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INFINITE COMPOSITIONS OF MÖBIUS TRANSFORMATIONS[1]

BY

JOHN GILL

ABSTRACT. A sequence of Möbius transformations $\{t_n\}_{n=1}^{\infty}$, which converges to a parabolic or elliptic transformation t, may be employed to generate a second sequence $\{T_n\}_{n=1}^{\infty}$ by setting $T_n = t_1 \circ \cdots \circ t_n$. The convergence behavior of $\{T_n\}$ is investigated and the ensuing results are shown to apply to continued fractions which are periodic in the limit.

This paper treats the convergence behavior of sequences of Möbius transformations $\{T_n(z)\}$ which are generated in the following way:

Let $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, where $t = \lim t_n$ is either parabolic or elliptic. Set $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$, $n = 2, 3, \cdots$.

Our approach is essentially the same as that of Magnus and Mandell [1], who investigated the cases in which the t_n and t are hyperbolic or loxodromic, and in which the t_n and t are all elliptic. They established conditions on the fixed points $\{u_n\}$ and $\{v_n\}$ of $\{t_n\}$ that insure behavior of $\{T_n(z)\}$ very much like that observed in the special case $t_n = t$ for all n [2]. Convergence is in the extended plane, so that divergence is of an oscillatory nature only.

The present paper consists of results concerning the two remaining possible combinations of t_n and t:

(1) t_n any type and t parabolic, and (2) t_n elliptic or loxodromic and t elliptic. The principal result obtained in the investigation of case (2) is an extension and sharpening of the main theorem in [1].

The parabolic case. First consider the case in which $t = \lim t_n$ is parabolic, with a finite fixed point v. Some conditions on the rates at which u_n and v_n approach v are necessary, as the following example illustrates.

Example 1. Let $t_n = [n/(n+1)]^s z + 1$, where $s = 1 + iy$, $y \neq 0$. Then $t = z + 1$, which is parabolic with fixed point $v = \infty$. We have

$$T_n(z) = z/(n+1)^s + \zeta_n(s),$$

where $\zeta_n(s)$ is the truncated Riemann-Zeta function.

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It can be shown, [3, p. 235], that $\zeta_n(s)$ oscillates finitely as $n \to \infty$ for the prescribed values of s.

Set $X(z) = z/(z-1)$. Then $X^{-1} \circ t_n \circ X(z) = t_n^*(z)$ and $t_n(z)$ are the same type of transformation [1], and $t^* = X^{-1} \circ t \circ X$ has the fixed point $v^* = 1$. Obviously

$$T_n^*(z) = t_1^* \circ \cdots \circ t_n^*(z) = X^{-1} \circ T_n \circ X(z)$$

has the same convergence behavior as $T_n(z)$.

Theorem 1. *Let $\{t_n\}$ be a sequence of Möbius transformations converging to a parabolic transformation t, having a finite fixed point v. If there exists an ordering of u_n and v_n, the fixed points of t_n, such that $\Sigma|u_n - v_n|$ and $\Sigma n|v_{n+1} - v_n|$ both converge, then the sequence $\{T_n(z)\}$ converges in the extended plane for every z.*

Proof. Assume the t_n's and t have been normalized so that $a_n d_n - b_n c_n = ad - bc = 1$, and that $a + d = 2$.

First observe that any t_n may be written implicitly

$$\frac{1}{t_n(z) - v_n} = \frac{k_n}{z - v_n} + q_n, \tag{1}$$

where

$$k_n = 1 \quad \text{if } t_n \text{ is parabolic,}$$
$$= (a_n - c_n u_n)/(a_n - c_n v_n) \quad \text{if } t_n \text{ is nonparabolic}$$

and

$$q_n = c_n \quad \text{if } t_n \text{ is parabolic,}$$
$$= (k_n - 1)/(v_n - u_n) \quad \text{if } t_n \text{ is nonparabolic.}$$

It may easily be shown that $\lim k_n = 1$ and $\lim q_n = c \neq 0$.

Next, set

$$Y_n(z) = 1/(z - v_n), \qquad K_n(z) = k_n \cdot z, \qquad Q_n(z) = q_n + z.$$

Then

$$t_n(z) = Y_n^{-1} \circ Q_n \circ K_n \circ Y_n(z).$$

Set

$$w_n(z) = Q_n \circ K_n \circ Y_n \circ Y_{n+1}^{-1}(z), \quad S_n(z) = Q_n \circ K_n \circ Y_n(z), \qquad n = h, h+1, \cdots,$$

where h will be chosen later. Thus

$$T_n(z) = T_{h-1} \circ Y_h^{-1} \circ w_h \circ \cdots \circ w_n \circ S_n(z).$$

Direct computation shows that $w_n(z) = (p_n z + q_n)/(r_n z + 1)$ where $r_n = v_{n+1} - v_n$ and $p_n = k_n + q_n r_n$.

Set $W_n^b(z) = w_b \circ \cdots \circ w_n(z)$, and consider the convergence behavior of $\{W_n^b \circ S_n(z)\}_{n=b+1}^{\infty}$ for a fixed value of b.

Let $W_n^b(z) = (A_n^b z + B_n^b)/(C_n^b z + D_n^b)$, where

$$A_n^b = p_n A_{n-1}^b + r_n B_{n-1}^b, \tag{2}$$

$$B_n^b = q_n A_{n-1}^b + B_{n-1}^b, \tag{3}$$

$$C_n^b = p_n C_{n-1}^b + r_n D_{n-1}^b, \tag{4}$$

$$D_n^b = q_n C_{n-1}^b + D_{n-1}^b. \tag{5}$$

It follows from (2) and (3) that

$$A_n^b = \prod_b^n p_i + \sum \left(\prod p_i\right) q_{k_1} r_{k_2} + \sum \left(\prod p_i\right) q_{k_1} r_{k_2} q_{k_3} r_{k_4} + \cdots + \sum \left(\prod p_i\right) q_{k_1} r_{k_2} \cdots q_{k_{2j-1}} r_{k_{2j}}, \tag{6}$$

where $b < k_1 < \cdots < k_l \le b + m = n$, $1 < l \le 2j$. The q- and r-factors alternate, and (Πp_i) designates finite p-products with $i \geq b$.

Lemma 1. *Suppose* $\{r_{b+k_j}\}_{j=1}^{l}$ *are the r-factors in a term of* A_n^b. *Then there are no more than s terms having this specific set of r-factors in* A_n^b, *where* $s \le \Pi_{i=1}^{l} k_i$.

Proof. The proof is by induction on the auxiliary recurrence relations:

$$A_{b+m}^b = A_{b+m-1}^b + r_{b+m} B_{b+m-1}^b \quad \text{and} \quad B_{b+m}^b = A_{b+m-1}^b + B_{b+m-1}^b.$$

We observe that

$$p_i = k_i + q_i r_i = 1 + (v_i - u_i) q_i + q_i r_i,$$

so that, by hypothesis, Πp_i converges, and there exists a positive number M such that both $|\Pi p_i|$ and $|q_i|$ are less than M for i greater than some b.

Fix $\epsilon > 0$ and choose b so large that the following conditions are met, in addition to those described above: $|\Pi_b^n p_i - 1| < \epsilon/2$, for $n \geq b$, and $\sum_{m=1}^{\infty} m|r_{b+m}| < l/M$, where $l < \min\{1, M, \epsilon/(2M + \epsilon)\}$.

Consequently, by the preceding remarks and Lemma 1,

$$\left|A_n^b - \prod_b^n p_i\right| \le \sum |(\prod p_i) q_{k_1} r_{k_2}| + \cdots + \sum |(\prod p_i) q_{k_1} \cdots r_{k_{2j}}|$$
$$< M^2(l/M) + \cdots + M^{j+1}(l/M)^j < \epsilon/2.$$

Hence $|A_n^b - 1| \leq |\Pi_b^n p_i - 1| + \epsilon/2 < \epsilon$.

In an entirely similar manner it may be shown that $|C_n^b| < \epsilon$, for a sufficiently large b.

(2) and (3) give

$$A_{b+m}^b - k_{b+m} A_{b+m-1}^b = q_{b+m} r_{b+m} A_{b+m-1}^b + r_{b+m} B_{b+m-1}^b,$$

from which we obtain

$$(7) \qquad A_{b+m}^b - A_{b+m-1}^b = (k_{b+m} - 1) A_{b+m-1}^b + r_{b+m} B_{b+m}^b.$$

Summing both sides of (7),

$$(8) \qquad A_{b+m}^b - p_b = \sum_{j=1}^m (k_{b+j} - 1) A_{b+j-1}^b + \sum_{j=1}^m r_{b+j} B_{b+j}^b.$$

Upon summing, (3) gives

$$(9) \qquad B_{b+m}^b = q_b + \sum_{j=1}^m q_{b+j} A_{b+j-1}^b.$$

Combine (8) and (9) to obtain

$$(10) \qquad A_{b+m}^b = p_b + \sum_{j=1}^m (k_{b+j} - 1) A_{b+j-1}^b + \sum_{j=1}^m r_{b+j} \left(q_b + \sum_{i=1}^m q_{b+i} A_{b+i}^b \right).$$

Thus, from (10), if $|q_{b+n}| < M$ and $|A_m^b| < 3$,

$$|A_{b+m+1}^b - A_{b+m}^b| < 3|k_{b+m+1} - 1| + M|r_{b+m+1}|[1 + 3(m+2)]$$
$$< 3[|k_{b+m+1} - 1| + M(m+3)|r_{b+m+1}|].$$

Therefore

$$|A_{b+m+n}^b - A_{b+m}^b| \leq \sum_{j=1}^n |A_{b+m+j}^b - A_{b+m+j-1}^b|$$
$$\leq 3M \left[\sum_{j=1}^n |v_{b+m+j} - u_{b+m+j}| + \sum_{j=1}^n (m+j+2)|r_{b+m+j}| \right].$$

The last expression on the right may be made arbitrarily small by choosing m sufficiently large and n a positive integer. The Cauchy criterion is satisfied and we have

$$(11) \qquad \lim_{n \to \infty} A_n^b = l(A, b) \approx 1.$$

Similarly,

$$(12) \qquad \lim_{n \to \infty} C_n^b = l(C, b) \approx 0.$$

It is obvious, from (9), that

$$(13) \qquad \lim_{n \to \infty} B_n^b = \infty.$$

Also,

$$A_n^h D_n^h - B_n^h C_n^h = \det W_n^h = \prod_h^{n-2} (\det w_j) = \prod_h^{n-2} k_j = \prod_h^{n-2} [1 + q_j(v_j - u_j)].$$

The hypothesis implies the convergence of this product to some number close to one, as $n \to \infty$. Hence

$$\lim_{n\to\infty} (D_n^h/B_n^h) = l_h \not\approx 0. \tag{14}$$

It is now possible to complete the proof of Theorem 1 for $z \neq v$. We have, from (11), (12), (13), and (14),

$$\lim_{n\to\infty} [W_n^h \circ S_n(z)] = \lim_{n\to\infty} \frac{(A_n^h/B_n^h)S_n(z) + 1}{(C_n^h/B_n^h)S_n(z) + (D_n^h/B_n^h)} = \frac{1}{l_h}.$$

Thus, $\lim_{n\to\infty} T_n(z) = T_{h-1} \circ Y_h(1/l_h)$, $z \neq v$.

We divide numerator and denominator of $W_n^h \circ S_n(v)$ by $S_n(v)$ and find, after some computation, that

$$\lim_{n\to\infty} T_n(v) = T_{h-1} \circ Y_h(1/l_h).$$

Corollary 1. *Let $\{t_n\}$ be a sequence of normalized Möbius transformations converging to t, which is parabolic and has a finite fixed point. If $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, then the convergence of the following four series imply the convergence of $\{T_n(z)\}$ for every z: $\Sigma n|\sqrt{[(a_{n+1} + d_{n+1})^2 - 4]}|$, $\Sigma n|a_{n+1} - a_n|$, $\Sigma n|c_{n+1} - c_n|$, $\Sigma n|d_{n+1} - d_n|$.*

The following example shows that the hypotheses of Theorem 1, although sufficient, are not necessary.

Example 2. Let

$$t_n(z) = [(v_n + 1)z - v_n^2]/[z + (1 - v_n)],$$

where $v_1 = 0$ and $v_n = \Sigma_{k=1}^{n-1}(-1)^k/k$ for $n \geq 2$. Then $\lim v_n = v = -\log 2$, and both t_n and t are parabolic. An intricate investigation, somewhat similar to the proof of Theorem 1, shows that $\{T_n(z)\}$ converges for every $z \neq v$.

The elliptic case. We next consider the case in which $t = \lim t_n$ is elliptic.

Theorem 2. *Let $\{t_n\}$ be a sequence of Möbius transformations having fixed points $\{u_n\}$ and $\{v_n\}$, chosen so that $|k_n| \leq 1$. Let $t = \lim t_n$ be an elliptic transformation having finite fixed points u and v.*

(i) *If $\Sigma|u_n - u_{n-1}| < \infty$, $\Sigma|v_n - v_{n-1}| < \infty$, and $\Pi k_n \to 0$, then $\{T_n(z)\}$ converges for every z except perhaps $z = v$.*

(ii) *If $\Sigma|u_n - u_{n-1}| < \infty$, $\Sigma|v_n - v_{n-1}| < \infty$, and $\Pi|k_n|$ converges, then*

$\{T_n(z)\}$ diverges by oscillation for $z \neq u$, v and converges to distinct values for $z = u$ and $z = v$.

Proof. Set $Y_n(z) = (z - u_n)/(z - v_n)$, $K_n(z) = k_n z$, $w_{n-1}(z) = K_{n-1} \circ Y_{n-1} \circ Y_n^{-1}(z)$, $S_n(z) = K_n \circ Y_n(z)$, and $W_n^h(z) = w_h \circ \cdots \circ w_{n-1}(z) = (A_n^h z + B_n^h)/(C_n^h z + D_n^h)$. Then

$$t_n(z) = Y_n^{-1} \circ K_n \circ Y_n(z),$$

and

$$T_n(z) = T_{h-1} \circ Y_h^{-1} \circ W_n^h \circ S_n(z).$$

As before, $w_n(z) = (p_n z + q_n)/(r_n z + 1)$, where $p_n = k_n(v_{n+1} - u_n)/(v_n - u_{n+1})$, etc.

We choose a positive ϵ and find an h such that $|A_n^h - \Pi_h^n p_j| < \epsilon$ and $|C_n^h| < \epsilon$ for $n > h$. Thus $\lim_{n\to\infty} B_n^h = l(B, h) \approx 0$ and $\lim_{n\to\infty} D_n^h = l(D, h) \approx 1$.

The following formula is established by induction:

$$A_n^h = \prod_h^n p_j + \sum_{m=h}^{n-2} \left(\prod_{m+1}^n p_j \right) r_{m+1} B_m^h + r_n B_{n-1}^h. \tag{15}$$

We observe that $\Pi_h^n |p_j| = \Pi_h^n |k_j| \cdot \Pi_h^n (1 + s_j)$, where $\Sigma |s_j| < \infty$. Therefore, in case (i), $\Pi_h^n |p_j| \to 0$, as $n \to \infty$. The three terms in (15) tend to zero, as $n \to \infty$. Hence, $\lim_{n\to\infty} A_n^h = 0$. In similar fashion, $\lim_{n\to\infty} C_n^h = 0$.

Consequently,

$$\lim_{n\to\infty} T_n(z) = T_{h-1} \circ Y_h^{-1} \circ \lim_{n\to\infty} W_n^h(S_n(z)) = T_{h-1} \circ Y_h^{-1} \circ \frac{l(B, h)}{l(D, h)}$$

for $z \neq v$.

The hypotheses of case (ii), and the observed behavior of the coefficients of W_n^h provide a straightforward proof of the next lemma.

Lemma 2. *For a fixed $z \neq v$, there exist finite numbers M and h_0 such that $h > h_0$, $n \geq h$, $m \geq h - 1$ imply $|S_n(z)| < M$ and $|T_n^h(z) - v_m| > |u - v|/4(1 + M)$.*

Using (1) and the fact that

$$\frac{1}{t_{n+1}(z) - v_n} = \frac{1}{t_{n+1}(z) - v_{n+1}} + \frac{v_n - v_{n+1}}{(t_{n+1}(z) - v_n)(t_{n+1}(z) - v_{n+1})},$$

the following formula may be established by induction on n:

$$\frac{1}{T_n^h(z) - v_h} = \frac{\Pi_h^n k_j}{z - v_n} + \sum_{m=h}^{n-1} \left(\prod_h^m k_j \right) \frac{v_m - v_{m+1}}{(T_n^{m+1}(z) - v_m)(T_n^{m+1}(z) - v_{m+1})} + \sum_{m=h-1}^{n-1} \left(\prod_h^m k_j \right) \frac{k_{m+1} - 1}{v_{m+1} - u_{m+1}}, \tag{16}$$

where $\Pi_h^{h-1} k_j \equiv 1$.

We may rewrite (16) in the form

$$\frac{1}{T_n^b(z)-v_b}=\frac{(\Pi_b^n k_j)(z-u_n)}{(z-v_n)(v_n-u_n)}$$

$$(17)\qquad +\sum_b^{n-1}\left(\prod_b^m k_j\right)\frac{v_m-v_{m+1}}{(T_n^{m+1}(z)-v_m)(T_n^{m+1}(z)-v_{m+1})}$$

$$+\sum_{b+1}^{n-1}\left(\prod_b^m k_j\right)\frac{v_{m+1}-v_m+u_m-u_{m+1}}{(v_m-u_m)(v_{m+1}-u_{m+1})}+\frac{k_b-1}{v_b-u_b}-\frac{k_b}{v_{b+1}-u_{b+1}}.$$

Set

$$\prod_b^n k_j=\exp\left(i\sum_b^n\theta_j\right)\prod_b^n|k_j|,$$

$F=F(z)=(z-u)/(z-v)(v-u)$, $R=|F|\sin(|\theta'|/4)$, where $\arg k=\theta=\theta' \pmod{2\pi}$, $|\theta'|\le\pi$.

We choose b so large that the following conditions are satisfied, in addition to previous stipulations:

$$(18)\qquad |f_1|<\frac{R}{6},\quad \text{where } F+f_1=\frac{z-u_n}{(z-v_n)(v_n-u_n)},$$

$$(19)\qquad |f_2|<\frac{R}{6},\quad \text{where } \frac{k_b-1}{v_b-u_b}-\frac{k_b}{v_{b+1}-u_{b+1}}=f_2+\frac{1}{u-v},$$

$$(20)\qquad |f_3|<\min\{1,\ R/6|F|\},\quad \text{where } \prod_b^n|k_j|=1+f_3,$$

$$(21)\qquad \sum_b|v_{m+1}-v_m|<\frac{R|v-u|^2}{96(1+M)^2},$$

$$(22)\qquad \sum_b|u_{m+1}-u_m|<\frac{R|v-u|^2}{48},$$

$$(23)\qquad |v_m-u_m|>\frac{|v-u|}{2},\qquad m\ge b-1.$$

Then, from (17), we obtain

$$(24)\qquad \frac{1}{T_n^b(z)-v_b}=|F|\exp\left[i\left(\arg F+\sum_b^n\theta_j\right)\right]+\frac{1}{u-v}+H(b,n),$$

where $|H(b,n)|<R$.

The sum of the first two terms of (24) is a point on a circle C with center $1/(u-v)$ and radius $|F|$. Hence $1/(T_n^b(z)-v_b)$ lies in a disc $U(b,m)$ at radius R with center g_n on C. R has been chosen so that three tangent discs of radius

R with centers on C can be constructed if the centers of the two end discs are separated by a central angle of θ'.

Clearly, the sequence $\{1/(T_n^b(z) - v_b)\}_{n=b}^{\infty}$ diverges by oscillation, so that $\{T_n^b(z)\}_{n=b}^{\infty}$ must do likewise. The pattern of divergence bears a close resemblence to that observed when $t_n = t$ for all n. In this special case

$$\frac{1}{T_n(z) - v} = |F| \exp[i(\arg F + n\theta)] + \frac{1}{u - v}.$$

Convergence at $z = u$ is easily established, since $S_n(u) \to 0$. We return to the beginning of the proof of case (ii) and interchange the u_n's and v_n's, in order to show convergence at $z = v$. The development in [1] can be paraphrased to show that $\lim T_n(u) \neq \lim T_n(v)$.

Corollary 2. *If the transformations* t_n *converge to the elliptic transformation* t, *where* $a_n d_n - b_n c_n = ad - bc = 1$ *and* $\Sigma|a_n - a_{n-1}|$, $\Sigma|b_n - b_{n-1}|$, $\Sigma|c_n - c_{n-1}|$, *and* $\Sigma|d_n - d_{n-1}|$ *all converge, then* $\{T_n(z)\}$

(i) *converges for* $z \neq v$, *if* $\Pi k_n \to 0$,

(ii) *diverges for* $z \neq u, v$, *and converges to distinct values at* u *and* v, *if* $\Pi|k_n|$ *converges.*

Continued fractions may be interpreted as compositions of Möbius transformations, and may be written so as to display the fixed points. Set $t_n(z) = -u_n v_n/(-(u_n + v_n) + z)$, to obtain

$$\frac{-u_1 v_1}{-(u_1 + v_1)} + \frac{-u_2 v_2}{-(u_2 + v_2)} + \cdots, \tag{25}$$

whose nth approximant is $T_n(0)$.

The following two examples are applications of Theorems 1 and 2 to continued fractions which are periodic in the limit.

Example 3. Let $u_n = |u_n|\exp(i\theta_n)$, $v_n = |v_n|\exp(i\phi_n)$, where $\lim|u_n| = \lim|v_n| = c \neq 0$, $\lim \theta_n = \theta$, $\lim \phi_n = \phi$, $\theta \not\equiv \phi \pmod{2\pi}$. Then

$$\lim k_n = \lim|u_n/v_n|\exp[i(\theta_n - \phi_n)] = k = \exp[i(\theta - \phi)],$$

so that t is elliptic. Theorem 2, case (i) guarantees the convergence of (25), provided $|u_n|$ and $|v_n|$ are chosen so that $\Pi|u_n/v_n| \to 0$, (e.g., $|u_n| = 1 - 1/n^2$, $|v_n| = 1 + 1/n$).

Example 4. Let $u_n = c + \epsilon_n$, $v_n = c + \delta_n$, where $\lim \epsilon_n = \lim \delta_n = 0$, $c \neq 0$, $\Sigma|\epsilon_n - \delta_n| < \infty$, $\Sigma n|\delta_{n+1} - \delta_n| < \infty$, (e.g., $u_n = -\frac{1}{2} - i/n^2$, $v_n = -\frac{1}{2} + i/n^2$). Then t is parabolic, and Theorem 1 insures the convergence of (25).

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DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80521

Current address: Department of Mathematics, Southern Colorado State College, Pueblo, Colorado 81005