(k^{1/p^n} \cap K)^{p^i}$ and k are linearly disjoint for all positive i.

Case 1. $n > i$. $(k^{1/p^n} \cap K)^{p^i} = k^{1/p^{n-i}} \cap K^{p^i}$. If $\{x_1, \cdots, x_n\} \subset k$ is linearly independent over $K^{p^i} \cap k = (k^{1/p^{n-i}} \cap K^{p^i}) \cap k$, by the modularity of K over k, $\{x_1, \cdots, x_n\}$ is linearly independent over K^{p^i}, hence over $k^{1/p^{n-i}} \cap K^{p^i}$.

Case 2. $n \leq i$. $(k^{1/p^n} \cap K)^{p^i} = k^{p^{i-n}} \cap K^{p^i}$. Since $k^{p^{i-n}} \cap K^{p^i} \cap k = k^{p^{i-n}} \cap K^{p^i}$, $(k^{1/p^n} \cap K)^{p^i}$ and k are trivially linearly disjoint.

Conversely, if $k^{1/p^n} \cap K$ is modular over k for any positive n, then $\bigcup(k^{1/p^n} \cap K) = K$ is modular over k by Lemma 2. Q.E.D.

Sweedler has shown in [2, p. 408] that any purely inseparable extension K over k of finite exponent is contained in a unique minimal field extension L, where L is modular over k. L is called the *modular closure* of K over k. We now extend this definition to the infinite exponent case.

Theorem 4. *Let K be a purely inseparable field extension of k.*

(1) *There exists a unique minimal field extension $L \supset K$ where L is modular over k.*

(2) *L is purely inseparable over k.*

(3) *If K is of exponent n, then L has exponent n over k.*

Proof. Since $k^{1/p^i} \cap K$ has exponent p^i over k, by [2, p. 408] $k^{1/p^i} \cap K$ has a corresponding modular closure L_i satisfying (1)–(3). The minimality of each L_i guarantees that $L_1 \subset L_2 \subset L_3 \subset \cdots$. By Lemma 2, $\bigcup L_i = L$ is modular over k. L clearly contains K, since $k^{1/p^i} \cap K \subset L_i$ and $K = \bigcup (k^{1/p^i} \cap K)$. Any field extension of K that is modular over k must contain all the L_i (by the minimality of each L_i). Thus L must be the unique minimal such field. Q.E.D.

Naturally L is called the *modular closure* of K over k.

Corollary 5. *Let $\{M_\alpha\}$ be a set of fields such that each M_α is modular over a common ground field k. Then $\bigcap_\alpha M_\alpha$ is modular over k.*

Proof. Let N be the modular closure of $\bigcap_{\alpha \in A} M_\alpha$. By minimality, $N \subset M_\alpha$ for all $\alpha \in A$. Hence $N \subset \bigcap_\alpha M_\alpha$. Q.E.D.

2. **A structure theorem for field extensions of unbounded exponent.**

Lemma 6. *If K over k is modular, then $k(k^{1/p^i} \cap K^p) = k^{1/p^i} \cap kK^p$ for any positive i.*

Proof. $K = \bigcup (k^{1/p^i} \cap K) \Rightarrow K^p = \bigcup (k^{1/p^i} \cap K^p) \Rightarrow kK^p = \bigcup k(k^{1/p^i} \cap K^p)$.

If $\{x_n\}$ is a p-basis for $k^{1/p^n} \cap K^p$ over $k^{1/p^{n-1}} \cap K^p$, $\{x_n\}$ is p-independent over $k^{1/p^{n-1}} \cap kK^p \Leftrightarrow \{x_n^{p^{n-1}}\}$ is p-independent over $k \cap k^{p^{n-1}} K^{p^n}$. $\{x_n^{p^{n-1}}\}$ $(\subset (k^{1/p} \cap K^{p^n}) \subset K^{p^n})$ is by assumption p-independent over $(k^{1/p^{n-1}} \cap K^p)^{p^{n-1}} = k \cap K^{p^n}$, so by the linear disjointness of k and K^{p^n}, $\{x_n^{p^{n-1}}\}$ is p-independent over k, hence p-independent over $k \cap k^{p^{n-1}} K^{p^n}$. So $\{x_n\}$ is p-independent over $k^{1/p^{n-1}} \cap kK^p$. We also have that $\{x_n^p\} \subset (k \cap K^p)[\{x_1\}, \{x_2\}, \cdots, \{x_{n-1}\}] \subset k[\{x_1\}, \cdots, \{x_{n-1}\}]$ where each $\{x_i\}$ is a p-basis for $k^{1/p^i} \cap K^p$ over $k^{1/p^{i-1}} \cap K^p$.

$$kK^p = \bigcup k(k^{1/p^i} \cap K^p) = k[\{x_1\}, \{x_2\}, \cdots]$$

where each $\{x_i\} \in k^{1/p^i} \cap kK^p$ is p-independent over $k^{1/p^{i-1}} \cap kK^p$,

$\{x_i^p\} \subset k[\{x_1\}, \{x_2\}, \cdots, \{x_{i-1}\}]$

$\Rightarrow$ each $\{x_i\}$ is a p-basis for $k^{1/p^i} \cap kK^p$ over $k^{1/p^{i-1}} \cap kK^p$

$\Rightarrow k^{1/p^i} \cap kK^p = k[\{x_1\}, \{x_2\}, \cdots, \{x_i\}]$

$= k[(k \cap K^p)[\{x_1\}, \cdots, \{x_i\}] = k[k^{1/p^i} \cap K^p]$. Q.E.D.

Proposition 7. *If K over k is a modular extension then kK^{p^n} is modular over k for any positive n.*

Proof. Let $\{y_1\}, \{y_2\}, \cdots, \{y_{i+1}\}$ be a modular basis for $k^{1/p^{i+1}} \cap K$ over k, i.e., $k^{1/p^{i+1}} \cap K = k[\{y_1\}] \otimes k[\{y_2\}] \otimes \cdots \otimes k[\{y_{i+1}\}]$ where, for each $y_j \in \{y_j\}$, y_j has exponent j over k. Then

$$\begin{aligned}k^{1/p^i} \cap kK^p &= k[k^{1/p^i} \cap K^p] \text{ (by Lemma 6)}\\ &= k[(k^{1/p^{i+1}} \cap K)^p] = k[\{y_1^p\}, \cdots, \{y_{i+1}^p\}]\\ &= k[\{y_2^p\}] \otimes \cdots \otimes k[\{y_{i+1}^p\}]\end{aligned}$$

$\Rightarrow k^{1/p^i} \cap kK^p$ is modular over k for any $i \geq 0$,

$\Rightarrow$ (by Proposition 3) that kK^p is modular over k.

Now replacing K by kK^p, we have that $k(kK^p)^p = kK^{p^2}$ is modular over k, hence in general that kK^{p^n} is modular over k. Q.E.D.

Lemma 8. *If K is relatively perfect over k, i.e. $K = kK^p = kK^{p^i}$ for all positive i, and K is modular over k, then $k^{1/p} \cap K = k^{1/p} \cap kK^{p^n} = k[k^{1/p} \cap K^{p^n}]$ for any positive n.*

Proof. Since K is modular over k, we have from Lemma 6 that $k^{1/p^i} \cap kK^p = k[k^{1/p^i} \cap K^p]$ for any $i \geq 0$. Using this and the fact that $K = kK^{p^n}$ for any $n \geq 0$, we have that

$$\begin{aligned}k^{1/p} \cap K &= k^{1/p} \cap kK^p = k[k^{1/p} \cap K^p]\\ &= k[(k^p)^{1/p^2} \cap k^pK^{p^2}] = k \cdot k^p[(k^p)^{1/p^2} \cap k^{p^2}K^{p^3}] = k[k^{1/p} \cap K^{p^2}].\end{aligned}$$

Applying this argument repeatedly we get that $k^{1/p} \cap K = k^{1/p} \cap kK^{p^n} = k[k^{1/p} \cap K^{p^n}]$. Q.E.D.

Proposition 9. *If K over k is a modular extension then the relatively perfect subfield $\bigcap kK^{p^i}$ is modular over k. If $\{x\}$ is a p-basis for $k^{1/p^{n+1}} \cap (\bigcap kK^{p^i})$ over $k^{1/p^n} \cap (\bigcap kK^{p^i})$, then $\{x^{p^n}\}$ is a p-basis for $k^{1/p} \cap (\bigcap kK^{p^i})$ over k.*

Proof. Apply Corollary 5 and Proposition 7 to obtain modularity. For simplicity denote $\bigcap kK^{p^i}$ by L. The modularity of L over $k \Rightarrow \{x^{p^n}\}$ is p-independent over k.

$$\begin{aligned}k^{1/p} \cap L &= k^{1/p} \cap kL^{p^n} = k[k^{1/p} \cap L^{p^n}] \text{ (by Lemma 6)}\\ &= k[k^{1/p^{n+1}} \cap L]^{p^n} = k[(k^{1/p^n} \cap L)[\{x\}]]^{p^n}\\ &= k(k \cap L^{p^n})[\{x^{p^n}\}] = k[\{x^{p^n}\}]. \quad \text{Q.E.D.}\end{aligned}$$

We note that a relatively perfect field extension K over k ($K = kK^p = \bigcap kK^{p^i}$) is not necessarily modular, as the following example shows.

Let $k = \mathbf{Z}_3(x^p, y^p, z^{p^2})$, x, y, z indeterminates. Denote any set $\{w, w^{1/p}, w^{1/p^2}, \ldots\}$ by w^{1/p^∞}. Set $K = k[z^{1/p^\infty}, (xz + y)^{1/p^\infty}]$. $k^{1/p^2} \cap K$ over k has diagram

z, $xz + y$
z^p

(cf. [2, p. 402]) $\Rightarrow k^{1/p^2} \cap K$ over k is not a modular extension. By Proposition 3, K is not modular over k.

Lemma 10. *If A, B are field extensions of k, such that A is modular over k and $k^{1/p} \cap A$, B are linearly disjoint over k, then A, B are linearly disjoint over k.*

Proof. If $\{x_n\}$ is a p-basis for $k^{1/p^{n+1}} \cap A$ over $k^{1/p^n} \cap A$, then $\{x_n\}$ is p-independent over $(k^{1/p^n} \cap A)(B)$. Otherwise $\{x_n^{p^n}\}$ $(\subset k^{1/p} \cap A)$, which is p-independent over k, would be p-dependent over $(k \cap A^{p^n})(B^{p^n}) \subset B$, contradicting the linear disjointness of $k^{1/p} \cap A$ and B. So the basis for A over k consisting of monomials of finite products of the x_i is a basis for $A \cdot B$ over $B \Rightarrow A \otimes_k B \cong AB \Rightarrow A, B$ are linearly disjoint over k. Q.E.D.

Theorem 11. *Any modular field extension K over k, where for some finite N, $kK^{p^N} = kK^{p^{N+1}}$, is isomorphic to $\bigcap kK^{p^i} \otimes M$ where M is a modular subfield of K of finite exponent.*

Proof. Let $\{x_1\}$ be a p-basis for $k^{1/p^2} \cap K$ over $k^{1/p} \cap K$. By modularity $\{x_1^p\}$ is p-independent/k. Let A_1 be a completion of $\{x_1^p\}$ to a p-basis for $k^{1/p} \cap K$ over k. A_1 is not unique, but the choice of A_1 is independent of the choice of $\{x_1\}$, since A_1 is precisely a p-basis for $k^{1/p} \cap K$ over $k[k^{1/p} \cap K^p]$ (which equals $k^{1/p} \cap kK^p$ by Lemma 6). Hence $k^{1/p} \cap K = (k^{1/p} \cap kK^p) \otimes k[A_1] \Rightarrow kK^p \otimes k[A_1] \hookrightarrow K$ (by Lemma 10).

Let $\{x_2\}$ be a p-basis for $k^{1/p^3} \cap K$ over $k^{1/p^2} \cap K$. $\{x_2^p\}$ is p-independent over $k^{1/p} \cap K$. Choose A_2 as any completion of $\{x_2^p\}$ to a p-basis for $k^{1/p^2} \cap K$ over $k^{1/p} \cap K$, i.e., a p-basis for $k^{1/p^2} \cap K$ over $(k^{1/p} \cap K)(k^{1/p^2} \cap K^p)$. $k[\{x_2^{p^2}\}] = k[k^{1/p} \cap K^{p^2}] = k^{1/p} \cap kK^{p^2}$, so $k^{1/p} \cap K = (k^{1/p} \cap kK^{p^2}) \otimes k[A_2^p] \otimes k[A_1] \Rightarrow kK^{p^2} \otimes k[A_2] \otimes k[A_1] \hookrightarrow K$.

In general, we let A_n be a p-basis for $k^{1/p^n} \cap K$ over $(k^{1/p^{n-1}} \cap K)(k^{1/p^n} \cap K^p)$. $k^{1/p} \cap K = (k^{1/p} \cap kK^{p^n}) \otimes k[A_n^{p^{n-1}}] \otimes \cdots \otimes k[A_1] \Rightarrow kK^{p^n} \otimes k[A_n] \otimes \cdots \otimes k[A_1] \hookrightarrow K$. So for any finite n, kK^{p^n} and $k[A_1 \cup \cdots \cup A_n]$ are linearly disjoint over $k \Rightarrow \bigcap kK^{p^i}$ and $k[A_1 \cup \cdots \cup A_n]$ are linearly disjoint over k. By Lemma 1, $\bigcup k[A_1 \cup \cdots \cup A_n] = k[A_1 \cup A_2 \cup \cdots]$ and $\bigcap kK^{p^i}$ are linearly disjoint over k, i.e., $\bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]$ imbeds in K.

By assumption, there exists a finite N such $kK^{p^N} = kK^{p^{N+1}}$, so $A_i = \emptyset$ for all $i > N$. Set $M = k[A_1 \cup \cdots \cup A_N]$. M is modular since $k[A_1 \cup \cdots \cup A_N] = k[A_1] \otimes k[A_2] \otimes \cdots \otimes k[A_N]$ (cf. §3). We assert that $K = \bigcap kK^{p^i} \otimes M$. Let $\{x_1, \cdots, x_s\}$ be a p-basis for $k^{1/p^{i+1}} \cap K$ over $k^{1/p^i} \cap K$, where $i > N$. Then $k^{1/p} \cap K = k[x_1^{p^i}, \cdots, x_s^{p^i}] \otimes k[A_N^{p^{N-1}}] \otimes \cdots \otimes k[A_1]$, where $x_1^{p^i}, \cdots, x_s^{p^i}$ are p-independent over k. $k[x_1^{p^i}, \cdots, x_s^{p^i}] = k[k^{1/p} \cap K^{p^i}] = k^{1/p} \cap kK^{p^i} = k^{1/p} \cap (\bigcap kK^{p^i})$ (since $i > N$). Proposition 9 says that we may assume that $\{x_1, \cdots, x_s\} \subset k^{1/p^{i+1}} \cap (\bigcap kK^{p^i})$. Q.E.D.

Remark 12. Either condition that $[k^{1/p} \cap K: k] < \infty$ or $[k : k^p] < \infty$ implies that, for some N, $kK^{p^N} = kK^{p^{N+1}}$. Since if $[k^{1/p} \cap K : k] < \infty$ (note $[k : k^p] < \infty$ implies that $[k^{1/p} \cap K : k] < \infty$), then $A_N = \varnothing$ for some $N \Rightarrow kK^{p^N} = kK^{p^{N+1}}$.

With only the restriction that K over k is modular we always have that $\bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]$ imbeds in K. Since there may be infinitely many non-empty A_i's, $k[A_1 \cup A_2 \cup \cdots]$ is not, in general, of finite exponent. We can describe $k[A_1 \cup A_2 \cup \cdots]$ as being of locally finite exponent.

Definition. K/k, a purely inseparable extension, is said to be of locally finite exponent if, for any field L, $k \subset L \subset K$, where $[k^{1/p} \cap L : k] < \infty$, then L has finite exponent over k.

$k[A_1 \cup A_2 \cup \cdots]$ is easily seen to be of locally finite exponent. Given any intermediate field L, such that $[k^{1/p} \cap L : k] < \infty$, there must exist a finite n such that $L \subset k[A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}^p \cup A_{n+2}^{p^2} \cdots] \Rightarrow L$ is of exponent n over k.

Though we have $\bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots] \hookrightarrow K$, counterexample (d) in §4 shows that we need not have $\bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots] = K$. We do have the following:

Proposition 13. *Given K over k modular, set $Q = \bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]$. Then K is modular and relatively perfect over Q (i.e., $K = \bigcap QK^{p^i}$).*

Proof. Let $\{x_n\}$ be a p-basis for $k^{1/p^n} \cap Q$ over $k^{1/p^{n-1}} \cap Q$. $\{x_n\}$ is p-independent over $k^{1/p^{n-1}} \cap K$ (since Q is a modular subfield of K). If $\{y_n\}$ is a completion of $\{x_n\}$ to a p-basis for $k^{1/p^n} \cap K$ over $k^{1/p^{n-1}} \cap K$, then $\{y_n^p\}$ is a possible choice for $\{y_{n-1}\}$ for all positive n and hence $\{y_1\} \cup \{y_2\} \cup \{y_3\} \cup \cdots$ is a modular basis for K over Q. Q.E.D.

Counterexample (e) in §4 shows that Theorem 11 cannot be refined by "tensoring off" $k[\bigcap K^{p^i}]$ from $\bigcap kK^{p^i}$.

3. A generalization of simple extensions and a tensor product theorem. Sweedler has shown in [2, p. 403] that any purely inseparable extension K over k of finite exponent is modular if and only if it is isomorphic to the tensor product (over k) of simple extensions of k. In this section we will study the problem as to whether this theorem has an analogue in the infinite exponent case. We must first extend our definition of a simple extension to include those of the form $k[x, x^{1/p}, x^{1/p^2}, \ldots] \equiv k[x^{1/p^\infty}]$. We may now ask whether a modular extension of infinite exponent is modular if and only if it is isomorphic to the tensor product of simple extensions of k.

The following theorem gives us the implication in one direction.

Theorem 14. *If K is isomorphic to the tensor product of simple extensions (of k), then K is modular over k.*

Proof. By assumption $K \cong (\bigotimes_\alpha k[x_\alpha^{1/p^\infty}]) \otimes (\bigotimes_\beta k[y_\beta])$. To prove that K is modular we need to show that K^{p^n} and k are linearly disjoint for all positive n. Given any $\{z_1, \cdots, z_r\} \subset K$, $\{z_1, \cdots, z_r\} \subset [(\bigotimes_{i=1}^s k[x_i^{1/p^u}]) \otimes (\bigotimes_{j=1}^t k[y_j])] (=Q)$ for some finite s, t and u. Q is modular since it is the tensor product of simple (in the old sense) extensions. Hence if $\{z_1^{p^n}, \cdots, z_r^{p^n}\} \subset K^{p^n}$ is linearly dependent over k, then, by the linear disjointness of Q^{p^n} and k, $\{z_1^{p^n}, \cdots, z_r^{p^n}\}$ is linearly dependent over $Q^{p^n} \cap k$, and hence over $K^{p^n} \cap k$ $(\supset Q^{p^n} \cap k)$. Thus K^{p^n} and k are linearly disjoint. Q.E.D.

It is not true, however, that if K is modular, that it is the tensor product of simple extensions, as counterexamples (a) and (e) of §4 demonstrate. Counterexamples (b) and (c) show that even with the added restriction that $[k : k^p] < \infty$, K may still not be the tensor product of simple extensions.

If we restrict K over k such that $kK^{p^N} = kK^{p^{N+1}}$ for some N (or $[k^{1/p} \cap K : k] < \infty$ or $[k : k^p] < \infty$) by Theorem 11 we know that $K \cong \bigcap kK^{p^i} \otimes M$, where M is modular (and of finite exponent). Hence M is the tensor product of simple extensions. Since $\bigcap kK^{p^i}$ is modular, the following structure theorem shows that if $\bigcap kK^{p^i} = k[\bigcap K^{p^i}]$ then $\bigcap kK^{p^i}$ is the tensor product of simple extensions. Thus K will be the tensor product of simple extensions.

Theorem 15. *Given a field extension K over k, if $k[\bigcap K^{p^i}]$ is modular, then $k[\bigcap K^{p^i}] \cong \bigotimes_\alpha k[z_\alpha^{1/p^\infty}]$ for some $\{z_\alpha\} \subset \bigcap K^{p^i}$.*

Proof. Let $\{z_\alpha\} \subset \bigcap K^{p^i}$ be a p-basis for $k^{1/p} \cap k[\bigcap K^{p^i}]$ over k. Then $\{z_\alpha^{1/p^n}\}$ must be p-independent over $k^{1/p^n} \cap k[\bigcap K^{p^i}]$. If $\{z_\alpha^{1/p^n}\}$ were not a p-basis for $k^{1/p^{n+1}} \cap k[\bigcap K^{p^i}]$ over $k^{1/p^n} \cap k[\bigcap K^{p^i}]$, then the modularity of $k[\bigcap K^{p^i}]$ would imply that $\{z_\alpha\}$ was not a p-basis for $k^{1/p} \cap k[\bigcap K^{p^i}]$ over k. Hence $k^{1/p^n} \cap k[\bigcap K^{p^i}] \cong \bigotimes_\alpha k[z_\alpha^{1/p^{n-1}}] \Rightarrow k[\bigcap K^{p^i}] \cong \bigotimes_\alpha k[z_\alpha^{1/p^\infty}]$. Q.E.D.

Theorem 16. *Given K over k a modular extension, K is the tensor product of simple extensions of k if and only if (in the terminology of Theorem 11) $K \cong \bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]$ and $\bigcap kK^{p^i} = k[\bigcap K^{p^i}]$.*

Proof. "$\Leftarrow$". The remarks preceding Theorem 15 showed that if $K \cong k[\bigcap K^{p^i}] \otimes k[A_1 \cup A_2 \cup \cdots]$, then K is the tensor product of simple extensions.

"$\Rightarrow$". Assume that $K \cong (\bigotimes_\alpha k[x_\alpha^{1/p^\infty}]) \otimes (\bigotimes_\beta k[y_\beta])$. Since each y_β has finite exponent over k, $k[\bigcap K^{p^i}] \subset \bigcap kK^{p^i} \cong \bigotimes_\alpha k[x_\alpha^{1/p^\infty}] \subset k[\bigcap K^{p^i}]$, i.e. $\bigcap kK^{p^i} = k[\bigcap K^{p^i}]$.

By Proposition 13 we have that

$$K = \bigcap_n \left(\bigcap_i kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]\right) \left(\left(\bigotimes_\alpha k[x_\alpha^{1/p^\infty}]\right) \otimes \left(\bigotimes_\beta k[y_\beta]\right)\right)^{p^n}$$

$$= \left(\bigcap_i kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]\right) \left(\bigotimes_\alpha k[x_\alpha^{1/p^\infty}]\right)$$

$$= \bigcap_i kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]$$

(since $\bigotimes_\alpha k[x_\alpha^{1/p^\infty}] \subset k[\bigcap_i K^{p^i}] \subset \bigcap_i kK^{p^i}$). Q.E.D.

We have shown (Remark 12) that if $kK^{p^N} = kK^{p^{N+1}}$ for some finite N, or $[k^{1/p} \cap K : k] < \infty$, or $[k : k^p] < \infty$, then we will have $K \cong \bigcap kK^{p^i} \otimes k[A_1 \cup A_2 \cup \cdots]$. The following theorem gives some conditions under which $\bigcap kK^{p^i} = k[\bigcap K^{p^i}]$.

Theorem 17. *Let K over k be a modular extension.*

(1) *If K over $\bigcap K^{p^i}$ is finitely generated as a field, then $\bigcap kK^{p^i} = k[\bigcap K^{p^i}]$.*

(2) *If there exists a generating set for K over K^p that generates K over $\bigcap K^{p^i}$, then $\bigcap kK^{p^i} = k[\bigcap K^{p^i}]$.*

Proof. (1) Assume $K = (\bigcap_i K^{p^i})(x_1, \cdots, x_m)$. There exists a finite n such that $x_i^{p^n} \in k$, $i = 1, \cdots, m$. Hence $K^{p^j} = (\bigcap_i K^{p^i})(x_1^{p^j}, \cdots, x_m^{p^j}) \subset k(\bigcap_i K^{p^i})$ for any $j \geq n \Rightarrow kK^{p^j} = k(\bigcap_i K^{p^i})$ for any $j \geq n$. Therefore $\bigcap_i kK^{p^i} = k(\bigcap_i K^{p^i})$.

(2) Assume that $\{x\}$ is a generating set for K over K^p such that $K = (\bigcap_i K^{p^i})(\{x\})$. For any positive n, $kK^{p^n} = k[(\bigcap_i K^{p^i})[\{x^{p^n}\}]]$. Hence if $\alpha \in \bigcap_n kK^{p^n}$, $\alpha \in \bigcap_n (k[(\bigcap_i K^{p^i})[\{x^{p^n}\}]]) = k[\bigcap_i K^{p^i}]$. Q.E.D.

4. **Some counterexamples.** (a) The following counterexample illustrates that a modular field extension of unbounded exponent need not be a tensor product of simple extensions. In particular we will construct a modular relatively perfect field extension K over k (i.e. $K = \bigcap kK^{p^i}$), where $K \neq k[\bigcap K^{p^i}]$.

Let $\mathbf{Z}_p$ be the prime field of characteristic p, and $x_1, x_2, x_3, \cdots$ be indeterminates.

Let

$$k = \mathbf{Z}_p(x_1, x_2, x_3, \cdots)$$
$$k^{1/p} \cap K = k[x_1^{1/p}]$$
$$k^{1/p^2} \cap K = k[x_1^{1/p^2} x_2^{1/p}]$$

$$k^{1/p^n} \cap K = k[x_1^{1/p^n} x_2^{1/p^{n-1}} \cdots x_n^{1/p}]$$

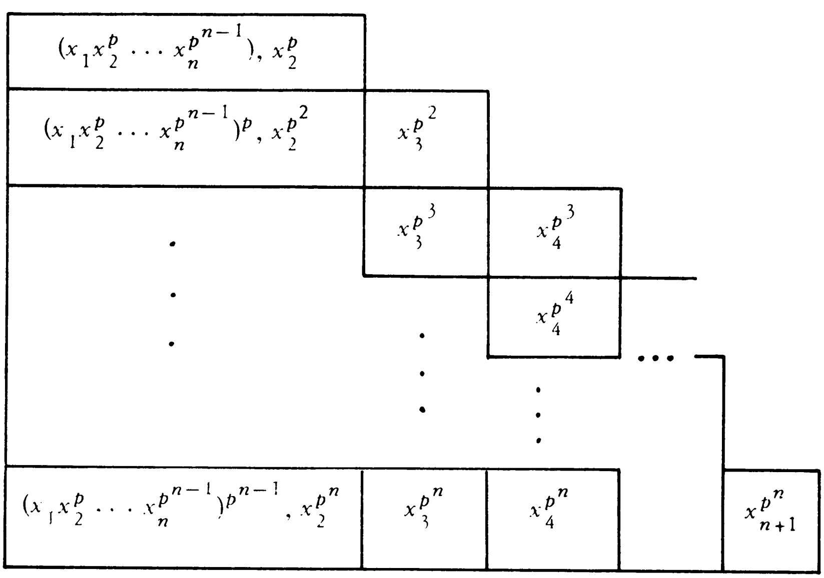
It is easy to verify that K is of the form $\bigcap kK^{p^i}$. We claim that $K \neq k[\bigcap K^{p^i}]$, in particular that $\bigcap K^{p^i} \subset k$.

Assume not. Then there exists a $y \in \bigcap K^{p^i}$ where $y \in k - k^p$ (set difference). Then y will be in all of the following fields:

$$k \cap K^p = \mathbf{Z}_p(x_1, x_2^p, x_3^p, \ldots)$$
$$k \cap K^{p^2} = \mathbf{Z}_p(x_1x_2^p, x_1^p, x_2^{p^2}, x_3^{p^2}, \ldots)$$
$$\vdots$$
$$k \cap K^{p^i} = \mathbf{Z}_p(x_1x_2^p \cdots x_i^{p^{i-1}}, x_1^{p^{i-1}}, x_2^{p^i}, x_3^{p^i}, \ldots)$$
$$\vdots$$

$\bigcap(k \cap K^{p^i}) \subset \mathbf{Z}_p(x_1, x_2^p, x_3^{p^2}, \ldots)$, so $y \in \mathbf{Z}_p(x_1, x_2^p, \ldots, x_n^{p^{n-1}})$ $(= F)$ for some finite n. $y \in k \cap K^{p^{n+1}} \Rightarrow y \in \mathbf{Z}_p(x_1x_2^p \cdots x_n^{p^{n-1}}x_{n+1}^{p^n}, x_1^{p^n}, x_2^{p^{n+1}}, \ldots, x_{n+1}^{p^{n+1}})$ $(= G)$.. We claim that $F \cap G = \mathbf{Z}_p(x_1^p x_2^{p^2} \cdots x_n^{p^n}, x_1^{p^n}, x_2^{p^{n+1}}, \ldots, x_n^{p^{n+1}})$. So $y \in F \cap G \subset k^p$, which contradicts our hypothesis that $y \in k - k^p$.

Consider the following diagram for $\mathbf{Z}_p(x_1, x_2^p, \ldots, x_n^{p^{n-1}}, x_{n+1}^{p^n})$ as a field extension of $\mathbf{Z}_p(x_1^{p^n}, x_2^{p^{n+1}}, \ldots, x_n^{p^{n+1}}, x_{n+1}^{p^{n+1}})$.



We want to describe F and G using the above diagram. Recall that the ground field for this diagram is $\mathbf{Z}_p(x_1^{p^n}, x_2^{p^{n+1}}, \ldots, x_n^{p^{n+1}}, x_{n+1}^{p^{n+1}})$. If we deleted $x_{n+1}^{p^n}$ from the diagram and restricted the ground field to $\mathbf{Z}_p(x_1^{p^n}, x_2^{p^{n+1}}, \ldots, x_n^{p^{n+1}})$, the above would be a diagram for F.

The elements of G are those in the span of monomials in $((x_1x_2^p \cdots x_n^{p^{n-1}}) \cdot x_{n+1}^{p^n})^i$ and $((x_1x_2^p \cdots x_n^{p^{n-1}})^p)^j$ where $i = 0, \cdots, p-1$, $j = 0, \cdots, p^{n-1} - 1$.

Hence $F \cap G$ can be described as polynomials in $(x_1x_2^p \cdots x_n^{p^{n-1}})^p$ with coefficients in $\mathbf{Z}_p(x_1^{p^n}, x_2^{p^{n+1}}, \cdots, x_n^{p^{n+1}})$, i.e.

$$F \cap G = \mathbf{Z}_p(x_1^p x_2^{p^2} \cdots x_n^{p^n}, x_1^{p^n}, x_2^{p^{n+1}}, \cdots, x_n^{p^{n+1}}). \text{ Q.E.D.}$$

(b) The following argument shows that even with the restriction that $[k : k^p] < \infty$, a modular extension may not always be a tensor product of simple extensions.

Let k be any field of characteristic $p \neq 0$ such that $[k : k^p] > p$. k by assumption is not perfect, hence in particular $|k|$ is infinite. Assume that $|k| = \aleph_0$, and let L be the perfect closure of k.

A well-known theorem from Galois theory states that, given any field extension K over k, K is primitive over $k \Leftrightarrow$ there are a finite number of intermediate fields. In the case when K is purely inseparable over k and of exponent one, there are a finite number of intermediate fields $\Leftrightarrow [K : k] = p$.

Lemma 18. *Any purely inseparable extension K over k of the form $\bigcap kK^{p^i}$, where $[k^{1/p^{n+1}} \cap K : k^{1/p^n} \cap K] = p$, is modular.*

Proof. Any such K is of the form $k[x_1, x_2, \cdots]$ where x_i has exponent i over k and $x_i^p \in k[x_1, \cdots, x_{i-1}]$. $k^{1/p^n} \cap K = k[x_1, x_2, \cdots, x_n] = k[x_n]$ is modular by [2, p. 403]. By Proposition 3, K is modular over k. Q.E.D.

Lemma 19. *There are at most $|k| = \aleph_0$ purely inseparable modular extensions K of the form $k[\bigcap K^{p^i}]$, where $[k^{1/p} \cap K : k] = p$ (i.e., of the form $k[z^{1/p^\infty}]$).*

Proof. Trivial, since we may associate any field of the form $k[z^{1/p^\infty}]$ $(z \in k)$ with z. Q.E.D.

Theorem 20. *There exist $2^{\aleph_0}$ purely inseparable modular field extensions of the form $\bigcap kK^{p^i}$, where $[k^{1/p} \cap K : k] = p$ and $[k : k^p] > p$.*

Proof. The field extensions under consideration are of the form $k[x_1, x_2, x_3, \cdots]$ where $[k[x_i] : k] = p^i$ and $x_i^p \in k[x_1, \cdots, x_{i-1}]$.

Given any $x_1 \in (k^{1/p} \cap L) - k$, there exists a $y \in (k^{1/p} \cap L) - k$ such that x_1 and y are p-independent over k (since $[k : k^p] > p$). Consider $M_1 = k[x_1^{1/p}, y]$ as a field extension of $k[x_1]$. Since $M_1^p \subset k[x_1]$ and $[M : k[x_1]] = p^2$, by an earlier remark there are an infinite number of intermediate fields, all of which (excepting $k[x_1, y]$) have exponent 2 over k. We associate with each such field N, an α such that $N = (k[x_1])[\alpha] = k[\alpha]$. For any given α, $[k[\alpha] : k] = p^2$, and if $\alpha \neq \alpha'$, $k[x_1, \alpha] = k[\alpha] \neq k[\alpha'] = k[x_1, \alpha']$. Clearly the α's provide an infinite number of candidates for an x_2.

Having chosen an x_2, we consider $M_2 = k[x_1, x_2^{1/p}, y]$ as a field extension of $k[x_1, x_2]$. $M_2^p \subset k[x_1, x_2]$, $[M_2 : k[x_1, x_2]] = p^2$, so there exist an infinite number of intermediate fields of degree p over $k[x_1, x_2]$ and exponent 3 over k. Each of these provides a candidate for x_3, such that if $x_3 \neq x_3'$, then $k[x_1, x_2, x_3] \neq k[x_1, x_2, x_3']$.

We proceed in the above manner to choose an x_4, x_5, etc. For each x_i there exist an infinite number of choices such that, if $x_i \neq x_i'$, $k[x_1, \cdots, x_i] \neq k[x_1, \cdots, x_i']$. Hence there exist at least $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ choices for modular field extensions K/k of the form $\bigcap kK^{p^i}$, where $[k^{1/p} \cap K : k] = p$ and $[k : k^p] > p$. Q.E.D.

(c) In the following example, we will explicitly construct a modular field extension K over k, where $K = \bigcap kK^{p^i} \neq k[\bigcap K^{p^i}]$ and $[k : k^p] < \infty$.

Let $k = \mathbf{Z}_p(x, y, z_1^{1/p^\infty}, z_2^{1/p^\infty}, \cdots)$ where $x, y, z_1, z_2, \cdots$ are indeterminates (note $[k : k^p] = p^2$).

Let

$$\begin{aligned}
k^{1/p} \cap K &= k[x^{1/p}] \\
k^{1/p^2} \cap K &= k[x^{1/p^2} + z_1 y^{1/p}] \\
k^{1/p^3} \cap K &= k[x^{1/p^3} + z_1^{1/p} y^{1/p^2} + z_2 y^{1/p}] \\
&\;\vdots \\
k^{1/p^n} \cap K &= k[x^{1/p^n} + z_1^{1/p^{n-2}} y^{1/p^{n-1}} + \cdots + z_{n-1} y^{1/p}] \\
&\;\vdots
\end{aligned}$$

Assume that $K = k[\bigcap K^{p^i}]$. Then there exists an $\alpha \in k - k^p$ such that $\alpha \in \bigcap K^{p^i}$. α will lie in all of the following fields.

$$\begin{aligned}
k \cap K^p &= \mathbf{Z}_p(x, y^p, z_1^{1/p^\infty}, z_2^{1/p^\infty}, \cdots) \\
k \cap K^{p^2} &= \mathbf{Z}_p(x + z_1^{p^2} y^p, y^{p^2}, z_1^{1/p^\infty}, z_2^{1/p^\infty}, \cdots) \\
&\;\vdots \\
k \cap K^{p^i} &= \mathbf{Z}_p(x + z_1^{p^2} y^p + \cdots + z_{i-1}^{p^i} y^{p^{i-1}}, y^{p^i}, z_1^{1/p^\infty}, z_2^{1/p^\infty}, \cdots) \\
&\;\vdots
\end{aligned}$$

There exists a finite n such that $\alpha \in \mathbf{Z}_p(x, y, z_1^{1/p^\infty}, \cdots, z_n^{1/p^\infty})$ $(= F)$. But α also $\in k \cap K^{p^{n+2}} = \mathbf{Z}_p(x + z_1^{p^2} y^p + \cdots + z_{n+1}^{p^{n+2}} y^{p^{n+1}}, y^{p^{n+2}}, z_1^{1/p^\infty}, z_2^{1/p^\infty}, \cdots)$ $(= G)$.

Consider the following diagram for $\mathbf{Z}_p(x, y, z_1^{1/p^\infty}, z_2^{1/p^\infty}, \cdots)$ as a field extension of $\mathbf{Z}_p(x^{p^{n+1}}, y^{p^{n+2}}, z_1^{1/p^\infty}, \cdots, z_n^{1/p^\infty}, z_{n+1}^{p^{n+3}}, z_{n+2}^{1/p^\infty}, \cdots)$.

$\vdots$		
$z_{n+1}^{1/p}$		
z_{n+1}		
z_{n+1}^{p}	y	
$z_{n+1}^{p^2}$	y^p	$x+z_1^{p^2}y^p+\cdots+z_n^{p^{n+1}}y^{p^n}$
$\vdots$	$\vdots$	$\vdots$
$z_{n+1}^{p^{n+2}}$	$y^{p^{n+1}}$	$(x+z_1^{p^2}y^p+\cdots+z_n^{p^{n+1}}y^{p^n})^{p^n}$

We would like to use the above diagram to describe F and G. If we deleted the left-hand column (i.e. all pth roots of $z_{n+1}^{p^{n+2}}$) and restricted the ground field to $\mathbf{Z}_p(x^{p^{n+1}}, y^{p^{n+2}}, z_1^{1/p^\infty}, \cdots, z_n^{1/p^\infty})$ the above would be a diagram for F.

G consists of the span of monomials in $\{((z_{n+1}^{p^{n+2}})^{1/p^\infty})^i\}$, $[(x+z_1^{p^2}y^p+\cdots+z_n^{p^{n+1}}y^{p^n})+(z_{n+1}^{p^{n+2}}y^{p^{n+1}})]^j$ and $((x+z_1^{p^2}y^p+\cdots+z_n^{p^{n+1}}y^{p^n})^p)^k$ where $i, j=0,\cdots,p-1$, $k=0,\cdots,p^n-1$. Hence $F\cap G$ can be described as polynomials in $(x+z_1^{p^2}y^p+\cdots+z_n^{p^{n+1}}y^{p^n})^p$ with coefficients in $\mathbf{Z}_p(x^{p^{n+1}}, y^{p^{n+2}}, z_1^{1/p^\infty},\cdots,z_n^{1/p^\infty})$, i.e., $F\cap G=\mathbf{Z}_p((x+z_1^{p^2}y^p+\cdots+z_n^{p^{n+1}}y^{p^n})^p, x^{p^{n+1}}, y^{p^{n+2}}, z_1^{1/p^\infty},\cdots,z_n^{1/p^\infty})\subset k^p \Rightarrow \alpha\in k^p$. Contradiction.

(d) The next counterexample shows (using the terminology of Theorem 11) that in general for a modular extension K over k, $\bigcap kK^{p^i}\otimes M\neq K$.

Set $k=\mathbf{Z}_p(x, a_1^p, a_2^{p^2}, a_3^{p^3},\cdots)$ where $x, a_1, a_2,\cdots$ are indeterminates. Set

$$k^{1/p}\cap K=k[x^{1/p}, a_1, a_2^p, a_3^{p^2},\cdots]$$
$$k^{1/p^2}\cap K=k[x^{1/p^2}a_1^{1/p}, a_1, a_2, a_3^p, a_4^{p^2},\cdots]$$
$$k^{1/p^3}\cap K=k[x^{1/p^3}a_1^{1/p^2}a_2^{1/p}, a_1, a_2, a_3, a_4^p, a_5^{p^2},\cdots]$$

K is modular since

$$k^{1/p^n}\cap K=k[x^{1/p^n}a_1^{1/p^{n-1}}\cdots a_{n-1}^{1/p}, a_1, a_2,\cdots,a_n, a_{n+1}^p, a_{n+2}^{p^2},\cdots]$$
$$=k[x^{1/p^n}a_1^{1/p^{n-1}}\cdots a_{n-1}^{1/p}]\otimes k[a_1]\otimes k[a_2]\otimes\cdots\otimes k[a_n]\otimes k[a_{n+1}^p]\otimes\cdots.$$

We will show that $\bigcap kK^{p^i}=k$. Using the terminology of Theorem 11, we set

$A_i = \{a_i\}$. Hence we will have $(\bigcap kK^{p^i}) \otimes k[A_1 \cup A_2 \cup \cdots] = k[a_1, a_2, \cdots] \neq K$.

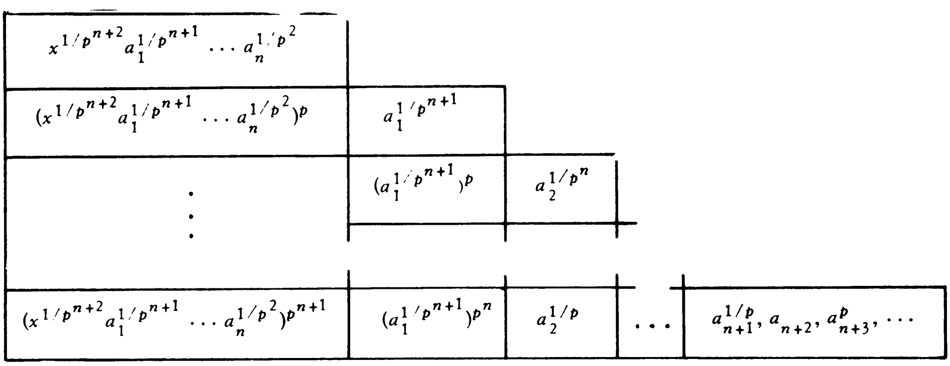
Since it takes only one element to complete $\{a_1, a_2^p, a_3^{p^2}, \cdots\}$ to a p-basis for $k^{1/p} \cap K$ over k, and since $\bigcap kK^{p^i}$ is modular (by Proposition 9), if $\bigcap kK^{p^i} \not\subset k$, then $\bigcap kK^{p^i} = k[x_1, x_2, \cdots]$ where x_i has exponent i over k, and $x_i^p \in k[x_1, \cdots, x_{i-1}]$. Consider K as lying in the perfect closure L of k. $x_1 \in k[x^{1/p}, a_1, a_2^p, \cdots, a_n^{p^{n-1}}]$ for some finite n, so

$$x_{n+2} \in (k^{1/p^{n+1}} \cap L)[x^{1/p^{n+2}}, a_1^{1/p^{n+1}}, a_2^{1/p^n}, \cdots, a_n^{1/p^2}]$$
$$= \mathbf{Z}_p(x^{1/p^{n+2}}, a_1^{1/p^{n+1}}, a_2^{1/p^n}, \cdots, a_n^{1/p^2}, a_{n+1}, a_{n+2}^p, a_{n+3}^{p^2}, \cdots) \quad (= F).$$

We also have that

$$x_{n+2} \in k^{1/p^{n+2}} \cap K = \mathbf{Z}_p(x^{1/p^{n+2}} a_1^{1/p^{n+1}} a_2^{1/p^n} \cdots a_n^{1/p^2} a_{n+1}^{1/p}, a_1, a_2, \cdots, a_{n+2}, a_{n+3}^p, a_{n+4}^{p^2}, \cdots) \quad (= G).$$

Consider the following diagram of $\mathbf{Z}_p(x^{1/p^{n+2}}, a_1^{1/p^{n+1}}, a_2^{1/p^n}, \cdots, a_n^{1/p^2}, a_{n+1}^{1/p}, a_{n+2}, a_{n+3}^p, \cdots)$ over $\mathbf{Z}_p(x, a_1, a_2, \cdots, a_{n+1}, a_{n+2}^p, a_{n+3}^{p^2}, \cdots)$.



The above would be for a diagram for F if we deleted the far right-hand box (containing $a_{n+1}^{1/p}, a_{n+2}, a_{n+3}^p, \cdots$).

G consists of the span of monomials in $((x^{1/p^{n+2}} a_1^{1/p^{n+1}} \cdots a_n^{1/p^2}) \cdot a_{n+1}^{1/p})^i$, $a_{n+2}^i, (a_{n+3}^p)^i, \cdots$ and $((x^{1/p^{n+2}} a_1^{1/p^{n+1}} \cdots a_n^{1/p^2})^p)^j$ where $i = 0, \cdots, p-1$ and $j = 0, \cdots, p^{n+1} - 1$. So $F \cap G = \mathbf{Z}_p((x^{1/p^{n+2}} a_1^{1/p^{n+1}} \cdots a_n^{1/p^2})^p, a_1, a_2, \cdots, a_{n+1}, a_{n+2}^p, a_{n+3}^{p^2}, \cdots)$. $[F \cap G : k] = p^{n+1} \Rightarrow [k[x_{n+2}] : k] \leq p^{n+1}$, with contradicts our initial assumption that $[k[x_{n+2}] : k] = p^{n+2}$

(e) This final counterexample shows that in general $k[\bigcap K^{p^i}]$ does not have a tensor complement over k in $\bigcap kK^{p^i}$ (K over k a modular extension).

We let $k = \mathbf{Z}_p(x, y, z_1, z_2, \cdots)$ where $x, y, z_1, z_2, \cdots$ are indeterminates. Let

$$k^{1/p} \cap K = k[\{z_j^{1/p}\}, x^{1/p}]$$
$$k^{1/p^2} \cap K = k[\{z_j^{1/p^2}\}, x^{1/p^2} + z_1^{1/p} y^{1/p}]$$
$$k^{1/p^3} \cap K = k[\{z_j^{1/p^3}\}, x^{1/p^3} + z_1^{1/p^2} y^{1/p^2} + z_2^{1/p} y^{1/p}]$$
$$\vdots$$

Clearly $k[\{z_j^{1/p^\infty}\}] \subset k[\bigcap K^{p^i}]$. We assert that $k[\{z_j^{1/p^\infty}\}]$ has no tensor complement in $\bigcap kK^{p^i}$. This will imply that $k[\{z_j^{1/p^\infty}\}] = k[\bigcap K^{p^i}]$, and in particular be another counterexample showing that in general $\bigcap kK^{p^i} \neq k[\bigcap K^{p^i}]$.

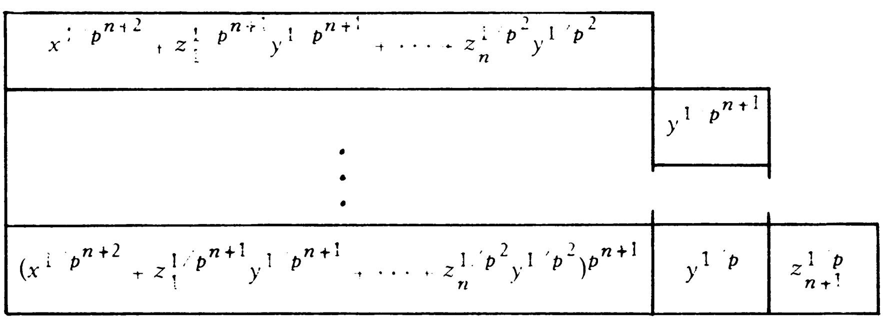
Assume that there exists an $M \subset K$ such that $M \otimes_k k[\{z_j^{1/p^\infty}\}] \cong K$. Since it takes only one element to complete $\{z_j^{1/p^n}\}$ to a p-basis for $k^{1/p^n} \cap K$ over $k^{1/p^{n-1}} \cap K$, M must be of the form $k[x_1, x_2, \cdots]$ where x_i has exponent i over k, and $x_i^p \in k[x_1, \cdots, x_{i-1}]$.

$x_1 \in \mathbf{Z}_p(x^{1/p}, y, z_1^{1/p}, \cdots, z_n^{1/p})$ for some finite $n \Longrightarrow x_{n+2} \in \mathbf{Z}_p(x^{1/p^{n+2}}, y^{1/p^{n+1}}, z_1^{1/p^{n+2}}, \cdots, z_n^{1/p^{n+2}})$ $(= F)$. But

$$x_{n+2} \in k^{1/p^{n+2}} \cap K = \mathbf{Z}_p(x^{1/p^{n+2}} + z_1^{1/p^{n+1}} y^{1/p^{n+1}} + \cdots + z_{n+1}^{1/p} y^{1/p}, y, z_1^{1/p^{n+2}}, z_2^{1/p^{n+2}}, \cdots)$$

$$\Longrightarrow x_{n+2} \in \mathbf{Z}_p(x^{1/p^{n+2}} + z_1^{1/p^{n+1}} y^{1/p^{n+1}} + \cdots + z_{n+1}^{1/p} y^{1/p}, y, z_1^{1/p^{n+2}}, \cdots, z_n^{1/p^{n+2}}, z_{n+1}^{1/p}) \quad (= G).$$

Consider the following diagram of $\mathbf{Z}_p(x^{1/p^{n+2}}, y^{1/p^{n+1}}, z_1^{1/p^{n+2}}, \cdots, z_n^{1/p^{n+2}}, z_{n+1}^{1/p})$ over $\mathbf{Z}_p(x, y, z_1^{1/p^{n+2}}, \cdots, z_n^{1/p^{n+2}}, z_{n+1})$.



F consists of the $\mathbf{Z}_p(x, y, z_1^{1/p^{n+2}}, \cdots, z_n^{1/p^{n+2}})$ span of monomials in which (in terms of the above modular basis) no power of $z_{n+1}^{1/p}$ occurs. G consists of polynomials in $[(x^{1/p^{n+2}} + z_1^{1/p^{n+1}} y^{1/p^{n+1}} + \cdots + z_n^{1/p^2} y^{1/p^2}) + z_{n+1}^{1/p} y^{1/p}]^i$, $(z_{n+1}^{1/p})^i$ $(i = 0, \cdots, p-1)$ and $(x^{1/p^{n+2}} + z_1^{1/p^{n+1}} y^{1/p^{n+1}} + \cdots + z_n^{1/p^2} y^{1/p^2})^{p \cdot j}$ $(j = 0, \cdots, p^{n+1} - 1)$. Hence

$$F \cap G = \mathbf{Z}_p((x^{1/p^{n+2}} + z_1^{1/p^{n+1}} y^{1/p^{n+1}} + \cdots + z_n^{1/p^2} y^{1/p^2})^p,$$
$$y, z_1^{1/p^{n+2}}, \ldots, z_n^{1/p^{n+2}}).$$

Then

$$x_{n+2}^{p^{n+1}} \in k[z_1^{1/p}, \ldots, z_n^{1/p}] \Rightarrow x_1 \in k[z_1^{1/p}, \ldots, z_n^{1/p}] \Rightarrow x_1 \in k,$$

which contradicts our assumption that $[k(x_1):k] = p$.

BIBLIOGRAPHY

1. N. Jacobson, *Lectures in abstract algebra*. Vol. 3: *Theory of fields and Galois theory*, Van Nostrand, Princeton, N. J., 1964. MR 30 #3087.

2. M. Sweedler, *Structure of inseparable extensions*, Ann. of Math. (2) 87 (1968), 401–410. MR 36 #6391.

DEPARTMENT OF MATHEMATICS, BOSTON STATE COLLEGE, BOSTON, MASSACHUSETTS 02115