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BANACH SPACES WHOSE DUALS CONTAIN $l_1(\Gamma)$ WITH APPLICATIONS TO THE STUDY OF DUAL $L_1(\mu)$ SPACES(1)

BY

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ABSTRACT. THEOREM I. *If E is a separable Banach space such that E' has a complemented subspace isomorphic to $l_1(\Gamma)$ with Γ uncountable then E' contains a complemented, $\sigma(E', E)$ closed subspace isomorphic to $M(\Delta)$, the Radon measures on the Cantor set.*

THEOREM II. *If E is a separable Banach space such that E' has a subspace isomorphic to $l_1(\Gamma)$ with Γ uncountable, then E contains a subspace isomorphic to l_1.*

THEOREM III. *Let E be a Banach space. The following are equivalent:*
(i) *E' is isomorphic to $l_1(\Gamma)$;*
(ii) *every absolutely summing operator on E is nuclear;*
(iii) *every compact, absolutely summing operator on E is nuclear;*
(iv) *if X is a separable subspace of E, then there exists a subspace Y such that $X \subseteq Y \subseteq E$ and Y' is isomorphic to l_1.*

THEOREM IV. *If E is a $\mathcal{L}_\infty$ space then* (i) *E' is isomorphic to $l_1(\Gamma)$ for some set Γ or* (ii) *E' contains a complemented subspace isomorphic to $M(\Delta)$.*

COROLLARY. *If E is a separable $\mathcal{L}_\infty$ space, then E' is* (i) *finite dimensional, or* (ii) *isomorphic to l_1, or* (iii) *isomorphic to $M(\Delta)$.*

COROLLARY. *If $L_1(\mu)$ is isomorphic to the conjugate of a separable Banach space, then $L_1(\mu)$ is isomorphic to l_1 or $M(\Delta)$.*

Introduction. In the past few years, a great deal of information has been obtained about Banach spaces E such that E', the dual of E (the space of continuous linear functionals on E), is *isometric* to some $L_1(\mu) = L_1(S, \Sigma, \mu)$, the Banach space of (equivalence classes of) measurable, absolutely integrable functions on some abstract measure space (S, Σ, μ). See [9] for the most recent results in this area. There apparently have been very few results characterizing those Banach spaces E such that E' is isomorphic to some $L_1(S, \Sigma, \mu)$. Closely related to this problem is the problem raised by Dieudonné [2] of determining those $L_1(\mu)$ spaces which are isomorphic to dual spaces. The study of this problem began with Gelfand who proved that $L_1[0, 1]$ is not isomorphic to a dual space [3]; recently, Pełczyński proved that if μ is σ-finite and not purely atomic then $L_1(\mu)$ is not isomorphic to a dual space [16]. (See [8] and the references of [16]

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for other results.) A positive result is that of D. R. Lewis and the author [8]: if E is isomorphic to a separable dual space and isomorphic to a complemented subspace of some $L_1(\mu)$ space then E is finite dimensional or isomorphic to l_1 (the space of absolutely summable sequences). A complemented subspace is a closed subspace which is the image of a continuous linear projection.

Combining the results given here with those of [8] we give a complete solution of the separable case of the problem of Dieudonné: *if E is an infinite dimensional, separable Banach space such that E' is isomorphic to a complemented subspace of some $L_1(\mu)$, then E' is isomorphic to l_1 or to the normalized functions of bounded variation on the unit interval.*

Preliminaries and statements of results. The single most important technique in the isometric study has been the use of extreme point structure; a technique, of course, not available in the isomorphic study. More importantly, what the isometric and isomorphic have in common is the finite dimensional geometry of the spaces involved. To make this more explicit we repeat the definitions first given by Lindenstrauss and Pełczyński of those spaces which have (at least isomorphically) the same finite dimensional structure as the $L_p(\mu)$ spaces, for some p, $1 \leq p < \infty$, or the $C(K)$ spaces, the continuous functions on a compact Hausdorff space K. By Δ we denote the usual Cantor set.

Recall that if E and F are isomorphic Banach spaces then $d(E, F) = \inf\{\|T\| \cdot \|T^{-1}\| : T$ is an isomorphism from E to $F\}$ is the Banach-Mazur distance between them. For $1 \leq p \leq \infty$, $l_p(\Gamma)$ denotes the Banach space of functions f from the set Γ to the real (or complex) numbers such that $\Sigma_{\gamma \in \Gamma} |f(\gamma)|^p$ is finite if $1 \leq p < \infty$ and f is bounded if $p = \infty$; the norm of f is $(\Sigma_{\gamma\in\Gamma} |f(\gamma)|^p)^{1/p}$ if $1 \leq p < \infty$ and $\sup_{\gamma\in\Gamma} |f(\gamma)|$ if $p = \infty$. By l_p^n (respectively, l_p) we mean $l_p(\Gamma)$ when Γ is the set of the first n positive integers (respectively, Γ is the set of positive integers).

By $M(K)$ we denote the Banach space of Radon measures on the compact Hausdorff space K. By the Riesz representation theorem we know that $C(K)'$ is isometric to $M(K)$. It is a well-known result of Kakutani that $M(K)$ is isometric to some $L_1(\mu)$ space. In the particular case $K = [0, 1]$, the unit interval, the classical Riesz representation theorem tells us that $M(K)$ is isometric to the normalized functions of bounded variation on the unit interval.

Definition [11]. For $1 \leq p \leq \infty$ and $\lambda \geq 1$ a Banach space E is an $\mathcal{L}_{p,\lambda}$ space if for each finite dimensional subspace X of E there exists a finite dimensional subspace Y of E such that $X \subseteq Y \subseteq E$, and $d(Y, l_p^n) \leq \lambda$ where n is the dimension of Y.

A Banach space is an $\mathcal{L}_p$ space if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda \geq 1$. It is easy to see that $L_p(\mu)$ for $1 \leq p \leq \infty$ is an $\mathcal{L}_p$ space and $C(K)$ is an $\mathcal{L}_\infty$ space.

The converse is false for $1 \leq p < \infty$, $p \neq 2$ [11, Example 8.2], and true for $p = 2$ [11, Theorem 7.1] and unknown for $p = \infty$. For $1 \leq p < \infty$, it is known that if E is an $\mathcal{L}_{p,1+\epsilon}$ space for all $\epsilon > 0$, then E is isometric to an $L_p(\mu)$ ([11, p. 309] and [20]). Also, it follows from local reflexivity (see below) that E is an $\mathcal{L}_{p,1+\epsilon}$ space for all $\epsilon > 0$ if and only if E' is an $\mathcal{L}_{q,1+\epsilon}$ space for all $\epsilon > 0$ where $p^{-1} + q^{-1} = 1$ and $1 \leq p,\ q \leq \infty$; E is an $\mathcal{L}_p$ space if and only if E' is an $\mathcal{L}_q$ space for $p^{-1} + q - 1 = 1$ and $1 \leq p,\ q \leq \infty$ [13]. The above theorems tell us that E is an $\mathcal{L}_{\infty,1+\epsilon}$ space for all $\epsilon > 0$ if and only if E' is isometric to an $L_1(\mu)$ space. See [9] for more precise results.

It is a consequence of the results of [13] that E *is an* $\mathcal{L}_\infty$ *space if and only if* E' *is isomorphic to a complemented subspace of some* $L_1(\mu)$. See [11], [12] and [8] for other results about $\mathcal{L}_p$ spaces.

We shall denote Banach spaces by $D, E, F, \cdots, X, Y, Z$, and by $T: E \to F$ we mean an operator (continuous linear function) from E to F. If $\{x_\alpha\}$ is a subset of a Banach space E, then $[x_\alpha]_{\alpha\in\Gamma}$ denotes the smallest subspace of E containing $\{x_\alpha\}$. A subspace of a Banach space is meant to be a closed, linear subspace.

We shall require in addition to the theory of tensor products as given in [4] the following very important result:

Theorem (Principle of local reflexivity [13], [6]). *If* E *is a Banach space regarded as a subspace of* E'', *and* X, Y *finite dimensional subspaces of* E'' *and* E' *respectively and* P *is a projection from* E'' *onto* X, *then for any* $\epsilon > 0$, *there exists an operator* $T: X \to E$ *and a projection* Q on E *such that*

(i) $Tx = x$ *for all* $x \in E \cap X$,

(ii) $\langle Tx, y\rangle = \langle x, y\rangle$ *for all* $x \in X$ *and all* $y \in Y$,

(iii) $\|T\| \cdot \|T^{-1}\| < 1 + \epsilon$,

(iv) Q *is a projection of* E *onto* $T(X)$ *and* $\|Q\| < (1 + \epsilon)\|P\|$.

Of our main results, Theorems III and IV concern $\mathcal{L}_\infty$ spaces and Theorems I and II are more general results because they apply to Banach spaces other than $\mathcal{L}_\infty$ spaces. We now state these results.

Theorem I. *Suppose* E *is a separable Banach space such that* E' *has a complemented subspace isomorphic to* $l_1(\Gamma)$. *Suppose* $K = \{e_\gamma\}_{\gamma\in\Gamma}$ *denotes a set of unit vectors in* E' *equivalent to the usual basis of* $l_1(\Gamma)$, *such that the closed span of* K *is complemented, and we regard* K *with the* $\sigma(E', E)$ *topology. If* K *has an infinite subset dense in itself, then there exists an onto operator* $T: E \to C(\Delta)$ *such that* $T'(M(\Delta))$ *is complemented in* E'.

We should point out that our *hypothesis on* K *is satisfied if* Γ *is uncountable.* Our next theorem is an improvement of a result of Pełczyński [16, Theorem 3.4].

Theorem II. *Suppose E is a separable Banach space such that E' contains a subspace isomorphic to $l_1(\Gamma)$. If K is defined as above, and has an infinite subset that is $\sigma(E', E)$ dense in itself then E has a subspace isomorphic to l_1.*

We also obtain the following improvement of a theorem of D. R. Lewis and the author [8, Theorem 1].

Theorem III. *Let E be a Banach space. The following are equivalent:*

(i) *E' is isomorphic to $l_1(\Gamma)$;*

(ii) *if F is any Banach space and $T: E \to F$ is absolutely summing then T is nuclear;*

(iii) *if F is any Banach space and $T: E \to F$ is absolutely summing and compact, then T is nuclear;*

(iv) *if X is a separable subspace of E, then there exists a space Y such that $X \subseteq Y \subseteq E$ and Y' is isomorphic to l_1.*

From the above we obtain our most general result on the conjugates of the $\mathcal{L}_\infty$ spaces:

Theorem IV. *If E is an $\mathcal{L}_\infty$ space then* (i) *E' is isomorphic to $l_1(\Gamma)$ for some set Γ, or* (ii) *E' contains a complemented subspace isomorphic to* $M(\Delta)$.

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Proofs of results. We begin by giving a proof of Theorem II. This proof is a modification of a proof of Pełczyński [16, Proposition 2.2].

Proof (of Theorem II). Suppose $\{e_\gamma\}_{\gamma \in \Gamma} \subseteq E'$, $\|e_\gamma\| \leq 1$, and $\Sigma_\gamma |t_\gamma| \leq C \|\Sigma_\gamma t_\gamma e_\gamma\|$ for all scalars (t_γ). The unit ball of E' in the $\sigma(E', E)$ topology is a compact metric space and $K = \{e_\gamma\}$ has a subset K_0 that is dense in itself; that is, each nonempty, open subset of K_0 is infinite. We shall construct a sequence of infinite subsets of K_0, $\{A_{ni}\}_{n=1;i=0}^{\infty;2n-1}$, $A_{n+1,2i} \cup A_{n+1,2i+1} \subseteq A_{n,i}$, and a sequence $\{x_n\}_{n=1}^\infty \subseteq E$, $\|x_n\| \leq M$, such that

$$\{e_\gamma \in K_0 : (-1)^k \langle x_n, e_\gamma \rangle > \tfrac{1}{2}\} \supseteq A_{n+1,2k} \cup A_{n+1,2k+1}.$$

Choose A_{10}, A_{11} infinite, disjoint subsets of K_0. Each has an accumulation point, x'_{10}, x'_{11} respectively. There exists a constant $M > 0$ and an $x_1 \in E$, $\|x_1\| \leq M$, such that $\langle x_1, x'_{10} \rangle > \tfrac{1}{2}$ and $\langle x_1, x'_{11} \rangle < -\tfrac{1}{2}$. (This follows from the fact that there exists an element x'' of E'', $\|x''\| \leq C$, such that $\langle x'', x'_{10} \rangle = 1$, $\langle x'', x'_{11} \rangle = -1$, and since the unit ball of E is $\sigma(E'', E')$ dense in the unit ball of E'', there exists the constant M and the x_1 as above.) Since x'_{10}, x'_{11} are accumulation points of A_{10}, A_{11}, respectively, the sets $\{x' \in A_{10}: \langle x, x'\rangle > \frac{1}{2}\} = B_{10}$ and $\{x' \in A_{11}: -\langle x, x'\rangle > \frac{1}{2}\} = B_{11}$ are infinite. Write B_{10} and B_{11} as the disjoint union of infinite sets. Continue by induction: assume we have constructed $\{A_{n,i}\}$, $n = 1, \cdots, m$, $i = 0, \cdots, 2^m - 1$ and $x_1, \cdots x_{m-1}$. Choose $x'_{m,0} \cdots x'_{m,2^m-1}$ accumulation points of $A_{m,0}, \cdots, A_{m,2^m-1}$ respectively. There exists an $x_m \in E$, $\|x_m\| \le M$, $(-1)^k \langle x_m, x'_{m,k}\rangle > \frac{1}{2}$, $k = 0, \cdots, 2^m - 1$, and since the $\{x'_{m,k}\}$ are accumulation points, the sets $B_{m,k} = \{x' \in A_{m,k}: (-1)^k\langle x_m, x'\rangle > \frac{1}{2}\}$ are infinite for each $k = 0, \cdots, 2^m - 1$. Write $B_{m,k} = A_{m+1,2k} \cup A_{m+1,2k+1}$ where these sets are disjoint and infinite. This establishes the sequence $\{x_n\}_{n=1}^\infty$ and the sets $\{A_{n,i}\}_{n=1;i=0}^{\infty;2^n-1}$. We shall show that the sequence $\{x_n\}_{n=1}^\infty$ is equivalent to the usual unit vector basis of l_1. Since we have

$$M\sum_{n=1}^m |s_n| \ge \left\| \sum_{n=1}^m s_n x_n \right\| \ge \sup_{x' \in K_0} \left| \sum_{n=1}^m s_n \langle x_n, x'\rangle \right|$$

we have only to choose x' in the appropriate $A_{m,k}$ such that $s_n\langle x_n, x'\rangle > |s_n|/2$, $1 \le n \le m$, which can obviously be done.

We shall make use of the following definition and lemma in the proof of Theorem I.

Definition. Let E, F be Banach spaces and $T: E \to F$. We shall say that T admits local selections if there exists a constant $\lambda \ge 1$ such that if G is any finite dimensional Banach space, $S: G \to F$, then there exists $\hat{S}: G \to E$, $\|\hat{S}\| \le \lambda \|S\|$, and $T\hat{S} = S$.

Lemma 1. *If F has the bounded approximation property and $T: E \to F$ then the following are equivalent*:

(i) *For each Banach space G, if I denotes the identity operator on G, then $I \otimes T: G \hat{\hat{\otimes}} E \to G \hat{\hat{\otimes}} F$ is onto*;

(ii) *T admits local selections*;

(iii) *T admits approximate local selections*; *that is, there exists a constant $\lambda \ge 1$ such that if G is any finite dimensional Banach space, $S: G \to F$, then for $\epsilon > 0$, there exists $\hat{S}: G \to E$, $\|\hat{S}\| \le \lambda\|S\|$ such that $\|T\hat{S} - S\| < \epsilon$.*

(iv) *T is onto and $T'(F')$ is complemented in E'.*

Proof. To prove that (i) implies (ii) we need only take $G = (\Sigma_\alpha G_\alpha)_{l_2}$ where $\{G_\alpha\}$ is the set of all finite dimensional subspaces of l_∞. Then $I' \otimes T: G' \hat{\hat{\otimes}} E \to G' \hat{\hat{\otimes}} F$ is onto where $I': G' \to G'$ is the identity. By the open mapping theorem there exists a constant $\lambda > 0$ such that if $u \in G' \hat{\hat{\otimes}} F$ then there is a $v \in G' \hat{\hat{\otimes}} E$ such that $\|v\| < \lambda\|u\|$ and $(I \otimes T)v = u$. If we regard each G_α as a subspace of G and u as an operator S from G to F and assume for some α that $S(G_\beta) = 0$ for all $\beta \neq \alpha$, then we know that there is an operator $\hat{S}_1: G \to E$ such that $\|\hat{S}_1\| < \lambda \|S\|$, $T\hat{S}_1|_{G_\alpha} = S$. We need only let $\hat{S} = \hat{S}_1 |_{G_\alpha}$, $\hat{S}(G_\beta) = 0$ for $\beta \neq \alpha$ where $\hat{S}_1$ corresponds to v. This proves (ii). Since (ii) implies (iii) is obvious, we prove that (iii) implies (iv). We know that F has a net of finite rank operators $\{T_\alpha\}$, $\|T_\alpha\| \leq M$, such that T_α converges strongly to the identity. Denote by I_α: $T_\alpha(F) \hookrightarrow F$ the identity operator, then for each α there exists $\hat{I}_\alpha: T_\alpha(F) \to E$, $\|\hat{I}_\alpha\| \leq \lambda$, and $\|T\hat{I}_\alpha - I_\alpha\| < (1 + \dim T_\alpha(F))^{-2}$. The net of operators $\{(\hat{I}_\alpha T_\alpha)'\}$ is uniformly bounded in $\mathfrak{L}(E', F')$ and hence has a cluster point S in the topology of pointwise convergence on $E' \times F$ (that is the $\sigma(\mathfrak{L}(E', F'), E' \hat{\otimes} F)$ topology). We shall show that ST' is the identity operator on F' which will show that $T'(F')$ is complemented in E'. Let $y \in F$, $y' \in F'$: since $\langle y, T'_\alpha y'\rangle$ converges to $\langle y, y'\rangle$, $\|(T\hat{I}_\alpha)' - I'_\alpha\|$ converges to zero, and $\langle y, ST'y'\rangle$ is a cluster point of $\langle y, (\hat{I}_\alpha T_\alpha)'T'y'\rangle$, we have $\langle y, y'\rangle = \lim_\alpha\langle y, T'_\alpha y'\rangle = \lim_\alpha\langle y, (I_\alpha T_\alpha)'y'\rangle = \lim_\alpha\langle y, (T\hat{I}_\alpha T_\alpha)'y'\rangle = \lim_\alpha\langle y, (\hat{I}_\alpha T_\alpha)'T'y'\rangle$. Thus we have proved $\langle y, y'\rangle = \langle y, ST'y'\rangle$ or ST' is the identity on F'. Finally, for (iv) implies (i), we have only to observe that if $T'(F')$ is complemented in E', the canonical operator from $J(G, F')$ to $J(G, E')$ (the space of integral operators from G to F' and from G to E' respectively) is an isomorphism. This operator is the adjoint of $I \otimes T$: $G \hat{\hat{\otimes}} E \to G \hat{\hat{\otimes}} F$; thus $I \otimes T$ is onto.

Before giving the proof of Theorem I we make an elementary observation about the statement of local reflexivity. If X is a finite dimensional subspace of E'', then we know there is an operator $S: X \to E$ such that $\|S\| \cdot \|S^{-1}\| < 1 + \epsilon$ and $Sx = x$ if $x \in X \cap E$. If $X \cap E$ is not trivial then $\|S\| \geq 1$ and $\|S^{-1}\| \geq 1$; if $X \cap E = \{0\}$ we have only to take $x \in E$, $x \neq 0$, and construct $S_1: [x] + X \to E$ such that $S_1 x = x$ and $\|S_1\| \cdot \|S_1^{-1}\| < 1 + \epsilon$. Then we have $\|S_1\| \geq 1$ and $\|S_1^{-1}\| \geq 1$. In either case we may make the formally stronger statement in local reflexivity that $1 \leq \|S\| < 1 + \delta$ and $1 \leq \|S^{-1}\| < 1 + \delta$ where $\delta > 0$ is small enough so that $(1 + \delta)^2 < 1 + \epsilon$. (This follows also from the proof of local reflexivity as given in [6] and [13].)

Proof (of Theorem I). There exists an into isomorphism $\phi: l_1(\Gamma) \to E'$, $\|\phi\| \leq 1$, and a closed subspace Y of E' such that $E' = \phi(l_1(\Gamma)) \oplus Y$. Define the norm on $l_1(\Gamma) \times Y$ to be $\|(\zeta, y)\| = \|\zeta\| + \|y\|$. Define $\psi: l_1(\Gamma) \times Y \to E'$ by $\psi((\zeta, y)) = \phi\zeta + y$. It follows from the open mapping theorem that ψ^{-1} exists and if X is a subspace of $l_1(\Gamma) \times Y$, then $d(X, \psi(X)) \leq \|\psi^{-1}\| = \lambda$, or if X is a subspace of $[l_1(\Gamma) \times Y]'$, then $d(X, (\psi')^{-1}(X)) \leq \lambda$. Let $a_{\gamma_1}, \cdots, a_{\gamma_m}$ be elements of the usual basis of $l_1(\Gamma)$ and $b_{\gamma_1}, \cdots, b_{\gamma_m}$ be the unit vectors of $l_\infty(\Gamma) = [l_1(\Gamma)]'$ such that $\langle a_{\gamma_j}, b_{\gamma_i}\rangle = \delta_{ij}$ for $1 \leq i, j \leq m$, $\langle a_\gamma, b_{\gamma_i}\rangle = 0$ for $\gamma \notin \{\gamma_1, \cdots, \gamma_m\}$. We know that $d([b_{\gamma_i}]_{i=1}^m, l_\infty^m) = 1$ and the elements $(b_{\gamma_i}, 0)$ of $[l_1(\Gamma) \times Y]'$ also have the property that $d([(b_{\gamma_i}, 0)]_{i=1}^m, l_\infty^m) = 1$. Define $e_\gamma = \psi((a_\gamma, 0)) = \phi(a_\gamma)$ and $d_\gamma = (\psi')^{-1}(b_\gamma, 0)$. We have $\langle e_\gamma, d_\beta\rangle = \delta_{\gamma\beta}$ and for any collection of scalars (t_γ) the following inequalities hold:

$$\lambda^{-1}\sum_{\gamma}|t_\gamma| \le \left\|\sum t_\gamma e_\gamma\right\| \le \sum |t_\gamma|, \qquad \max|t_\gamma| \le \left\|\sum_\gamma t_\gamma d_\gamma\right\| \le \lambda \max|t_\gamma|.$$

Since $\|e_\gamma\| \le 1$ for all γ we have that $\{e_\gamma\}$ in the $\sigma(E', E)$ topology is a separable metric space. By hypothesis there exists a $K_0 \subseteq \{e_\gamma\}$ such that K_0 is infinite and dense in itself. By induction we construct the following: a sequence $\{x'_{ni}\}_{n=0;i=0}^{\infty;2^n-1} \subseteq K_0$ with $x'_{n+1,2k} = x'_{n,k}$ for all n, k; a sequence $\{x_{ni}\}_{n=0;i=0}^{\infty;2^n-1} \subseteq E$ with $d([x_{ni}]_{i=0}^{2^n-1}, l_\infty^{2^n}) < (1+2^{-n})\lambda$ and $\sigma(E', E)$ open sets $\{W_{ni}\}_{n=0;i=0}^{\infty;2^n-1}$ in K_0 such that the diameter of W_{ni} is less than 2^{-n}, $W_{n+1,2i} \cup W_{n+1,2i+1} \subseteq W_{n,i}$ and $x'_{n,i} \in W_{n,i}$; for each pair (n, j), if $x' \in W_{n,j} \cap K_0$ then $|\langle x_{n,j}, x'\rangle - 1| < 2^{-2n}$ and if $x' \in W_{n,i} \cap K_0$ for $i \ne j$ then $|\langle x_{n,j}, x'\rangle| < 2^{-2n}$. The first step is very easy: Let x'_{00} be any element of K_0. Since x'_{00} is some e_{γ_0}, there is by the local reflexivity principle (Goldstine's theorem will suffice) an operator S_0: $[d_{\gamma_0}] \to E$ such that $\|S_0\| \cdot \|S_0^{-1}\| < 2$ and $\langle S_0 d_{\gamma_0}, x'_{00}\rangle = \langle S_0 d_{\gamma_0}, e_{\gamma_0}\rangle = \langle d_{\gamma_0}, c_{\gamma_0}\rangle = 1$. Let $x_{00} = S_0 d_{\gamma_0}$ and let W_{00} be an $\sigma(E', E)$ open set in K_0 of diameter less than 1 such that $x'_{00} \in W_{00}$ and $W_{00} \subseteq \{e_\gamma \in K_0 : |\langle x_{00}, e_\gamma\rangle - 1| < 1\}$. Suppose we have made the construction up to some nonnegative integer n: Let $x'_{n+1,2i} = x'_{n,i}$ for $0 \le i \le 2^n - 1$. Choose $x'_{n+1,2i+1} \in K_0 \cap W_{n,i}$, $x'_{n+1,2i+1} \ne x'_{n,i}$ for $0 \le i \le 2^n - 1$. Choose $d_{\gamma_j} \in E''$, $0 \le j \le 2^{n+1} - 1$ as in the first step; that is, $\langle d_{\gamma_j}, x'_{n+1,i}\rangle = \delta_{ji}$, $0 \le i, j \le 2^{n+1} - 1$. By local reflexivity, there exists S_{n+1}: $[d_{\gamma_j}] \to E$, $\|S_{n+1}\|$ and $\|S_{n+1} - 1\|$ are less than $1 + 2^{-n-1}$, $\langle S_{n+1} d_{\gamma_j}, x'_{n+1,i}\rangle = \delta_{ji}$. Let $x_{n+1,j} = S_{n+1} d_{\gamma_j}$ for $0 \le j \le 2^{n+1} - 1$.

From the inequalities

$$\max_{0\le j\le 2^{n+1}-1} |t_j| \le \left\|\sum_{j=0}^{2^{n+1}-1} t_j d_{\gamma_j}\right\| \le \lambda \max_{0\le j\le 2^{n+1}-1} |t_j|,$$

$$(1+2^{-n-1})^{-1}\left\|\sum_{j=0}^{2^{n+1}-1} t_j d_{\gamma_j}\right\| \le \left\|\sum_{j=0}^{2^{n+1}-1} t_j x_{n+1} d_{\gamma_j}\right\|$$
$$\le (1+2^{-n-1})\left\|\sum_{j=0}^{2^{n+1}-1} t_j d_{\gamma_j}\right\|$$

we have that

$$(1+2^{-n-1})^{-1} \max_{0\le j\le 2^{n+1}-1} |t_j| \le \left\|\sum_{j=0}^{2^{n+1}-1} t_j x_{n+1,j}\right\|$$
$$\le \lambda(1+2^{-n-1}) \max_{0\le j\le 2^{n+1}-1} |t_j|.$$

Choose $W_{n+1,i} \subseteq W_{n,[i/2]}$ to be an open subset of K_0 of diameter less than 2^{-n-1}, with $x'_{n+1,i} \in W_{n+1,i}$, and for each $x_{n+1,j}$ if $x' \in W_{n+1,j} \cap K_0$ then $|\langle x_{n+1,j}, x'\rangle - 1| < 2^{-2n-2}$, and if $x' \in W_{n+1,i} \cap K_0$ for $i \neq j$, then $|\langle x_{n+1,j}, x'\rangle| < 2^{-2n-2}$. Such open sets clearly exist by the $\sigma(E', E)$ continuity of $x_{n+1,j}$ on K_0 and the biorthogonality of $\{x_{n+1,i}\}_{i=0}^{2^{n+1}-1}$ and $\{x'_{n+1,i}\}_{i=0}^{2^{n+1}-1}$. This completes the construction.

As everyone knows, if Δ denotes the $\sigma(E', E)$ closure of $\{x'_{ni}\}$, then Δ is homeomorphic to the Cantor set. Define $T: E \longrightarrow C(\Delta)$, $(Tx)(e) = \langle x, e\rangle$ for $e \in \Delta$. We shall show T admits approximate local selections. Let g_{ni} denote the characteristic function of the $\sigma(E', E)$ closure of W_{ni} intersected with Δ; g_{ni} is continuous on Δ. For fixed n, $\{g_{ni}\}_{i=0}^{2^n-1}$ forms a (disjointly supported) partition of unity in $C(\Delta)$. Let G be a finite dimensional Banach space and $S: G \longrightarrow C(\Delta)$ be a continuous linear operator. Suppose $\{z_q, z'_q\}_{q=1}^{p}$ is an orthonormal system for G. For $\delta > 0$, it follows that there exists an n such that $\|S\|\lambda 2^{-n+4} < \epsilon$ and $h_1, \cdots, h_p \in [g_{ni}]$ such that $\|Sz_q - h_q\| < \delta$, $1 \leq q \leq p$. We know $\|Tx_{ni} - g_{ni}\| < 2^{-2n}$ and

$$(1 + 2^{-n})^{-1} \max_{0\le i\le 2^n-1} |t_i| \leq \left\| \sum_{i=0}^{2^n-1} t_i x_{ni} \right\| \leq \lambda(1 + 2^{-n}) \max_{0\le i\le 2^n-1} |t_i|.$$

We have a representation $h_q = \sum_{i=0}^{2^n-1} t_{qi} g_{ni}$ for $1 \leq q \leq p$; define $\hat{S}: G \longrightarrow E$ such that $\hat{S}(z_q) = \sum_{i=0}^{2^n-1} t_{qi} x_{ni}$: let $z = \sum_{q=1}^{p} s_q z_q$; then

$$\begin{aligned}
\|\hat{S}z\| &= \left\| \hat{S}\left(\sum_{q=1}^{p} s_q z_q\right)\right\| = \left\| \sum_{i=0}^{2^n-1} \left(\sum_{q=1}^{p} s_q t_{qi}\right) x_{ni} \right\| \\
&\leq \lambda(1 + 2^{-n}) \max_{0\le i\le 2^n-1} \left| \sum_{q=1}^{p} s_q t_{qi} \right| \\
&= \lambda(1 + 2^{-n}) \left\| \sum_{i=0}^{2^n-1} \left(\sum_{q=1}^{p} s_q t_{qi}\right) g_{ni} \right\| = \lambda(1 + 2^{-n}) \left\| \sum_{q=1}^{p} s_q h_q \right\| \\
&\leq \lambda(1 + 2^{-n}) \left\| \sum_{p=1}^{q} s_q S z_q \right\| + \lambda(1 + 2^{-n}) \sum_{q=1}^{p} |s_q| \, \|h_q - Sz_q\| \\
&\leq \lambda(1 + 2^{-n}) \|Sz\| + \lambda p \delta (1 + 2^{-n}) \max_{1\le q\le p} |s_q| \\
&\leq \lambda(1 + 2^{-n}) \|Sz\| + \lambda p \delta (1 + 2^{-n}) \|z\|.
\end{aligned}$$

If

$$0 < \delta < \epsilon \, \|S\| (4\lambda p)^{-1}/(1+\epsilon)(1+\|S\|)$$

then we have $\|\hat{S}\| < 2\lambda\|S\|$.

Also,

$$\|Sz - T\hat{S}z\| = \left\| \sum_{q=1}^{p} s_q(Sz_q - T\hat{S}z_q) \right\| = \left\| \sum_{q=1}^{p} s_q \left(Sz_q - \sum_{i=0}^{2^n-1} t_{qi} T x_{ni} \right) \right\|$$

$$\leq \sum_{q=1}^{p} |s_q| \, \|Sz_q - h_q\| + \left\| \sum_{q=1}^{p} \left(\sum_{i=0}^{2^n-1} s_q t_{q_i} (g_{ni} - Tx_{ni}) \right) \right\|$$

$$\leq p\delta \max |s_q| + 2^{-n} \max_{0 \leq i \leq 2^n - 1} \left| \sum_{q=1}^{p} s_q t_{qi} \right|$$

$$\leq \frac{\epsilon}{2} \|z\| + 2^{-n}(1+2^{-n}) \left\| \sum_{i=0}^{2^n-1} \left(\sum_{q=1}^{p} s_q t_{qi} \right) x_{ni} \right\|$$

$$= \frac{\epsilon}{2} \|z\| + 2^{-n+1} \|\hat{S}z\| \leq \frac{\epsilon}{2} \|z\| + 2^{-n+1} 2\lambda \|S\| \|z\| < \epsilon \|z\|.$$

Hence we have proved $\|T\hat{S} - S\| < \epsilon$ so $T'(C(\Delta)') = T'(M(\Delta))$ is complemented in E'.

To prove Theorems III and IV we shall need the following known facts which we shall state as lemmas.

Lemma 2. *If E is a Banach space, then the following are equivalent*:

(1) *E is a $\mathcal{L}_\infty$ space.*

(2) *For any Banach space X and for any into isomorphism $I: E \to X$, the kernel of I' is complemented in X'.*

(3) *For any Banach spaces X, Y and any into isomorphism $I: E \to X$, the canonical operator $Y \hat{\otimes} E \to Y \hat{\otimes} X$ is an into isomorphism.*

(4) *For any Banach spaces X, Y and any into isomorphism $I: E \to X$ and any operator $T: Y \to E'$, there exists an operator $\tilde{T}: Y \to X'$ such that $I'\tilde{T} = T$.*

Proofs of Lemma 2 may be found in [19] and [13].

Lemma 3 [18, Lemma 1.3]. *If K is a compact Hausdorff space and E is a subspace of $M(K)$, then either there exists a positive element of $M(K)$, such that each $x \in E$ is absolutely continuous with respect to μ (that is, $E \subseteq L_1(\mu)$) or E contains a subspace F, complemented in $M(K)$, such that F is isomorphic to $l_1(\Gamma)$ with Γ uncountable.*

Proof (of Theorem III). The equivalence of (i) and (ii) is proved in [8, Theorem 1]. We begin by showing (i) implies (iv). Let F be a closed, separable subspace of E, where E is an $\mathcal{L}_{\infty,\lambda}$ space. Then there exists a separable, infinite dimensional subspace G, $F \subseteq G \subseteq E$, such that G is an $\mathcal{L}_{\infty,\lambda+\epsilon}$ space for each $\epsilon > 0$. It follows from Lemma 2 that G' is isomorphic to a complemented subspace of E', so G' is isomorphic to some $l_1(\Lambda)$ ([7], [14]). By Pełczyński's theorem [16, Theorem 3.4], Λ must be countable since G does not contain an isomorphic copy of $L_1[0, 1]$. To see that (iv) implies (ii), we first observe that E is an $\mathcal{L}_{\infty,\lambda}$ space (each separable subspace is an $\mathcal{L}_{\infty,\rho}$ space for some $\rho > 0$). From [19] we know that an absolutely summing operator on E is integral. Suppose we have an operator $T: E \to L_\infty(\mu)$, μ a Radon measure on some compact Hausdorff space. We shall show JT is nuclear where J is the canonical operator from $L_\infty(\mu)$ to $L_1(\mu)$. First we shall show JT is compact: suppose there exist a $\delta > 0$ and a sequence $\{x_n\} \subseteq E$, $\|x_n\| \leq 1$, such that $\|JT(x_n - x_m)\| \geq \delta > 0$ for $n \neq m$. Let G be a subspace of E such that $\{x_n\} \subseteq G$ and G' is isomorphic to l_1. The restriction of JT to G is nuclear (it is integral and G' is isomorphic to l_1 [4, Proposition 9, p. 64]) so the restriction of JT to G is compact or $\{JTx_n\}$ has a Cauchy subsequence. This is a contradiction. Choose a sequence $\{y_n\} \subseteq E$, $\|y_n\| \leq 1$, so that $\{y_n\}$ is dense in $\{JTx: \|x\| \leq 1\}$. Choose G a subspace of E such that $G \supseteq \{x_n\}$, G' isomorphic to l_1. Again, the restriction of JT to G is nuclear; that is, the set $A = \{JTx: x \in G, \|x\| \leq 1\}$ is equimeasurable [4, Proposition 9] so the closure of A is equimeasurable, but the closure of A contains $\{JTx: \|x\| \leq 1\}$ so $\{JTx: \|x\| \leq 1\}$ is equimeasurable and JT is nuclear. This proves that every absolutely summing operator on E is nuclear. Since (ii) implies (iii) is obvious, we have only to show (iii) implies (iv). First we have that E is an $\mathcal{L}_\infty$ space because by (iii) we know that each quasi-nuclear operator on E is nuclear [19, Theorem III.3]. Let F be a separable subspace of E, such that F is an $\mathcal{L}_\infty$ space. If F' is not separable, then by Lemma 3 there exists a complemented subspace isomorphic to $l_1(\Gamma)$ with Γ uncountable. By Theorem I there exists an onto operator $T_1: F \to C(\Lambda)$ such that $T_1'(M(\Lambda))$ is complemented in F'. If we denote $I: F \to E$ the containment operator, then by Lemma 2 there exists an operator $R: M(\Lambda) \to E'$ such that $I'R = T_1'$. Let μ be Haar measure on the Cantor set, considered as the countable product of the discrete group $\{0, 1\}$. Let $g_{ni} \in C(\Lambda)$ be characteristic functions such that $\sum_{i=0}^{2^n-1} g_{ni}(e) = 1$ for all $e \in \Lambda$, $g_{n+1,2k} + g_{n+1,2k+1} = g_{n,k}$, and $\int g_{nk}\, d\mu = 2^{-n}$. Define $S: C(\Lambda) \to c_0$ by $S(f) = \int f g_{ni}\, d\mu$ for $n = 1, 2, \cdots$ and $i = 0, \cdots, 2^n - 1$. Let μ_{ni} be the element of $M(\Lambda)$ corresponding to $g_{n,i}\, d\mu$ and let $x'_{ni} = R\mu_{ni}$. The space $G = [\mu_{ni}]$ is isometric to $L_1(\Lambda, \mu)$ and is complemented in $M(\Lambda)$ (Lebesgue decomposition). Define $T: E \to c_0$ such that $Tx = (\langle x, x'_{ni}\rangle)$; since $\|x'_{ni}\| = \|R\mu_{ni}\| \leq \|R\| \cdot \|\mu_{ni}\| = \|R\| 2^{-n}$, T is a well defined compact operator. To see that T is integral, we shall show $T' = RS'$ is integral. Since $S': l_1 \to M(\Delta)$ and $\{S'\xi: \|\xi\| \leq 1\}$ is contained in the closed, convex, balanced hull of $\{\mu_{ni}\}$ which is a compact, lattice bounded, but not equimeasurable subset of G, S' is integral but not nuclear. Since $T_1'(G)$ is complemented in F', $R(G)$ is complemented in E'. Thus $T' = RS'$ is nuclear if and only if S' is nuclear.

Proof (of Theorem IV). Suppose E is an $\mathfrak{L}_\infty$ space. Using Theorem III, if E' is not isomorphic to $l_1(\Gamma)$ for some Γ, then there exists a separable $\mathfrak{L}_\infty$ space $F \subseteq E$, such that F' is not isomorphic to l_1. We know that F' is not separable. Suppose $I: F \to C(\Delta)$ is an isomorphism. By Lemma 2, there exists a subspace $G \subseteq M(\Delta)$ such that the restriction of I' to G is an isomorphism from G onto F'. Then G is a nonseparable subspace of $M(\Delta)$, hence by Lemma 3 there exists a subspace H isomorphic to $l_1(\Gamma)$, with Γ uncountable, and H is complemented in $M(\Delta)$. Hence we have $I'(H)$ is complemented in F'. By Theorem I there exists a complemented $\sigma(F', F)$ closed subspace $D \subseteq F'$, such that D is isomorphic to $M(\Delta)$. Hence $(I')^{-1}(D) \cap G$ is isomorphic to $M(\Delta)$ and is complemented in G. But G itself is complemented in $M(\Delta)$, so we have

$$(I')^{-1}(D) \cap G \subseteq G \subseteq M(\Delta)$$

and each space is complemented in the one above it and $M(\Delta)$ and $(I')^{-1}(D) \cap G$ are isomorphic. Since $M(\Delta)$ satisfies the condition of Proposition 4 of [14] (that is, $l_1 \hat{\otimes} M(\Delta)$ is isomorphic to $M(\Delta)$) we have that G is isomorphic to $M(\Delta)$ and F' is also. By Lemma 2 we know that F' is isomorphic to a complemented subspace of E', hence E' has a complemented subspace isomorphic to $M(\Delta)$.

Corollary 1. *If E is a Banach space such that E' is isomorphic to a complemented subspace of $M(K)$ for K a compact metric space, then E' is either isomorphic to l_1 or $M(\Delta)$.*

Proof. If K is countable, then $M(K)$ is isomorphic to l_1, and a complemented subspace of l_1 is finite dimensional or is isomorphic to l_1 [15]. By the Banach-Mazur theorem there is an operator $J: C(K) \to C(\Delta)$ that is an isometry into. Then since $M(K)$ is isometrically isomorphic to some $L_1(\mu)$, we know that the canonical operator $I' \otimes J: M(K) \hat{\otimes} C(K) \to M(K) \hat{\otimes} C(\Delta)$ is an isometry into [4, Theorem 2, p. 59]. Hence, by the Hahn-Banach theorem, the adjoint $(I' \otimes J)': \mathfrak{L}(M(K), M(\Delta)) \to \mathfrak{L}(M(K), M(L))$ carries the unit ball onto the unit ball. That is, the identity operator on $M(K)$ has a lifting $\hat{I}: M(K) \to M(\Delta)$, $\|\hat{I}\| = 1$, such that $J'\hat{I} = I$, which implies $M(\Delta)$ has a subspace, complemented by a projection of norm one, isometric to $M(K)$. If K is uncountable, then K contains a subset Δ_1 homeomorphic to Δ. There is a natural embedding of $M(\Delta_1)$ into $M(K)$ that is complemented in $M(K)$

by a projection of norm one. Since $M(\Delta)$ satisfies [15, Proposition 4], we know that $M(K)$ and $M(\Delta)$ are isomorphic if K is an uncountable, compact metric space. So we may assume E' is isomorphic to a complemented subspace G, of $M(\Delta)$. If E' is nonseparable then, by Lemma 3, G contains a subspace isomorphic to $l_1(\Gamma)$, with Γ uncountable, complemented in $M(\Delta)$, so certainly in G. By Theorem I, E' has a complemented subspace isomorphic to $M(\Delta)$. Again, G contains a complemented subspace isomorphic to $M(\Delta)$, and $M(\Delta)$ satisfies [15, Proposition 4]. Thus G is isomorphic to $M(\Delta)$, and by assumption, E' is isomorphic to $M(\Delta)$.

Remark. If E is a separable $\mathcal{L}_\infty$ space, then E satisfies the hypothesis of the above corollary. We have only to consider E as a subspace of $C(\Delta)$. There are, however, nonseparable spaces which satisfy the hypothesis of the corollary: the simplest example, due to Pełczyński, is $C(\Delta) \oplus c_0(\Gamma)$ where Γ has the cardinality of the continuum.

Corollary 2. *Let E be a $\mathcal{L}_\infty$ space; for E' to be isomorphic to $l_1(\Gamma)$ it is necessary and sufficient that any one of the following conditions be satisfied:*

(i) *E does not have l_2 as a quotient space;*

(ii) *E does not have a reflexive quotient space;*

(iii) *E does not contain l_1;*

(iv) *E' has the Schur property: (that is a sequence is weakly Cauchy if and only if it is norm Cauchy);*

(v) *E' does not contain $M(\Delta)$;*

(vi) *every bounded sequence in E has a weakly Cauchy subsequence.*

Proof. These equivalences are easily established by Theorems III and IV and Banach's theorem that $l_1(\Gamma)$ has no infinite dimensional reflexive spaces (see [15] for a proof) and the fact that $L_1[0, 1]$ contains l_2 (the classical Khinchin inequality) so any nonpurely atomic measure space contains l_2; thus $C(\Delta)$ has l_2 as a quotient space.

Remarks, examples and problems. We have of course the obvious problem of generalizing our results to the nonseparable case; in particular, if E is a nonseparable $\mathcal{L}_\infty$ space, does there exists a compact Hausdorff space K such that E' is isomorphic to $M(K)$?

Since we have extensive knowledge of the structure of the $\mathcal{L}_{\infty,1+\epsilon}$ spaces (see [9]) it would be very useful to determine if an $\mathcal{L}_\infty$ space is isomorphic to an $\mathcal{L}_{\infty,1+\epsilon}$ space. This is a well-known problem. What we have shown, in the separable case, is the weaker result that they have the same duals.

Another problem has been to decide if a complemented subspace E of $L_1(\mu)$ is isomorphic to some $L_1(\nu)$. What we have shown is that if $L_1(\mu)$ is $M(K)$, K

an uncountable, compact metric space, and E is also isomorphic to a dual space, then E is isomorphic to $M(K)$ or l_1.

We should point out that there exist Banach spaces that are not $\mathcal{L}_\infty$ spaces which satisfy Theorem I: there exists a separable conjugate space Y and an operator $T: Y \to C(\Delta)$ that is onto and such that $T'(M(\Delta))$ is complemented in Y' [10]. A separable conjugate space is not an $\mathcal{L}_\infty$ space unless it is finite dimensional since a conjugate $\mathcal{L}_\infty$ space Y is injective (this follows from Lemma 2: Y' is a complemented subspace of some $M(K)$, so Y'' is a complemented subspace of some $M(K)'$ which is injective but Y is complemented in Y'' so Y is injective) and an infinite dimensional injective space contains l_∞ [17] and thus cannot be separable.

Note also that l_1 satisfies the hypothesis of Theorem II but not of Theorem I.

Combining Theorem II with a result of James Hagler (see [5] for a proof) that (1) and (4) below are equivalent we have the following improvement of a theorem of Pełczyński [16, Theorem 3.4]:

Theorem. *Let E be a separable Banach space. The following are equivalent*:
(1) *E contains a subspace isomorphic to l_1*;
(2) *$C[0, 1]$ is isomorphic to a quotient space of E*;
(3) *E' contains a subspace isomorphic to $M[0, 1]$*;
(4) *E' contains a subspace isomorphic to $L_1[0, 1]$*;
(5) *E' contains a subspace isomorphic to $l_1(\Gamma)$ with Γ uncountable.*

By an example of Amir [1] there exists a subspace E of $C[0, 1]$ that is isomorphic to $C(\omega^\omega)$ but E is not complemented in $C[0, 1]$. We know from Lemma 2 that the quotient map from $C[0, 1]$ to $C[0, 1]/E$ admits local selections, but does not admit a linear selection (this is equivalent to the kernel, E, being complemented in E).

Some of the previous arguments (e.g., the corollaries after Theorem IV) could have been simplified by the use of the theorem of Milutin [14] that if K is an uncountable, compact metric space, then $C(K)$ is isomorphic to $C(\Delta)$. As Milutin's theorem was not necessary we did not use it; however, we feel that Milutin's theorem can be combined with Theorem I and the results of [12] to prove the following: if E is an infinite dimensional, complemented subspace of $C(K)$, K a compact metric space, then E' is isomorphic to l_1 or E is isomorphic to $C(\Delta)$. More generally, we feel that if E is an $\mathcal{L}_\infty$ space such that E' is not isomorphic to $l_1(\Gamma)$ then E contains a subspace isomorphic to $C(\Delta)$ (an isometric version of this is known [9, Theorem 2.3]). These conjectures are certainly well known; we do feel solutions should follow from previously known facts and Theorem I.

Another possible application of our results might be in relation to the following well-known problem (at least some formulations of it are well known; there are many obviously equivalent ways of stating the problem): If E and F are $\mathfrak{L}_\infty$ spaces and $T: E \to F$ is onto, is the kernel of T an $\mathfrak{L}_\infty$ space? Suppose E is an $\mathfrak{L}_\infty$ space, G a subspace of E such that E/G is isomorphic to $C(\Delta)$. From Lemma 3 and Theorem I we have the following: there exists an $\mathfrak{L}_\infty$ space F, $G \subseteq F \subseteq E$, such that E/F is isomorphic to $C(\Delta)$. Since G^0 (the annihilator of G in E') is not separable (it is isomorphic to $M(\Delta)$), by Lemma 3 there exists $H \subseteq G^0$, H complemented in E', such that H is isomorphic to $l_1(\Gamma)$ with Γ uncountable. From the proof of Theorem I there exists an $\sigma(E', E)$ closed subspace D isomorphic to $M(\Delta)$, complemented in E', and D is contained in the $\sigma(E', E)$ closure of H, so $D \subseteq G^0$. Let $F = D^0$, the annihilator of D in E. Then E/F is isomorphic to $C(\Delta)$. Since F' is isomorphic to E'/D and D is complemented in E', E'/D is an $\mathfrak{L}_1$ space and F is an $\mathfrak{L}_\infty$ space [13]. Also if E is a separable $\mathfrak{L}_\infty$ space such that E' is nonseparable then for any separable $\mathfrak{L}_\infty$ space D, there exist $\mathfrak{L}_\infty$ spaces G, F such that $G \subseteq F \subseteq E$ and F/G is isomorphic to D. We have only to take an onto operator $T: E \to C(\Delta)$ such that kernel $(T) = G$ is an $\mathfrak{L}_\infty$ space, which exists by Theorem I. Suppose we consider D as a subspace of $C(\Delta)$ and let $F = T^{-1}(D)$. It is clear that since T admits local selections F is an $\mathfrak{L}_\infty$ space.

The differences between (ii) and (iii) of Theorem III may seem to be unimportant, but we feel that statement (iii) may be useful in solving the following problem: if E is a complemented subspace of $L_1(\mu)$ and E has the Schur property (weak sequential convergence is equivalent to norm convergence), is E isomorphic to $l_1(\Gamma)$? This problem is related to the following conjecture of A. Lazar and J. R. Retherford: If E is an $\mathfrak{L}_1$ space, then E contains Hilbert space or E is contained in $l_1(\Gamma)$ for some Γ.

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