

## MONADS OF INFINITE POINTS AND FINITE PRODUCT SPACES

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**ABSTRACT.** The notion of "monad" is generalized to infinite (i.e. non-near-standard) points in arbitrary nonstandard models of completely regular topological spaces. The behaviour of several such monad systems in finite product spaces is investigated and we prove that for paracompact spaces  $X$  such that  $X \times X$  is normal, the covering monad  $\mu$  satisfies  $\mu(x, y) = \mu(x) \times \mu(y)$  whenever  $x$  and  $y$  have the same "order of magnitude." Finally, monad systems, in particular non-standard models of the real line,  $R$ , are studied and we show that in a minimal nonstandard model of  $R$  exactly one monad system exists and, in fact,  $\mu(x) = \{x\}$  if  $x$  is infinite.

**0. Introduction.** One of the most intuitive applications of nonstandard analysis is the description of a topology on a set using the very natural idea of "infinitely close." In [14] Abraham Robinson defines the *monad*  $\mu(x)$  of a standard point  $x$  in a nonstandard model of a metric space to be the set of all points whose distance from  $x$  is infinitesimal. A point  $y$  in the nonstandard model is said to be *near-standard* if it is in the monad of some standard point. Two such near-standard points are said to be *infinitely close* if they are in the monad of the same standard point  $x$ . In this setting the standard continuous functions are precisely those standard functions which preserve the relationship "infinitely close." When one looks at *infinite* (i.e. non-near-standard) points it is no longer so obvious what should be meant by "infinitely close." For example in the non-standard reals, the most immediate relationship,  $x$  is infinitely close to  $y$  if and only if  $x - y$  is infinitesimal, is not preserved by the continuous function  $f(x) = x^2$ .

In [19] we began an investigation of this problem and described several possible ways of extending the notion of "infinitely close" to infinite points in a nonstandard model of a topological space. The most intuitive of these extensions, the covering monad, was limited to paracompact spaces. In the first half of this paper we continue this investigation and generalize the results of [19] in several important ways. First, we describe a series of extensions of the

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notion of "infinitely close" to infinite points in nonstandard models of completely regular spaces. In particular, the covering monad generalizes nicely. Second, except for one or two examples all of our present results are obtained in arbitrary nonstandard models as opposed to the highly saturated ([1], [7], [8], [9], [13]) models which were necessary for the proofs in [19]. In the third section of this paper we investigate the behaviour of monads in the product of two spaces. In the final section we make some observations about monad systems in nonstandard models of the reals.

Throughout this paper we will deal only with Hausdorff spaces. When we are working with several topological spaces  $X$ ,  $Y$ , and  $Z$ , their extensions will all be taken in a single nonstandard model  ${}^*\mathbb{M}$ . That is, we let  $\mathbb{M}$  be the complete higher order structure on  $X \cup Y \cup Z$  and let  ${}^*\mathbb{M}$  be a higher order elementary extension of  $\mathbb{M}$  ([10], [14]). The sets  ${}^*X$ ,  ${}^*Y$ , and  ${}^*Z$  will then all live in  ${}^*\mathbb{M}$ . If  $P$  denotes an object in  $\mathbb{M}$  then the corresponding object in  ${}^*\mathbb{M}$  will be denoted by  ${}^*P$ . Except where explicitly stated, we make no assumptions about  ${}^*\mathbb{M}$  other than that it is a proper elementary extension of  $\mathbb{M}$ . We assume the axiom of choice throughout and the generalized continuum hypothesis or Martin's axiom only when necessary to obtain particular kinds of models.

**1. Generalities on extensions of monad systems.** For convenience we recall some definitions from [14] and [19].

**Definition 1.1.** Suppose that  $T$  is a topological space (Hausdorff by convention) and that  ${}^*T$  is a nonstandard model of  $T$ . For each standard point  $t \in T$ , we define the *monad* of  $t$ ,  $\mu(t)$ , by

$$\mu(t) = \bigcap_{t \in U} {}^*U, \text{ where } U \text{ ranges over standard open sets.}$$

A point  $x \in {}^*T$  is said to be *near-standard* if it belongs to the monad of some standard point. Two near-standard points  $x$  and  $y$  are said to be *infinitely close* (denoted by  $x \approx y$ ) provided they belong to the monad of the same standard point. These monads have the following properties.

- (a) If  $x$  and  $y$  are two distinct points of  $T$  then  $\mu(x) \cap \mu(y) = \emptyset$ .
- (b) A standard function  $f: X \rightarrow Y$  is continuous if and only if, for each  $x \in X$ ,  ${}^*f(\mu(x)) \subset \mu(f(x))$ .

Property (a) implies that each near-standard point  $x$  lies in the monad of a unique standard point which we denote  $\text{St}(x)$ . Hence, for each near-standard point we can define  $\mu(x) = \mu(\text{St}(x))$ . In general, for any point  $x$ ,  $\mu(x)$  is an external subset of  ${}^*T$ . In fact,  $\mu(x)$  is internal if and only if  $x$  is a discrete point [10]. In [14] Robinson shows that if a Hausdorff space  $T$  is compact then every point of  ${}^*T$  is near-standard.

**Definition 1.2.** Suppose that  $\mathcal{T}$  is a class of topological spaces. A (total) *monad system*,  $m$ , for the class  $\mathcal{T}$  is a partition of each nonstandard model of a space in  $\mathcal{T}$ , which satisfies conditions (a) and (b) below. If  $x \in {}^*T$  and  $T \in \mathcal{T}$  then we denote by  $m(x)$  the equivalence class of the partition given by  $m$  of  ${}^*T$  which contains  $x$ .

(a) For each near-standard point  $x$ ,  $m(x) = \mu(x)$ .

(b) For each  $X, Y \in \mathcal{T}$  and each standard continuous function  $f: X \rightarrow Y$ , for every  $x \in {}^*X$ ,  ${}^*f(m(x)) \subseteq m({}^*f(x))$ .

In particular, we recall the following monad systems defined in [19].

**Examples 1.3.** (i) Let  $\mathcal{T}$  be the class of all Hausdorff spaces and define the discrete monad system  $t$  for  $\mathcal{T}$  by

$$t(x) = \begin{cases} \mu(x) & \text{if } x \text{ is near-standard,} \\ \{x\} & \text{otherwise.} \end{cases}$$

(ii) Let  $\mathcal{T}$  be the class of all metric spaces and define the metric monad system  $m$  for  $\mathcal{T}$  by

$$m(x) = \{y \text{ s.t. for every standard continuous function } f: T \rightarrow (0, 1), {}^*d(x, y) < {}^*f(x)\}.$$

(iii) Let  $\mathcal{T}$  be the class of all normal spaces and define the *coarse monad system*  $c$  for  $\mathcal{T}$  by

$$c(x) = \bigcap_{x \in {}^*F \subseteq {}^*U} {}^*U, \quad \text{where } F \text{ ranges over standard closed sets} \\ \text{and } U \text{ ranges over standard open sets.}$$

(iv) Let  $\mathcal{T}$  be the class of all normal spaces. We define the *covering monad*  $k$  for  $\mathcal{T}$  as follows:

First, a *locally finite family of pairs*  $\mathcal{U} = \{(U_\alpha, F_\alpha)\}_{\alpha \in \mathcal{A}}$  is a collection of pairs  $(U_\alpha, F_\alpha)$  such that

(a) Each  $F_\alpha$  is closed and  $U_\alpha$  is open and  $F_\alpha \subseteq U_\alpha$ .

(b) For every point  $t \in T$  there is an open neighborhood of  $t$  which intersects only finitely many  $U_\alpha$ 's.

A *quasi-standard pair* (q.s.p.) is a pair  $(U, F)$  which occurs in the nonstandard extension  ${}^*\mathcal{U} = \{(U_\alpha, F_\alpha)\}_{\alpha \in {}^*\mathcal{A}}$  of some locally finite family of pairs,  $\mathcal{U}$ .

Finally, we define  $k$  by

$$k(x) = \bigcap_{x \in F \subseteq U} U, \quad \text{where } (U, F) \text{ ranges over q.s.p.'s.}$$

In [19] the covering monad was only defined for paracompact spaces since we were unable to show that  $\{k(x)\}$  was a partition of  ${}^*T$  for nonparacompact  $T$ .

Also, in [19] it is shown that the coarse monad is the coarsest possible monad system for normal spaces and that for metric spaces the metric monads and covering monads are identical.

## 2. Induced monad systems.

**Definition 2.1.** A *prototype for a monad system* is a topological space  $A$  in which the unit interval  $[0, 1]$  is embedded together with a partition of  ${}^*A$  into equivalence classes such that if the equivalence class containing  $x$  is denoted by  $p(x)$ , then for each standard point  $x \in A$ ,  $p(x) = \mu(x)$ .

**Examples 2.2.** In this paper  $A$  will always be one of the following spaces.

(i) The unit interval  $I = [0, 1]$ .

(ii) The real line  $R$ .

(iii) If  $\lambda$  is any cardinal we denote by  $R^\lambda$  the set of all finitely nonzero functions  $f: \lambda \rightarrow R$ . This set is given the metric,  $d(f, g) = \sup_{\alpha \in \lambda} |f(\alpha) - g(\alpha)|$ .

The partition  $p$  will always be one of the monad systems of Examples 1.3 or the partition:

$$p(x) = \{y \text{ s.t. } {}^*d(x, y) < {}^*\epsilon \text{ for every standard } \epsilon < 0\}.$$

**Definition 2.3.** Suppose that  $(A, p)$  is a prototype for a monad system. Then we define the *induced monad system*  $p^*$  on the class of all completely regular spaces by

$$p^*(x) = \{y \text{ s.t. for every standard continuous function } f: T \rightarrow A, {}^*f(y) \in p({}^*f(x))\} \\ = \bigcap {}^*f^{-1}(p({}^*f(x))) \text{ where } f \text{ ranges over standard continuous functions } f: T \rightarrow A.$$

**Proposition 2.4.** (i) In the situation above  $p^*$  is a monad system.

(ii) If, in addition, property (#) below is satisfied, then, for each  $a \in A$ ,  $p^*(a) = p(a)$ .

(#) For every standard continuous function  $f: A \rightarrow A$  and for every  $a \in {}^*A$ ,  ${}^*f(p(a)) \subseteq p({}^*f(a))$ .

**Proof.** (i) Clearly  $p^*$  is a partition of  ${}^*T$  for any completely regular  $T$ . We must show that for each standard point  $a \in T$  that  $p^*(a) = \mu(a)$ . Since  $T$  is completely regular, for every open set  $U$  containing  $a$  there is a standard continuous function  $f: T \rightarrow I \subseteq A$  such that  $f(a) = 1$  and, for each  $x$  outside  $U$ ,  $f(x) = 0$ . Therefore  $x \notin U$  implies  $x \notin p^*(a)$  and hence  $p^*(a) \subseteq \mu(a)$ . Conversely, since  $a$  is standard, for each standard continuous function  $f: T \rightarrow A$ ,  ${}^*f(\mu(a)) \subseteq \mu(f(a)) = p(f(a))$ . Hence,  $\mu(a) \subseteq p^*(a)$ .

Finally, if  $f: X \rightarrow Y$  is a standard continuous function and  $x \in {}^*X$ , then, for any  $y \in p^*(x)$  and for any standard continuous  $b: X \rightarrow A$ ,  ${}^*b(y) \in p({}^*b(x))$ . In particular, for every standard continuous function  $k: Y \rightarrow A$ ,  ${}^*k({}^*f(y)) \in p({}^*k({}^*f(x)))$ . Therefore  ${}^*f(y) \in p^*({}^*f(x))$ .

(ii) The identity map is continuous and hence  $p^*(a) \subseteq p(a)$ . Property (#) implies  $p(a) \subseteq p^*(a)$ . This completes the proof.

**Proposition 2.5.** *Let  $\mu$  be the usual (and only) prototype for a monad system on the unit interval  $I$ . Then*

(i)  $\mu^*$  is the coarsest possible monad system on the class of all completely regular spaces.

(ii) On normal spaces  $\mu^*$  is the coarse monad  $c$ .

**Proof.** (i) is clear since  $I$  is completely regular and compact.

(ii) Let  $T$  be any normal space. Suppose first that  $x, y \in {}^*T$  and  $x \notin \mu^*(y)$ . Then there is a standard continuous function  $f: T \rightarrow I$  such that  ${}^*f(x) \notin \mu({}^*f(y))$ . We may assume without loss of generality that  ${}^*f(y) < {}^*f(x)$ . Let  $H$  be the internal set,  $H = \{\alpha \in [0, 1] \text{ s.t. } \alpha < ({}^*f(x) - {}^*f(y))\}$ .  $H$  is an internal interval which contains every infinitesimal and hence must also contain some standard  $\epsilon > 0$ . Therefore there are standard points  $\alpha$  and  $\beta$  such that  ${}^*f(y) < \alpha < \beta < {}^*f(x)$ . Let  $F = f^{-1}([0, \alpha])$  and  $U = f^{-1}([0, \beta])$ . Then  $y \in {}^*F$  but  $x \notin {}^*U$ . Hence  $x \notin c(y)$  and we have shown  $c(y) \subseteq \mu^*(y)$ .

Conversely, if  $x \notin c(y)$  there is a standard open set  $U$  and a standard closed set  $F$  such that  $y \in {}^*F$  and  $x \notin {}^*U$ . Then, since  $T$  is normal there is a standard continuous function  $f: T \rightarrow [0, 1]$  which is 0 on  $F$  and 1 off  $U$ . Hence,  ${}^*f(y) = 0$  and  ${}^*f(x) = 1 \notin \mu(0)$ . So  $x \notin \mu^*(y)$ .

The obvious next step is to investigate the monad system induced by the prototype  $(R, k) = (R, m)$ . One might expect that this prototype would induce a very natural monad system on the class of all completely regular spaces. However, the following proposition shows that even the much finer prototype  $(R, t)$  induces a monad system which is intuitively much too coarse even for metric spaces.

**Proposition 2.6.** *Consider the prototype  $(R, t)$  and the space  $R^\omega$ . If  ${}^*R^\omega$  is an enlargement of  $R$  then*

(i) *For every standard  $t \in [0, \infty)$  there are points  $x, y \in {}^*R^\omega$  such that  ${}^*d(x, y) = t$  and  $x \in t^*(y)$ .*

(ii) *Hence, the monad system  $t^*$  is neither finer nor coarser than the monad system  $m = k$ . Thus, in general, the set of monad systems is not linearly ordered.*

(iii) *Also, on  ${}^*R^\omega$  the monad system  $m = k$  is strictly finer than the monad system  $m^* = k^*$ , induced from  $(R, k) = (R, m)$ .*

**Proof.** (i) Consider the following relation:  $R(f, (a, b)) = f$  is a continuous function  $R^\omega \rightarrow R$  and  $(a, b)$  is a pair of points such that  $f(a) = f(b)$  and  $d(a, b) = t$ . It is sufficient to show that the relation  $R$  is concurrent (i.e. finitely satisfiable). Let  $f_1, f_2, \dots, f_k$  be continuous functions  $R^\omega \rightarrow R$  and define  $f: R^{k+1} \rightarrow R^k$  by

$$f(x_1, x_2, \dots, x_{k+1}) \\ = (f_1(x_1, x_2, \dots, x_{k+1}), f_2(x_1, x_2, \dots, x_{k+1}), \dots, f_k(x_1, x_2, \dots, x_{k+1})).$$

Let  $S^k \subseteq R^{k+1}$  be the sphere of diameter  $t$  with center at the origin. By the Borsuk-Ulam Theorem [18] there is a point  $x \in S^k$  such that  $f(x) = f(-x)$ . Hence,  $(x, -x)$  is the desired pair.

(ii) By (i)  $t^*(x)$  is, in general, not contained in  $m(x)$ . But clearly  $m(x)$  is not, in general, contained in  $t^*(x)$  since, if  $R$  is embedded in  $R^\omega$  in the obvious way,  $t^*(x) \cap R = \{x\}$  but  $m(x) \cap R \neq \{x\}$ .

(iii) Immediate.

The preceding proposition shows that the prototype,  $(R, m)$ , is not large enough to give us an intuitively reasonable monad system on the class of all completely regular spaces. Hence, we will consider the prototypes  $(R^\lambda, p_\lambda)$  and  $(R^\lambda, m_\lambda)$  where  $m_\lambda$  will denote the metric monad on  $R^\lambda$  and  $p_\lambda$  will denote the partition of  $R^\lambda$  given by

$$p_\lambda(x) = \{y \text{ s.t. } {}^*d(x, y) < \epsilon \text{ for every standard positive } \epsilon\}.$$

The following lemmas show that if the cardinal  $\lambda$  is large enough then these prototypes induce intuitively reasonable monad systems.

**Lemma 2.7.** *Suppose that  $T$  is normal and that  $\lambda$  is any cardinal number such that  $\lambda \geq \overline{\overline{T}}$ . Then for every  $x \in {}^*T$ ,  $k(x) = p_\lambda^*(x) = m_\lambda^*(x)$ . Hence,  $k$  is a monad system for the class of normal spaces.*

**Proof.** (i) First we show that  $p_\lambda^*(x) \subseteq k(x)$ . Suppose that  $y \notin k(x)$ . Then there is a locally finite family of pairs  $\mathcal{U} = \{(U_\alpha, F_\alpha)\}_{\alpha \in \mathcal{J}}$  such that, for some  $(U_\alpha, F_\alpha) \in {}^*\mathcal{U}$ ,  $x \in F_\alpha$  and  $y \notin U_\alpha$ . Since  $\mathcal{U}$  is locally finite,  $\overline{\overline{\mathcal{J}}} \leq \overline{\overline{T}} \leq \lambda$ . Let  $i: \mathcal{J} \rightarrow \lambda$  be any injection and define  $f: T \rightarrow R^\lambda$  by

$$f(t)(\xi) = \phi_\alpha(t) \quad \text{if } \xi = i(\alpha), \\ = 0 \quad \text{otherwise,}$$

where the functions  $\phi_\alpha$  are chosen (since  $T$  is normal) so that

$$\phi_\alpha(t) = 0 \quad \text{if } t \notin U_\alpha, \\ = 1 \quad \text{if } t \in F_\alpha;$$

then  $f$  is a standard continuous function and  ${}^*d({}^*f(x), {}^*f(y)) = 1$ . Hence  $y \notin p_\lambda^*(x)$ .

(ii) The proof is completed by showing that  $k(x) \subseteq m^*(x)$ . This follows immediately from the following two results from [19]. The proofs of these results are completely straightforward and are valid in any elementary extension. In ad-

dition the proof of the first result does not use paracompactness. Hence the results apply in the current situation.

(1) For every standard continuous function  $f: X \rightarrow Y$  between two normal spaces, for every  $x \in {}^*X$ ,  ${}^*f(k(x)) \subseteq k({}^*f(x))$ .

(2) For metric spaces,  $k(x) = m(x)$ .

**Lemma 2.8.** *Suppose that  $T$  is a completely regular space and that  $\lambda$  and  $\kappa$  are two cardinals such that  $\lambda, \kappa \geq \overline{\overline{T}}$  and  $\lambda, \kappa \geq 2^{\kappa_0}$ . Then for each  $t \in {}^*T$ ,  $p_\lambda^*(t) = p_\kappa^*(t) = m_\lambda^*(t) = m_\kappa^*(t)$ .*

**Proof.** We may assume that  $\lambda \leq \kappa$  so there is an obvious inclusion  $R^\lambda \subseteq R^\kappa$  and clearly  $p_\kappa^*(t) \subseteq p_\lambda^*(t)$ . Now given any standard continuous function  $f: T \rightarrow R^\kappa$ , for each  $t \in T$ ,  $f(t)$  has only finitely many nonzero coordinates. Hence if  $A = \{\xi \in \kappa: \exists t \in T(f(t)(\xi) \neq 0)\}$ , then  $\overline{\overline{A}} \leq \lambda$ . Thus we can think of  $f$  as a function  $T \rightarrow R^\lambda$  and hence  $p_\lambda^*(x) \subseteq p_\kappa^*(x)$  completing the proof of the first equality. The other equalities follow since  $R^\lambda = \lambda$ , so by Lemma 2.7 on  $R^\lambda$ ,  $p_\lambda^* = m$ , and by Proposition 2.4  $m_\lambda^* = m$ . Thus on  $T$ ,  $p_\lambda^*(t) \subseteq m_\lambda^*(t)$ . But trivially,  $m_\lambda^*(t) \subseteq p_\lambda^*(t)$ , completing the proof.

In view of Lemmas 2.7 and 2.8 and the intuitive feeling that the covering and metric monads are the "right" monads, we make the following definition. The main positive results of this section are then summarized by Theorem 2.10.

**Definition 2.9.** Suppose that  $\mathcal{T}$  is the class of completely regular spaces. Then we define the monad system  $\mu$  on  $\mathcal{T}$  by  $\mu(t) = p_\lambda^*(t)$ , for each  $t \in {}^*T$ , where  $\lambda$  is any cardinal such that  $\lambda \geq \overline{\overline{T}}$ .

**Theorem 2.10.** (i) *For near-standard points  $t \in {}^*T$ ,  $\mu(t)$  is the usual monad. Hence, our notation is consistent.*

(ii) *For metric spaces,  $\mu(t) = m(t) = k(t)$ .*

(iii) *For normal spaces,  $\mu(t) = k(t)$ .*

(iv)  *$\mu$  is a monad system on the class of completely regular spaces.*

The work in this section leaves us in the somewhat unsettling position of having a plethora of monad systems on the class of completely regular spaces. It would be of some interest to attempt to classify such monad systems or perhaps to study the partially ordered set of all such monad systems. In this connection it is interesting to note that Proposition 2.6 shows that the partial order is not linear and also that with the following definitions this set is actually a lattice.

(i)  $a \leq b$  if, for every completely regular space  $T$  and every  $t \in {}^*T$ ,  $a(t) \subseteq b(t)$ .

(ii)  $(a \wedge b)(x) = a(x) \cap b(x)$ .

(iii)  $(a \vee b)(x) = \{y \text{ s.t. there are points } x_0, x_1, \dots, x_n \text{ such that } x = x_0, y = x_n, \text{ and } x_{i+1} \in a(x_i) \cup b(x_i) \text{ for } i = 0, 1, 2, \dots, n-1\}$ .

However, there are also a number of reasons for believing that the monad system  $\mu$  is the "right" one.

(1) For metric spaces the metric monad,  $m$ , certainly seems very natural and  $\mu$  is the same as the metric monad on metric spaces. Similarly the covering monad seems very natural, at least for paracompact spaces, and again the monad  $\mu$  is the same as the covering monad on normal spaces.

(2) At least for paracompact spaces local properties hold even for infinite points. For example if  $T$  is paracompact and locally convex then  $\mu(x)$  is convex for every  $x \in {}^*T$ .

(3) The monad system  $\mu$  is the coarsest monad system consistent with the obvious partition  $p(x)$  of Examples 2.2.

In the next section we shall obtain some more evidence that the monad system  $\mu$  is the most appropriate one to use.

**3. Monads in finite products.** In this section we would like to investigate the following question. Suppose that  $a$  is a monad system and that  $(x, y) \in {}^*(X \times Y)$ . Then under what circumstances is  $a(x, y) = a(x) \times a(y)$ ? The following observation is immediate.

**Proposition 3.1.** *Suppose that  $a$  is a monad system on the class  $\mathcal{I}$  and that the spaces  $X$ ,  $Y$ , and  $X \times Y$  are all in  $\mathcal{I}$ . Then for every  $x \in {}^*X$  and  $y \in {}^*Y$ ,  $a(x, y) \subseteq a(x) \times a(y)$ .*

**Proof.** The projection maps  $p_1(u, v) = u$  and  $p_2(u, v) = v$  are continuous. Hence,  $a(x, y) \subseteq p_1^{-1}(a(x)) \cap p_2^{-1}(a(y)) = a(x) \times a(y)$ .

However, the following example shows that, in general, equality does not hold.

**Example 3.2.** If  $x$  is a standard point in  $R$  and  $y$  is an infinite point in  ${}^*R$  then  $\mu(x, y) \neq \mu(x) \times \mu(y)$ .

**Proof.** Clear.

The following proposition shows that for many monad systems even  $a(x, x) \neq a(x) \times a(x)$ .

**Proposition 3.3.** *Suppose that  $X$  and  $X \times X$  are paracompact and that  $x \in {}^*X$ . Then  $c(x, x) \cap (\{x\} \times {}^*X) \subseteq \{x\} \times \mu(x)$ .*

**Proof.** Suppose that  $y \notin \mu(x)$ . Then there is a locally finite family of pairs  $\mathcal{K} = \{(U_\alpha, F_\alpha)\}_{\alpha \in \mathcal{J}}$  such that for some  $(U_\gamma, F_\gamma)$ ,  $x \in F_\gamma$  and  $y \notin U_\gamma$ . Consider  $\mathcal{K} \cup \{(X, X)\}$ . This is a locally finite covering of  $X$  by pairs. By Lemma 2.15 of [19] there is another locally finite covering of  $X$  by pairs  $\{(V_\beta, H_\beta)\}_{\beta \in \mathcal{J}}$  such that, for every  $u, v \in X$  and every  $\beta \in \mathcal{J}$ ,  $u, v \in V_\beta$  implies that for every  $(U_\alpha, F_\alpha)$  if  $u \in F_\alpha$  then  $v \in U_\alpha$ . Hence, there is no  $V_\beta$  such that both  $x, y \in V_\beta$ . Now let



$$F = \Delta = \{(u, u) \in X \times X\}, \quad U = \bigcup_{\beta \in \mathcal{J}} V_\beta \times V_\beta.$$

Then,  $(x, x) \in {}^*F$  and  $(x, y) \notin {}^*U$ , so that  $(x, y) \notin c(x, x)$  which completes the proof.

**Corollary 3.4.** *Suppose  $a$  is a monad system which is defined for paracompact spaces and that  $X$  and  $X \times X$  are paracompact. Then  $a(x, x) \cap (\{x\} \times {}^*X) \subseteq (\{x\} \times \mu(x))$ .*

**Proof.** Since  $c$  is the coarsest possible monad system for normal spaces  $a(x, x) \subseteq c(x, x)$ .

Although many monad systems behave badly for products, the trivial monad system trivially behaves well, and the covering monad system behaves well at least for paracompact spaces.

**Proposition 3.5.** *If  $X$  and  $Y$  are any topological spaces and  $x$  is an infinite point of  ${}^*X$  and  $y$  is an infinite point of  ${}^*Y$  then  $t(x, y) = t(x) \times t(y)$ .*

**Proof.** Trivial.

**Theorem 3.6.** *Suppose that  $X$  is paracompact and that  $X \times X$  is normal. Let  $x$  be any element of  ${}^*X$ . Then  $\mu(x, x) = \mu(x) \times \mu(x)$ .*

**Proof.** (i) By Proposition 3.1,  $\mu(x, x) \subseteq \mu(x) \times \mu(x)$ .

(ii) Suppose that  $(u, v) \notin \mu(x, x)$ . We must show that either  $u \notin \mu(x)$  or  $v \notin \mu(x)$ . Let  $\mathcal{U} = \{(U_\alpha, F_\alpha)\}_{\alpha \in \mathcal{J}}$  be a locally finite family of pairs such that, for some  $(U_\beta, F_\beta) \in {}^*\mathcal{U}$ ,  $(x, x) \in F_\beta$  and  $(u, v) \notin U_\beta$ . By adding the pair  $(X \times X, X \times X)$ , if necessary we may assume that  $\mathcal{U}$  is a covering of  $X$ . Let  $\Delta = \{(t, t) \in X \times X\}$ . We will produce a locally finite covering of  $X$  by pairs by breaking each  $(U_\alpha \cap \Delta, F_\alpha \cap \Delta)$  up into a locally finite family as follows.

For each  $\alpha$  such that  $F_\alpha \cap \Delta \neq \emptyset$ , for each  $(t, t) \in F_\alpha$  there is an open subset  $A_{t, \alpha} \subseteq X$  such that  $(t, t) \in A_{t, \alpha} \times A_{t, \alpha} \subseteq U_\alpha$ . Since  $X$  is paracompact so is  $H_\alpha = \{t \in X: (t, t) \in F_\alpha\}$ . Hence,  $\{A_{t, \alpha}\}$  can be refined to a locally finite covering  $\{B_{\gamma, \alpha}\}$  of  $H_\alpha$ . Since  $X$  is normal we can find closed sets  $C_{\gamma, \alpha} \subseteq B_{\gamma, \alpha}$  such that  $\bigcup C_{\gamma, \alpha} = H_\alpha$ . Let  $\mathcal{V} = \{(B_{\gamma, \alpha}, C_{\gamma, \alpha})\}_{\alpha, \gamma}$ . Clearly,  $\mathcal{V}$  is a locally finite covering of  $X$  by pairs. Since  $(x, x) \in F_\beta$  and  $(u, v) \notin U_\beta$ , there is a  $\gamma$  such that  $(x, x) \in C_{\gamma, \beta} \times C_{\gamma, \beta} \subseteq B_{\gamma, \beta} \times B_{\gamma, \beta} \subseteq U_\beta$ . Therefore  $x \in C_{\gamma, \beta}$  but either  $u \notin B_{\gamma, \beta}$  or  $v \notin B_{\gamma, \beta}$  and hence either  $u \notin \mu(x)$  or  $v \notin \mu(x)$  which completes the proof.

In view of Example 3.2, whether or not  $\mu(x, y) = \mu(x) \times \mu(y)$  depends strongly on the relative "sizes" of  $\mu(x)$  and  $\mu(y)$  or, as we shall see, on the relative "orders of magnitude" of  $x$  and  $y$ . In order to make this precise we proceed as follows.

**Definition 3.7.** Suppose that  $X$  is a topological space and that  $K \subseteq {}^*X$ .  $K$  is said to be a *quasi-standard compact* subset of  ${}^*X$  if there is a standard locally finite family of compact subsets of  $X$ ,  $\mathcal{K} = \{K_\alpha\}$  such that  $K \in {}^*\mathcal{K}$ .

If  $x, y \in {}^*X$  we say that  $x$  and  $y$  are of the same *order* (written  $o(x) = o(y)$ ) if either

- (i)  $x = y$  or
- (ii) there is some quasi-standard compact set  $K$  such that  $\{x, y\} \subseteq K$ .

We say  $x$  is of *finite order* (written  $o(x) = 0$ ) if there is some standard point  $y$  such that  $o(x) = o(y)$ .

Clause (i) is necessary in the definition above since a straightforward enlargement argument shows that if  $X$  is not locally compact there are (near-standard) points in  ${}^*X$  which belong to no quasi-standard compact subsets of  ${}^*X$ .

**Lemma 3.8.** Suppose that  $K$  is a standard compact subset of  $X$  and that  $\{C_\alpha\}$  is a standard locally finite family of subsets of  $X$ . Then there is a neighborhood of  $K$  which intersects only finitely many  $C_\alpha$ 's.

**Proof.** For each  $x \in K$  there is an open set  $U_x$  containing  $x$  which intersects only finitely many  $C_\alpha$ 's. Since  $K$  is compact,  $K$  is contained in the union of finitely many  $U_x$ 's.

**Proposition 3.9.**  $o(x) = o(y)$  is an equivalence relation.

**Proof.** Symmetry and reflexivity are clear. Now suppose that  $o(x) = o(y)$  and that  $o(y) = o(z)$ . We must show that  $o(x) = o(z)$ . There are locally finite families of compact sets  $\mathcal{K} = \{K_\alpha\}$  and  $\mathcal{C} = \{C_\beta\}$  such that  $\{x, y\} \subseteq C \in {}^*\mathcal{C}$ . Let  $\mathcal{L}$  be the family  $\{K_\alpha \cup C_\beta : K_\alpha \cap C_\beta \neq \emptyset\}$ . We claim that  $\mathcal{L}$  is a locally finite family of compact sets. Let  $v$  be an arbitrary element of  $X$ . Since  $\mathcal{K}$  and  $\mathcal{C}$  are locally finite there is an open neighborhood  $W$  of  $v$  which intersects only finitely many  $K_\alpha$ 's and  $C_\beta$ 's. But by Lemma 3.8 each  $C_\beta$  intersects only finitely many  $K_\alpha$ 's and each  $K_\beta$  intersects only finitely many  $C_\alpha$ 's. So altogether  $W$  intersects only finitely many elements of  $\mathcal{L}$ . Therefore  $\mathcal{L}$  is locally finite and  $o(x) = o(z)$  since  $K \cap C \neq \emptyset$  and  $\{x, z\} \subseteq K \cup C$ .

**Proposition 3.10.** (i) If  $x$  and  $y$  are standard points in  $x$ , then  $o(x) = o(y)$ .

(ii) If  $o(x) = 0$ , then  $x$  is near-standard.

(iii) If  $X$  is locally compact, then  $o(x) = 0$  if and only if  $x$  is near-standard.

**Proof.** (i)  $\{x, y\}$  is compact and hence quasi-standard compact.

(ii) If  $\{x, y\} \subseteq K$ ,  $K$  is quasi-standard and  $y$  is standard, then  $K$  must also be standard and compact. Hence, since  $x$  is an element of a standard compact set  $x$  must be near-standard [14].

(iii) If  $x$  is near-standard let  $K$  be a standard compact neighborhood of  $\text{St}(x)$ . Hence,  $x \in {}^*K$  and  $\text{o}(x) = \text{o}(\text{St}(x)) = 0$ .

In general, order is not preserved by standard continuous functions as can easily be seen from consideration of the function  $f: R \rightarrow R$  given by  $f(x) = x \sin x$ . However, we do have the following partial result.

**Proposition 3.11.** *Suppose that  $f: X \rightarrow Y$  is a standard homeomorphism into (that is, as a function  $X \rightarrow f(X)$ ,  $f$  is a homeomorphism); then for every  $x, y \in {}^*X$ ,  $\text{o}(x) = \text{o}(y)$  implies  $\text{o}({}^*f(x)) = \text{o}({}^*f(y))$ . Furthermore, if  $f(X)$  is a closed subset of  $Y$  then for every  $x, y \in {}^*X$ ,  $\text{o}({}^*f(x)) = \text{o}({}^*f(y))$  implies  $\text{o}(x) = \text{o}(y)$ .*

**Proof.** (i) If  $\mathcal{K} = \{K_\alpha\}$  is a locally finite family of compact sets of  $X$  such that  $\{x, y\} \subseteq K \in {}^*\mathcal{K}$ , then  $\{{}^*f(x), {}^*f(y)\} \subseteq {}^*f(K) \in {}^*(f(\mathcal{K})) = {}^*\{f(K_\alpha)\}$  and, clearly,  $f(\mathcal{K})$  is a locally finite family of compact subsets of  $Y$ .

(ii) Suppose that  $\mathcal{K} = \{K_\alpha\}$  is a locally finite family of compact subsets of  $Y$  such that  $\{{}^*f(x), {}^*f(y)\} \subseteq K \in {}^*\mathcal{K}$ . Then  $\mathcal{K}' = \{K_\alpha \cap f(X)\}$  is also a locally finite family of compact sets since  $f(X)$  is closed and  $\{{}^*f(x), {}^*f(y)\} \subseteq K \cap {}^*f(X) \in \mathcal{K}'$ . Then,  $\{x, y\} \subseteq f^{-1}(K \cap {}^*f(X)) \in {}^*(f^{-1})(\mathcal{K}')$  which is a locally finite family of compact subsets of  $X$ .

Before proving the main result of this section we need three lemmas.

**Lemma 3.12.** *Suppose that  $K$  is a compact subset of  $X$  and that  $V$  is an open subset of  $X \times X$  such that  $K \times K \subseteq V$ . Then there is an open subset  $W$  of  $X$  such that  $K \subseteq W$  and  $W \times W \subseteq V$ .*

**Proof.** (i) Fix  $x \in K$ . For each  $y \in K$  there are open subsets of  $X$ ,  $A_y, B_y$  such that  $(x, y) \in A_y \times B_y \subseteq V$ . Since  $K$  is compact there is a finite subcover  $B_{y_1}, B_{y_2}, \dots, B_{y_k}$  of  $K$ . Therefore  $\{x\} \times K \subseteq \bigcup (A_{y_i} \times B_{y_i}) \subseteq V$ . In fact, if we let  $U_x = \bigcap A_{y_i}$  and  $T_x = \bigcup B_{y_i}$  we have  $U_x \times T_x \subseteq V$ .

(ii)  $U_x$  is an open cover of  $K$  and, hence, there is a finite subcover  $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ . Let  $W = (\bigcup U_{x_i}) \cap (\bigcap T_{x_i})$ ; then clearly  $W$  is the set we want.

**Lemma 3.13.** *Suppose that  $X$  is normal and that  $F$  is a compact subset of  $X \times X$ . Suppose, further, that  $A_1, A_2, \dots, A_n$ , and  $B_1, B_2, \dots, B_n$  are open subsets of  $X$  such that  $F \subseteq \bigcup (A_i \times B_i)$ . Then there are open subsets of  $X$ ,  $C_1, C_2, \dots, C_n$  and  $D_1, D_2, \dots, D_n$  such that  $C_i \subseteq \bar{C}_i \subseteq A_i$ ,  $D_i \subseteq \bar{D}_i \subseteq B_i$  and  $F \subseteq \bigcup (C_i \times D_i)$ .*

**Proof.** Let  $H = F - \bigcup_{i=2}^n (A_i \times B_i)$ .  $H$  is compact and  $H \subseteq A_1 \times B_1$ . Let  $p_1: X \times X \rightarrow X$  and  $p_2: X \times X \rightarrow X$  denote the projection maps on the first and second factors, respectively. Then  $p_1(H)$  is compact and  $p_1(H)$  is contained in

$A_1$ , so by normality there is an open set  $C_1$  such that  $p_1(H) \subseteq C_1 \subseteq \bar{C}_1 \subseteq A_1$ . Similarly there is an open set  $D_1$  such that  $p_2(H) \subseteq D_1 \subseteq \bar{D}_1 \subseteq B_1$ . Hence  $H \subseteq C_1 \times D_1$  and  $F \subseteq (C_1 \times D_1) \cup (\bigcup_{i=2}^n A_i \times B_i)$ . The remainder of the proof is similar.

**Lemma 3.14.** *Suppose that  $X$  is paracompact and that  $\{K_\alpha\}$  is a locally finite family of compact subsets of  $X$ . Then there are open sets  $U_\alpha \supseteq K_\alpha$  such that the family  $\{U_\alpha\}$  is locally finite.*

**Proof.** By Lemma 3.8 there are open sets  $S_\alpha \supseteq K_\alpha$  such that each  $S_\alpha$  intersects only finitely many  $K_\beta$ 's. Since  $\{K_\alpha\}$  is a locally finite family of closed sets  $\bigcup K_\alpha$  is closed and  $X - \bigcup K_\alpha$  is open. Thus,  $\mathcal{U} = \{S_\alpha\} \cup \{X - \bigcup K_\alpha\}$  is an open cover of  $X$ . Since  $X$  is paracompact there is a locally finite refinement  $\{T_\beta\}$  of  $\mathcal{U}$ .

For each  $K_\alpha$  there is a finite subcollection  $T_{\beta_1}, T_{\beta_2}, \dots, T_{\beta_k}$  such that  $K_\alpha \subseteq T_{\beta_1} \cup T_{\beta_2} \cup \dots \cup T_{\beta_k}$ . We may assume each of these  $T_{\beta_i}$ 's has a nonempty intersection with  $K_\alpha$ . Let  $U_\alpha = T_{\beta_1} \cup T_{\beta_2} \cup \dots \cup T_{\beta_k}$ . Clearly  $K_\alpha \subseteq U_\alpha$ . We claim that  $\{U_\alpha\}$  is locally finite and, hence, the family we seek. First, notice that each  $T_\beta$  occurs in only finitely many  $U_\alpha$ 's since, if it has a nonvoid intersection with some  $K_\alpha$ , then it is not contained in  $X - \bigcup K_\alpha$  and hence must be contained in some  $S_\alpha$ . But by the choice of the  $S_\alpha$ 's, each one intersects only finitely many  $K_\alpha$ 's. Now if  $x \in X$ , since  $\{T_\beta\}$  is a locally finite family, there is a neighborhood  $W$  of  $x$  which intersects only finitely many  $T_\beta$ 's and hence only finitely many  $U_\alpha$ 's, since each  $T_\beta$  is contained in only finitely many  $U_\alpha$ 's. This completes the proof.

We are now ready to prove the main result of this section.

**Theorem 3.15.** *Suppose that  $X$  is paracompact and that  $X \times X$  is normal. If  $x, y \in {}^*X$  and  $o(x) = o(y)$  then  $\mu(x, y) = \mu(x) \times \mu(y)$ .*

**Proof.** By Theorem 3.6 if  $x = y$ ,  $\mu(x, y) = \mu(x) \times \mu(y)$ . So we may assume that there is a standard locally finite family of compact subsets of  $X$ ,  $\mathcal{K} = \{K_\gamma\}$  such that there is a  $K_\tau \in {}^*\mathcal{K}$  such that  $\{x, y\} \subseteq K_\tau$ . By Proposition 3.1,  $\mu(x, y) \subseteq \mu(x) \times \mu(y)$ .

Now suppose that  $(u, v) \notin \mu(x, y)$  and that  $\mathcal{U} = \{(U_\alpha, F_\alpha)\}$  is a locally finite family of pairs such that, for some  $(U_\beta, F_\beta) \in {}^*\mathcal{U}$ ,  $(x, y) \in F_\beta$  and  $(u, v) \notin U_\beta$ . By Lemma 3.14 there are open sets  $S_\gamma \supseteq K_\gamma$  such that the collection  $\{S_\gamma\}$  is locally finite. Since each  $K_\gamma$  is compact, so is  $K_\gamma \times K_\gamma$ , and by Lemma 3.8 and Lemma 3.12 there are neighborhoods  $W_\gamma$  of each  $K_\gamma$  such that  $K_\gamma \subseteq W_\gamma \subseteq S_\gamma$ , and each  $W_\gamma \times W_\gamma$  intersects only finitely many  $F_\beta$ 's.

For each  $t \in F_\beta \cap (K_\gamma \times K_\gamma)$  we can find open sets  $A_{t,\beta,\gamma}$  and  $B_{t,\beta,\gamma}$  such that  $t \in A_{t,\beta,\gamma} \times B_{t,\beta,\gamma} \subseteq U_\beta \cap (W_\gamma \times W_\gamma)$ . Since each  $K_\gamma \times K_\gamma$  is com-

pact we can find a finite subcover of each  $F_\beta \cap (K_\gamma \times K_\gamma)$ ,  $A_{t_1, \beta, \gamma} \times B_{t_1, \beta, \gamma}$ ,  $A_{t_2, \beta, \gamma} \times B_{t_2, \beta, \gamma}$ ,  $\dots$ ,  $A_{t_k(\beta, \gamma), \beta, \gamma} \times B_{t_k(\beta, \gamma), \beta, \gamma}$ . By Lemma 3.13 we can find open sets  $C_{t_i, \beta, \gamma}$  and  $D_{t_i, \beta, \gamma}$  such that  $\bar{C}_{t_i, \beta, \gamma} \subseteq A_{t_i, \beta, \gamma}$  and  $\bar{D}_{t_i, \beta, \gamma} \subseteq B_{t_i, \beta, \gamma}$  and  $F_\beta \cap (K_\gamma \times K_\gamma) \subseteq \bigcup_i (C_{t_i, \beta, \gamma} \times D_{t_i, \beta, \gamma})$ .

Since each  $K_\gamma \times K_\gamma$  intersects only finitely many  $F_\beta$ 's, for each  $K_\gamma$  we get two finite collections of pairs  $(A_{t_1, \gamma}, \bar{C}_{t_1, \gamma}), \dots, (A_{t_n(\gamma), \gamma}, \bar{C}_{t_n(\gamma), \gamma})$  and  $(B_{t_1, \gamma}, \bar{D}_{t_1, \gamma}), \dots, (B_{t_n(\gamma), \gamma}, \bar{D}_{t_n(\gamma), \gamma})$ . Since each of these sets is contained in the corresponding  $S_\gamma$  and the collection  $\{S_\gamma\}$  is locally finite, the two collections of pairs  $\{(A_{t_i, \beta, \gamma}, \bar{C}_{t_i, \beta, \gamma})\}$  and  $\{(B_{t_i, \beta, \gamma}, \bar{D}_{t_i, \beta, \gamma})\}$  are locally finite.

Now consider  $(x, y) \in F_\beta$  and  $(u, v) \notin U_\beta$ . Since  $\{x, y\} \subseteq K_\tau \in {}^*K$  we have  $(x, y) \in F_\beta \cap (K_\tau \times K_\tau)$  and hence, for some  $i$ ,  $(x, y) \in \bar{C}_{t_i, \beta, \tau} \times \bar{D}_{t_i, \beta, \tau}$  but since  $(u, v) \notin U_\beta$ ,  $(u, v) \notin A_{t_i, \beta, \tau} \times B_{t_i, \beta, \tau}$  and hence either  $u \notin A_{t_i, \beta, \tau}$  or  $v \notin B_{t_i, \beta, \tau}$ , but since  $x \in \bar{C}_{t_i, \beta, \tau}$  and  $y \in \bar{D}_{t_i, \beta, \tau}$ , this means that either  $u \notin \mu(x)$  or  $v \notin \mu(y)$ , which is what we had to show.

The remainder of this section will be concerned with metric spaces and the metric monad system,  $\mu$ . Intuitively, if  $x$  is a standard point and  $y$  is an infinite point in  ${}^*R$  then  $\mu(x)$  is "larger" than  $\mu(y)$ . In fact, we can directly compare them by noticing that  $\mu(y) + (x - y)$  is properly contained in  $\mu(x)$ . In general, we make the following definition to capture this notion of the "size" of  $\mu(x)$ .

**Definition 3.16.** Suppose that  $X$  is a metric space and that  $x, y \in {}^*X$ . Then we say that  $\mu(x)$  and  $\mu(y)$  have the same *size*, provided that for every standard continuous function  $f: X \rightarrow (0, \infty)$  there is a standard continuous function  $g: X \rightarrow (0, \infty)$  such that  ${}^*g(y) \leq {}^*f(x)$ , and  ${}^*g(x) \leq {}^*f(y)$ .

**Theorem 3.17.** Suppose that  $X$  is a metric space, that  $x, y \in {}^*X$  and that  $o(x) = o(y)$ . Then  $\mu(x)$  and  $\mu(y)$  have the same size.

**Proof.** Let  $\mathcal{K} = \{K_\alpha\}_{\alpha \in \mathcal{I}}$  be a locally finite family of compact subsets of  $X$  such that, for some  $K_\gamma \in {}^*K$ ,  $\{x, y\} \subseteq K_\gamma$ . Assume that  $f: X \rightarrow (0, \infty)$  is a standard continuous function. The idea of the proof is to define  $g: \bigcup K_\alpha \rightarrow (0, \infty)$  such that  $g|_{K_\alpha} \leq \text{Min}\{f|_{K_\alpha}\}$ . Then, since  $\{K_\alpha\}$  is locally finite  $\bigcup K_\alpha$  is closed, so by the Tietze Extension Theorem  $g$  can be extended to all of  $X$ . This extension is then the function we want. In order to define  $g$  on  $\bigcup K_\alpha$  we assume that the index set  $\mathcal{I}$  is an ordinal and proceed by induction on  $\alpha \in \mathcal{I}$ . By Lemma 3.14 there are open sets  $U_\alpha \supseteq K_\alpha$  such that the collection  $\{U_\alpha\}$  is locally finite. We define functions  $g_\beta: \bigcup_{\alpha < \beta} K_\alpha \rightarrow (0, \infty)$  as follows:

- (1)  $g_1: K_0 \rightarrow (0, \infty)$  is given by  $g_1(x) = m$  where  $m = \text{Min}\{f(x): x \in K_0\}$ .
- (2) Given  $g_\beta: \bigcup_{\alpha < \beta} K_\alpha \rightarrow (0, \infty)$  we define  $g_{\beta+1}: \bigcup_{\alpha < \beta+1} K_\alpha \rightarrow (0, \infty)$  as follows: Let

$$m = \text{Min} \left( \{f(x): x \in K_\beta\} \cup \left\{g_\beta(x): x \in \left(\bigcup_{\alpha < \beta} K_\alpha \cap K_\beta\right)\right\} \right).$$

For each  $x \in \bigcup_{\alpha < \beta+1} K_\beta$ , let

$$a(x) = d\left(x, \bigcup_{\alpha < \beta} K_\alpha - U_\beta\right), \quad b(x) = d(x, K_\beta).$$

Define  $g_{\beta+1}(x) = (b(x)g_\beta(x) + a(x)m)/(a(x) + b(x))$ . Notice if  $x \notin U_\beta$  then  $a(x) = 0$  and  $g_{\beta+1}(x) = g_\beta(x)$ .

(3) For limit ordinals  $\lambda$  and in particular for  $\sup \mathcal{J}$  let  $g_\lambda(x) = \text{Min}_{\beta < \lambda} g_\beta(x)$ . It is straightforward to verify that  $g(x) = g_{\sup \mathcal{J}}(x)$  is the function we want and is continuous since  $\{U_\alpha\}$  is locally finite so in a neighborhood of any point there are only finitely many changes in  $g_\beta$ .

4. Some remarks about monads in  ${}^*R$ . In this section we make a few observations about the work of the preceding sections when it is particularized to the nonstandard reals,  ${}^*R$ .

**Theorem 4.1.** *Suppose  $x, y \in {}^*R$  and  $|x| < |y|$  then the following are equivalent:*

- (i)  $x$  and  $y$  have the same size;
- (ii)  $\mu(x) = \mu(y) + (x - y)$ ;
- (iii) there is a standard continuous function  $f: [0, \infty) \rightarrow [0, \infty)$  such that  $|x| < |y| < {}^*f(|x|)$ ;
- (iv)  $o(x) = o(y)$ .

**Proof.** We may assume  $0 < x < y$  since all the conclusions are easily symmetric in  $x, -x, y$  and  $-y$ .

(i) is immediately equivalent to (ii) since the map  $T(t) = t + (x - y)$  is an isometry.

(ii) implies (iii). Suppose that  $\mu(x) = \mu(y) + (x - y)$ . Clearly,  $y + 1/y \notin \mu(y)$  and, hence,  $x + 1/y \notin \mu(x)$ . Thus, there is a standard continuous function  $b: R \rightarrow (0, \infty)$  such that  ${}^*b(x) \leq 1/y$ . Let  $g(t) = \text{Min} \{b(s): 0 \leq s \leq t\}/2$ . Then  $g$  is also standard continuous and  $g$  is monotone decreasing. Notice  ${}^*g(x) < 1/y$ . Define  $f(t) = 1/g(t)$  for  $0 \leq t < \infty$ . Then  $f$  is standard, continuous and monotone increasing and  ${}^*f(x) > y$  which completes the proof of (ii) implies (iii).

(iii) implies (iv). Suppose  $f: [0, \infty) \rightarrow [0, \infty)$  is a standard continuous function such that  $x \leq y \leq {}^*f(x)$ . For each positive integer  $n$ , let  $K_n = [n, f(n+1)]$ . We may assume  $y$  is infinite since otherwise the theorem is trivial, and we may also assume by the proof of (ii) implies (iii) that  $f$  is monotone increasing. Hence,  $\text{Lim}_{x \rightarrow \infty} f(x) = \infty$ , and this implies that the family  $\{K_n\}$  is locally finite. Let  $\nu$  be the (infinite) integer such that  $\nu \leq x < \nu + 1$ . Then  $\{x, y\} \subseteq K_\nu$ .

(iv) implies (i) by Theorem 3.17.

This completes the proof of Theorem 4.1.

It is easy to show that if  ${}^*R$  is an enlargement then, for every infinite  $x \in {}^*R$ ,  $\mu(x) \neq \{x\}$ . However, for countable ultrapowers this is false. In fact, we have the following proposition.

**Proposition 4.2.** *Suppose that  $D$  is an ultrafilter on  $\omega$  and that  ${}^*R = D\text{-Prod } R$ . Let  $\nu$  be the infinite integer represented by the identity map. Then  $x \geq \nu$  implies  $\mu(x) = \{x\}$ .*

**Proof.** Suppose that  $y \neq x$  and  $y \in \mu(x)$ . Then since  $\nu \leq x$  it is easy to show that  $|y - x| + \nu \in \mu(\nu)$ . Let  $k$  be any integer such that  $k > 1/|y - x|$ . Let  $k$  be represented by the function  $g: \omega \rightarrow \omega$ . Then  ${}^*g(\nu) = k$ . We can assume  $g$  is non-zero since it is already nonzero on some set  $A \in D$ . Thus,  $|y - x| > 1/{}^*g(\nu)$ , and hence  $|y - x| \notin \mu(\nu)$ , which is a contradiction.

This proposition raises the disturbing possibility that for countable ultrapowers  ${}^*R$  of  $R$ , for every infinite  $x$ ,  $\mu(x) = \{x\}$ . In fact, we will show that this occurs if and only if the underlying ultrafilter is a  $P$ -point.  $P$ -points have been investigated under various names by Rudin [16], Rudin [17], Choquet ([5], [6]), Booth [4], Blass ([2], [3]) and others. We recall the definition:

**Definition 4.3.** Suppose that  $D$  is an ultrafilter on  $\omega$ . Then  $D$  is said to be a  $P$ -point if and only if for every function  $f: \omega \rightarrow \omega$  there is a set  $A \in D$  such that  $f|_A$  is either constant or finite-to-one.

There are lots of  $P$ -points if either the Continuum Hypothesis [16] or Martin's Axiom ([2], [3], [4]) holds.

**Theorem 4.4.** *Suppose that  $D$  is an ultrafilter on  $\omega$  and  ${}^*R = D\text{-Prod } R$ . Then the following are equivalent:*

- (i)  $D$  is a  $P$ -point.
- (ii) For every infinite  $x \in {}^*R$ ,  $\mu(x) = \{x\}$ .

**Proof.** (i) implies (ii). Let  $x$  be an infinite point in  ${}^*R$ . We may assume  $x$  is positive. If  $x \geq \nu$  we are done by Proposition 4.2. Hence we may assume  $x < \nu$ . Let  $k$  be the (infinite) integer such that  $k \leq x < k + 1$ . Let  $k$  be represented by the function  $f: \omega \rightarrow \omega$ . Then since  $D$  is a  $P$ -point there is a set  $A \in D$  such that  $f|_A$  is finite-to-one. We can assume that  $f$  is the identity on  $\omega - A$  without loss of generality. Thus,  $f$  is finite-to-one on all of  $\omega$ . Define  $g: \omega \rightarrow \omega$  by  $g(n) = \max\{i: f(i) = n\}$ . Then  $g(f(i)) \geq i$  for each  $i \in \omega$ . Hence,  ${}^*g(k) \geq \nu$ . We can extend  $g$  continuously to  $R$  and define  $b(t) = \text{Max}\{g(s): s \leq t\}$ . Thus,  ${}^*b(k) \geq \nu$ , and since  $b$  is monotone increasing,  ${}^*b(x) \geq \nu$ . This implies  $\mu(x) = \mu(\nu) + (x - \nu) = \{x\}$  by Theorem 4.1 and Proposition 4.2.

(ii) implies (i). Suppose  $D$  is not a  $P$ -point. Then there is a function  $f: \omega \rightarrow \omega$  such that  $f$  is not finite-to-one on any set  $A \in D$ . Let  $k$  be the infinite integer represented by  $f$ . We claim for every standard function  $g: \omega \rightarrow \omega$ ,  ${}^*g(k) < \nu$ . Hence, by Proposition 4.2 and Theorem 4.1,  $\mu(k) \neq \mu(\nu) + (k - \nu) = \{k\}$ .

**Proof of claim.** Suppose that  ${}^*g(k) \geq \nu$  then let  $A = \{i: g(i) \geq i\}$ . Since  ${}^*g(k) \geq \nu$ ,  $A \in D$ . Hence, there are a  $j \in \omega$  and an infinite subset  $B \subseteq A$  such that, for every  $r \in B$ ,  $f(r) = j$ , which implies  $g(f(r)) = g(j)$ . But since  $B$  is infinite there is an  $r \in B$  such that  $r > g(j)$ . Hence  $g(f(r)) = g(j) < r$ , which is a contradiction. This completes the proof.

It is perhaps even more surprising that the coarse monad of a point can be discrete. In fact, we will show that for minimal ultrafilters  $D$ , if  ${}^*R = D\text{-Prod } R$  then for every infinite point  $x \in {}^*R$ ,  $c(x) = \{x\}$ . Before proving results for the coarse monad analogous to those for the monad,  $\mu$ , we need some well-known definitions and results.

**Definition 4.5.** Suppose that  ${}^*\omega$  is a nonstandard model of the natural numbers and that  $k$  is an infinite integer in  ${}^*\omega$ . Let  $D_k = \{A \subseteq \omega: k \in {}^*A\}$ . Then  $D_k$  is an ultrafilter on  $\omega$ .

**Definition 4.6.** Suppose that  $D$  is an ultrafilter on  $\omega$  and  $f: \omega \rightarrow \omega$  is any function. Let

$$f(D) = \{A \subseteq \omega: f^{-1}(A) \in D\}.$$

Then  $f(D)$  is an ultrafilter.

**Proposition 4.7.** In the situation above  $f(D) = D$  if and only if  $\{i: f(i) = i\} \in D$ .

**Proof.** This result was obtained independently by several investigators. See, for example, [2], [3], or [4].

**Proposition 4.8.** Suppose that  $D$  is an ultrafilter on  $\omega$  and  ${}^*\omega = D\text{-Prod } \omega$ . Suppose  $a \in {}^*\omega - \omega$ , and that, as usual,  $\nu$  is the element of  ${}^*\omega$  represented by the identity map  $\omega \rightarrow \omega$ . Then

$$(i) \ D_\nu = D.$$

$$(ii) \ D_{*f(a)} = f(D_a).$$

**Proof.** (i)  $A \in D_\nu \leftrightarrow \nu \in {}^*A \leftrightarrow \{i: i \in A\} \in D \leftrightarrow A \in D$ .

(ii) Let  $a$  be represented by the function  $b: \omega \rightarrow \omega$ .

$$A \in D_{*f(a)} \leftrightarrow {}^*f(a) \in {}^*A \leftrightarrow \{i: fb(i) \in A\} \in D$$

$$\leftrightarrow \{i: b(i) \in f^{-1}(A)\} \in D \leftrightarrow f^{-1}(A) \in D_a \leftrightarrow A \in f(D_a).$$

**Proposition 4.9.** Suppose that  $D$  is an ultrafilter on  $\omega$  and  ${}^*R = D\text{-Prod } R$ . Then  $c(\nu) = \{\nu\}$ .



**Proof.** For each  $k \in {}^*\omega - \omega$  there is a function  $f: \omega \rightarrow \omega$  such that  $k = {}^*f(\nu)$ . If  $k \neq \nu$  by Proposition 4.7,  $f(D) \neq D$  and, hence,  $D_k = D_{*f(\nu)} \neq D_\nu$ . Thus, there is a set  $A \subseteq \omega$  such that  $\nu \in A$  but  $k \notin A$ .

Now suppose that  $f: R \rightarrow (0, \infty)$  is any standard continuous function and  $A$  is any subset of  $\omega$ . Define  $U(A, f) = \{y: \exists k \in {}^*A, {}^*d(k, y) < {}^*f(k)\}$ . Then  $c(\nu) \subseteq \bigcap_{\nu \in {}^*A, f} U(A, f) = \mu(\nu) = \{\nu\}$ , which completes the proof.

**Definition 4.10.** Suppose that  $D$  is an ultrafilter on  $\omega$ . Then  $D$  is said to be *minimal* provided for every function  $f: \omega \rightarrow \omega$  there is a set  $A \in D$  such that  $f|_A$  is either constant or one-to-one.

Notice that every minimal ultrafilter is a  $P$ -point. Minimal ultrafilters have been investigated by many of the same people who worked on  $P$ -points. If either the Continuum Hypothesis or Martin's Axiom holds there are many minimal ultrafilters.

**Theorem 4.11.** Suppose that  $D$  is a minimal ultrafilter on  $\omega$  and that  ${}^*R = D\text{-Prod } R$ . Then for every infinite  $x \in {}^*R$ ,  $c(x) = \{x\}$ .

**Proof.** (i) First we show that for every infinite integer,  $k$ , in  ${}^*R$ ,  $c(k) = \{k\}$ . Suppose that  $a, b$  are two infinite integers in  ${}^*R$ , represented respectively by  $f_a: \omega \rightarrow \omega$  and  $f_b: \omega \rightarrow \omega$ . Since  $D$  is minimal there is a set  $A \in D$  such that  $f_a|_A$  is one-to-one. By a straightforward argument we may assume that  $f_a$  is actually bijective. Let  $g = f_b f_a^{-1}$ . Then  $b = {}^*g(a)$  and the argument of Proposition 4.9 shows that, for every infinite integer  $k \in {}^*R$ ,  $c(k) = \{k\}$ .

(ii) Now assume that  $x$  is any infinite element of  ${}^*R$ . We will show that  $c(x) = \mu(x)$ , hence by Theorem 4.4  $c(x) = \{x\}$ . Suppose that  $f: R \rightarrow (0, \infty)$  is any standard continuous function. We will construct two standard continuous functions  $b, g: R \rightarrow (0, \infty)$  such that either  $g(x)$  or  $b(x)$  is an integer and  $g^{-1}(g(x)) \subseteq \{y: {}^*d(x, y) < {}^*f(x)\}$  and  $b^{-1}(b(x)) \subseteq \{y: {}^*d(x, y) < {}^*f(x)\}$ . By (i) this implies that  $\{y: {}^*d(x, y) < {}^*f(x)\} \supseteq c(x)$  which implies  $c(x) = \mu(x) = \{x\}$ .

For each positive integer  $n$  let  $e_n = \text{Min}\{f(t): 0 \leq |t| \leq n\}$ . Let  $k_n$  be any integer such that  $1/k_n < e_n$ . Let  $t_{n,i} = (n-1) + i/k_n$ , for  $i = 0, 1, 2, \dots, k_n$ . Let  $j(n, i) = 1 + i + \sum_{p=1}^{n-1} k_p$ . Now suppose  $y \in R$ . We may assume  $y$  is positive, since  $g, b$  will be defined so that  $g(-y) = -g(y)$  and  $b(-y) = -b(y)$ . Then there is a unique  $n$  and  $i$  such that

$$t_{n,i} \leq y < t_{n,i+1} \quad \text{or}$$

$$t_{n,k_n-1} \leq y < t_{n+1,0} = t_{n,k_n}.$$

Now let

$$\begin{aligned} b(y) &= j(n, i) \quad \text{if } j(n, i) \text{ is even,} \\ &= (j(n, i+2)(y - t_{n,i}) + j(n, i)(t_{n,i+1} - y))k_n \quad \text{if } j(n, i) \text{ is odd;} \end{aligned}$$

$$g(y) = j(n, i) \quad \text{if } j(n, i) \text{ is odd,}$$

$$= (j(n, i+2)(y - t_{n,i}) + j(n, i)(t_{n,i+1} - y))k_n \quad \text{if } j(n, i) \text{ is even.}$$

These functions are the ones we want and they complete the proof.

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