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LOCALLY B^*-EQUIVALENT ALGEBRAS. II[1]

BY

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ABSTRACT. Let A be a locally B^*-equivalent Banach *-algebra. Then A possesses a unique norm $|\cdot|$ with the property that $|a^*a| = |a|^2$ for all $a \in A$. Let B be the B^*-algebra which is the completion of A in the norm $|\cdot|$. In this paper it is shown that there exists a closed B^*-equivalent *-ideal of A which contains the maximal GCR ideal of B. In particular, when B is a GCR algebra, then $A = B$.

1. **Introduction.** Let A be a Banach *-algebra. A is B^*-equivalent if A is *-isomorphic to some B^*-algebra B. Then by [6, Theorem (4.1.20)] and the Closed Graph Theorem, the *-isomorphism between A and B is a homeomorphism. Thus A and B have the same algebraic and topological structure (but not necessarily the same geometric structure). We call A locally B^*-equivalent if for every selfadjoint element $a \in A$, the closed *-subalgebra of A generated by a is B^*-equivalent. Whether or not every locally B^*-equivalent Banach *-algebra is B^*-equivalent is not known. Some partial results on this question have been obtained in [2].

Now assume that A is a locally B^*-equivalent Banach *-algebra. Then A possesses a unique norm $|\cdot|$ with the property that $|a^*a| = |a|^2$ for all $a \in A$. Let B be the B^*-algebra which is the completion of A in the norm $|\cdot|$. In this paper, which is a sequel to [2], we prove that there exists a closed B^*-equivalent *-ideal of A which contains the maximal GCR ideal of B (Theorem 2). Thus if B is a GCR algebra, then $A = B$.

H. Behncke has considered some questions concerning locally B^*-equivalent algebras in [3]. However, there is almost no overlap of his paper with either this paper or [2].

2. **Preliminary remarks.** Throughout this paper A *is a locally B^*-equivalent Banach *-algebra with norm* $\|\cdot\|$. By [2, Proposition 2.1], there is a unique norm $|\cdot|$ on A with the B^*-property (that is, $|a^*a| = |a|^2$ for all $a \in A$). Throughout this paper B *is the completion of A with respect to the norm* $|\cdot|$. Then A is B^*-equivalent if and only if $A = B$.

By [6, Corollary (4.1.16)], there exists a constant $M > 0$ such that $|a| \leq M\|a\|$ for all $a \in A$. Since A is locally B^*-equivalent, the norms $\|\cdot\|$ and $|\cdot|$

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are equivalent on the closed *-subalgebra of A generated by a selfadjoint element of A. Therefore, if $a = a^* \in A$, there exists $K > 0$ such that $\|a^n\| \le K|a^n|$ for all $n \ge 1$. Then

$$\|a^n\|^{1/n} \le K^{1/n}|a^n|^{1/n} = K^{1/n}|a|.$$

It follows that the spectral radius of a selfadjoint element a is $|a|$.

The norms $\|\cdot\|$ and $|\cdot|$ are equivalent on every maximal commutative *-subalgebra of A by [2, Proposition 2.2]. This implies a fact that we use repeatedly: if $\{a_n\}$ is a sequence of selfadjoint elements of A and $a_n a_m = 0$ whenever $n \ne m$, then there exists $K > 0$ such that $\|a_n\| \le K|a_n|$ for all $n \ge 1$ (proof—the sequence $\{a_n\}$ is contained in some maximal commutative *-subalgebra of A).

3. **Preliminary results.** In this section we prove some basic lemmas. If I is a $|\cdot|$-closed ideal of A, we let $|\cdot|_q$ denote the usual quotient norm on A/I,

$$|a + I|_q = \inf\{|a - b| \mid b \in I\}.$$

Lemma 1. *Let I be a closed ideal of B. Then*

(1) $A \cap I$ is $|\cdot|$-dense in I, and

(2) if $\|\cdot\|$ and $|\cdot|$ are equivalent on $A \cap I$, then $I \subset A$.

Proof. Since I is a closed ideal in the B^*-algebra B, I is a *-ideal (see [6, Theorem (4.9.2)]). Then $A \cap I$ is a $|\cdot|$-closed (and therefore also $\|\cdot\|$-closed) *-ideal of A. Let J be the $|\cdot|$-closure of $A \cap I$ in B. Then $A \cap J = A \cap I$. The quotient algebra $A/A \cap I$ is naturally embedded in the quotient algebras B/I and B/J by the maps $a + A \cap I \to a + I$ and $a + A \cap I \to a + J$, respectively. Define two norms on $A/A \cap I$ by

$$|a + A \cap I|_1 = |a + I|_q \quad \text{and} \quad |a + A \cap I|_2 = |a + J|_q.$$

Since $|a + I|_q$ and $|a + J|_q$ are B^*-norms, then $|\cdot|_1$ and $|\cdot|_2$ are norms on $A/A \cap I$ that have the B^*-property. By [2, Proposition 2.3], $A/A \cap I$ is locally B^*-equivalent. Then by [2, Proposition 2.1], $|a + A \cap I|_1 = |a + A \cap I|_2$ for all $a \in A$. It follows that $|a + I|_q = |a + J|_q$ for all $a \in A$. Now assume $t \in I$, and choose a sequence $\{t_n\}$ in A such that $|t_n - t| \to 0$. Then $|(t_n - t) + I|_q \to 0$, and since $t \in I$, it follows that $|t_n + I|_q \to 0$. Therefore $|t_n + J|_q \to 0$. This means that there exists $\{s_n\} \subset J$ such that $|t_n - s_n| \to 0$. Then $|t - s_n| \to 0$, so that $t \in J$. This proves (1).

Now assume that $\|\cdot\|$ and $|\cdot|$ are equivalent on $A \cap I$. If $t \in I$, by (1) we can choose $\{t_n\} \subset A \cap I$ such that $|t_n - t| \to 0$. Then the sequence $\{t_n\}$ is $|\cdot|$-Cauchy, and therefore $\|\cdot\|$-Cauchy. Therefore there exists $s \in A$ such that $\|t_n - s\| \to 0$. Then $|t_n - s| \to 0$ so that $s = t$.

If $a = a^* \in A$, let $C(a)$ be the closed *-subalgebra of A generated by a. Let $\Phi_{C(a)}$ be the carrier space of $C(a)$, and if $b \in C(a)$, let $\hat{b}$ denote the Gelfand transform of b. Since A is locally B^*-equivalent, $b \rightarrow \hat{b}$ is a bicontinuous *-isomorphism of $C(a)$ onto the algebra of all complex continuous functions that vanish at infinity on $\Phi_{C(a)}$. Now let D be a *-subalgebra of A. By [2, Lemma 2.5], if there exists $K > 0$ such that $K|a| \geq \|a\|$ for all $a = a^* \in D$, then the $\|\cdot\|$-closure of D is B^*-equivalent. We use these results in the proof of the next lemma.

Lemma 2. *Let I be a *-subalgebra of A with the properties:*

(1) *if $t^* = t \in I$, then $C(t) \subset I$,*

(2) *if $t \in I$, then $(1 - t^*)I(1 - t) \neq \{0\}$.*

Then there exists $h \in I$ such that the $\|\cdot\|$-closure of $(1 - h^*)I(1 - h)$ is B^*-equivalent.

Proof. Suppose that there is no $h \in I$ such that the $\|\cdot\|$-closure of $(1 - h^*)I(1 - h)$ is B^*-equivalent. Then the $\|\cdot\|$-closure of I is not B^*-equivalent, so that there exists $h_1^* = h_1 \in I$ such that $|h_1| < \frac{1}{2}\|h_1\|$. Choose $g_1^* = g_1 \in C(h_1)$ such that $\hat{g}_1$ has compact support in $\Phi_{C(h_1)}$ and $|g_1| < \frac{1}{2}\|g_1\|$. Then choose $k_1^* = k_1 \in C(h_1)$ such that $\hat{k}_1 \equiv 1$ on the support of $\hat{g}_1$. Note that $g_1(1 - k_1) = (1 - k_1)g_1 = 0$. Let $I_1 = (1 - k_1)I(1 - k_1)$. By assumption the $\|\cdot\|$-closure of I_1 is not B^*-equivalent. Then choose $h_2^* = h_2 \in I_1$ such that $|h_2| < \frac{1}{4}\|h_2\|$. Choose as before $g_2^* = g_2$ and $k_2^* = k_2$ in $C(h_2)$ such that $|g_2| < \frac{1}{4}\|g_2\|$ and $g_2(1 - k_2) = (1 - k_2)g_2 = 0$. Since $g_1h_2 = h_2g_1 = 0$, then $g_1k_2 = k_2g_1 = 0$ and $g_1g_2 = g_2g_1 = 0$. Let $I_2 = (1 - k_2)(1 - k_1)I(1 - k_1)(1 - k_2)$. If $u = (1 - k_2)(1 - k_1)v(1 - k_1)(1 - k_2)$ for some $v \in I$, then $g_1u = ug_1 = 0$ and $g_2u = ug_2 = 0$. As before choose $g_3 = g_3^*$ and $k_3 = k_3^*$ in I such that $|g_3| < (\frac{1}{2})^3\|g_3\|$, $(1 - k_3)g_3 = g_3(1 - k_3) = 0$, and $g_ig_j = 0$ if $i \neq j$. Continuing in this fashion we can construct a sequence of self-adjoint elements $\{g_k\} \subset I$ such that $g_ig_j = 0$ whenever $i \neq j$ and $|g_k| < (\frac{1}{2})^k\|g_k\|$. This contradicts the fact that A is locally B^*-equivalent.

Lemma 3. *Let $\{I_\lambda\}$, $\lambda \in \Lambda$, be a collection of closed *-ideals of A, and let $I = \bigcup_{\lambda\in\Lambda} I_\lambda$. If I_λ is B^*-equivalent for each $\lambda \in \Lambda$, and I is a *-subalgebra of A, then the $\|\cdot\|$-closure of I is B^*-equivalent.*

Proof. Assume that there exists an element $t \in I$ such that $(1 - t^*)I(1 - t) = \{0\}$. Then for any $a \in I$, $(1 - t^*)a^*a(1 - t) = 0$, so that $a(1 - t) = 0$. There exists $\mu \in \Lambda$ such that $t \in I_\mu$. Then $a = at \in I_\mu$. Then $I = I_\mu$, which proves the lemma in this case.

Now assume that $(1 - t^*)I(1 - t) \neq \{0\}$ whenever $t \in I$. By Lemma 2 there exists $h \in I$ such that the $\|\cdot\|$-closure of $(1 - h^*)I(1 - h)$ is B^*-equivalent. Assume that $\{b_n\}$ is a $|\cdot|$-Cauchy sequence in the $\|\cdot\|$-closure of I. For each n,

$$b_n = [(1 - h^*)b_n(1 - h)] + [(1 - h^*)b_n h + h^* b_n].$$

There exists $\mu \in \Lambda$ such that $h \in I_\mu$. Then $\{(1 - h^*)b_n h + h^* b_n\}$ is a $|\cdot|$-Cauchy sequence in I_μ, and $\{(1 - h^*)b_n(1 - h)\}$ is a $|\cdot|$-Cauchy sequence in the $\|\cdot\|$-closure of $(1 - h^*)I(1 - h)$. Since both these $|\cdot|$-Cauchy sequences converge to some element of A, then $\{b_n\}$ converges to some element of A. It follows that the $\|\cdot\|$-closure of I is B^*-equivalent.

4. **Locally B^*-equivalent algebras of vector valued functions.** Let Ω be a compact Hausdorff space and denote by $C_R(\Omega)$ the algebra of all continuous real-valued functions on Ω. For each $\omega \in \Omega$, let B_ω be a B^*-algebra with norm $|\cdot|_\omega$. Define $C(\Omega, B_\omega)$ to be the algebra of all functions f defined on Ω such that $f(\omega) \in B_\omega$ for all $\omega \in \Omega$ and such that the function $\omega \longrightarrow |f(\omega)|_\omega$ is in $C_R(\Omega)$. The algebra $C(\Omega, B_\omega)$ is a B^*-algebra in the norm

$$|f| = \sup_{\omega \in \Omega} |f(\omega)|_\omega.$$

Throughout this section we assume that B is a closed $*$-subalgebra of $C(\Omega, B_\omega)$ which is closed under multiplication by functions in $C_R(\Omega)$.

Theorem 1. *Assume that A is a dense $*$-subalgebra of B such that A is a locally B^*-equivalent Banach algebra. Furthermore, assume that if $\omega_1, \cdots, \omega_n$ are any n distinct points in Ω and $b_1, \cdots, b_n$ are such that $b_k \in B_{\omega_k}$, $1 \le k \le n$, then there exists $g \in A$ such that $g(\omega_k) = b_k$, $1 \le k \le n$.*

Then $A = B$.

We proceed with the proof of Theorem 1 by establishing several lemmas. If V is a subset of Ω, let $J_V = \{f \in B \mid f(\omega) = 0 \text{ whenever } \omega \notin V\}$. We say that $\|\cdot\|$ and $|\cdot|$ are equivalent on A at $\omega \in \Omega$ if there exists an open neighborhood V of ω such that $\|\cdot\|$ and $|\cdot|$ are equivalent on the ideal $A \cap J_V$.

Lemma 4. *The norms $\|\cdot\|$ and $|\cdot|$ are equivalent on A at all but at most a finite number of points of Ω.*

Proof. Suppose that there is an infinite sequence of distinct points $\{\omega_n\} \subset \Omega$ such that $\|\cdot\|$ and $|\cdot|$ are not equivalent on A at ω_n for all $n \ge 1$. By choosing a suitable subsequence of $\{\omega_n\}$ if necessary, we may assume that there exists a disjoint sequence of open subsets of Ω, $\{U_n\}$, such that $\omega_n \in U_n$ for $n \ge 1$. Then for each $n \ge 1$ there exists $g_n^* = g_n \in J_{U_n} \cap A$ such that $n|g_n| < \|g_n\|$. Since $g_n g_m = g_m g_n = 0$ when $n \ne m$, this contradicts the assumption that A is locally B^*-equivalent.

If $|\cdot|$ and $\|\cdot\|$ are equivalent on A at every point in Ω, let $I = B$. In the case where there exist points in Ω at which $|\cdot|$ and $\|\cdot\|$ are not equivalent on A, then by Lemma 4 there must be only a finite number of such points. Let

$\{\omega_1, \cdots, \omega_n\}$ be this collection of points. Define I in this case to be the set of all $f \in B$ such that $f(\omega_k) = 0$, $1 \le k \le n$.

Lemma 5. $I \subset A$.

Proof. For each $\omega \in \Omega$ such that $\omega \ne \omega_k$, $1 \le k \le n$, there exists an open neighborhood U of ω in Ω such that $|\cdot|$ and $\|\cdot\|$ are equivalent on $A \cap J_U$. It follows from Lemma 1 that $J_U \subset A$. Let K be a closed subset of Ω disjoint from $\{\omega_1, \cdots, \omega_n\}$ (when $I = B$, let $K = \Omega$). Choose $\{U_1, \cdots, U_m\}$ a finite open cover of K such that $|\cdot|$ and $\|\cdot\|$ are equivalent on $A \cap J_{U_k}$, $1 \le k \le m$. Then, as noted above, $J_{U_k} \subset A$, $1 \le k \le m$. Let $U_0 = \Omega \setminus K$. Choose $f_k \in C_R(\Omega)$ such that $f_k(\omega) = 0$ whenever $\omega \notin U_k$, $0 \le k \le m$, and such that $1 = \sum_{k=0}^m f_k$. If $g \in J_K$, then $f_0 g = 0$. Then $g = \sum_{k=0}^m f_k g = \sum_{k=1}^m f_k g$. By hypothesis B is closed under multiplication by functions in $C_R(\Omega)$. Therefore $f_k g \in J_{U_k} \subset A$ for $1 \le k \le m$. (In the case where $I = B$, the proof is now complete.) There exist constants M_k such that $\|b\| \le M_k |b|$ whenever $b \in J_{U_k}$, $1 \le k \le m$. Then

$$\|g\| \le \sum_{k=1}^m \|f_k g\| \le \sum_{k=1}^m M_k |f_k g| \le \left(\sum_{k=1}^m M_k \|f_k\|_\infty \right) |g|.$$

Therefore $|\cdot|$ and $\|\cdot\|$ are equivalent on the closed *-ideal $J_K \subset A$. Let $I_0 = \bigcup J_K$ where the union is over all closed subsets K of Ω that are disjoint from $\{\omega_1, \cdots, \omega_n\}$. By Lemma 3, $|\cdot|$ and $\|\cdot\|$ are equivalent on the $\|\cdot\|$-closure of I_0. But I_0 is $|\cdot|$-dense in I. Therefore the same proof as given for part (2) of Lemma 1 shows that $I \subset A$.

Now we complete the proof of Theorem 1. Assume that $f \in B$. Let $\{\omega_1, \cdots, \omega_n\}$ be as above. By hypothesis there exists $g \in A$ such that $g(\omega_k) = f(\omega_k)$, $1 \le k \le n$. Then $f - g \in I$. But by Lemma 5, $I \subset A$, and therefore $f \in A$.

5. **The main result.** Assume that B is a CCR (liminaire) algebra with Hausdorff structure space Ω. To each point $P \in \Omega$ associate the B^*-algebra $B_P = B/P$. Each of the algebras B_P is *-isomorphic to the algebra of all compact operators on some Hilbert space. By [5, Lemma 4.3] Ω is locally compact in the Jacobson topology. If Ω is not compact, then compactify it by adding a point at ∞, P_∞, and associate with P_∞ the algebra $B_{P_\infty} = \{0\}$. Each $b \in B$ determines a function $b(P)$ defined on Ω by

$$b(P) = b + P \in B_P.$$

Let $|b(P)|_P$ be the quotient norm of $b + P$ in B/P ($b \in B$, $P \in \Omega$). Then $|b| = \sup\{|b(P)|_P \mid P \in \Omega\}$ for all $b \in B$. Furthermore, by [5, Theorem 4.1], $P \to |b(P)|_P$ is a continuous function on Ω. If $P_1, \cdots, P_n$ are n distinct points of Ω, and $b_1, \cdots, b_n$ are n elements of B, then there exists $b \in B$ such that

$b(P_k) = b + P_k = b_k + P_k$ for $1 \leq k \leq n$. This follows from [4, Proposition 4.2] for the case $n = 2$, but the result is easily extended to all positive integers n. Then [5, Theorem 3.3] applies to B, so that B is closed under multiplication by functions in $C_R(\Omega)$.

Now assume that A is a locally B^*-equivalent Banach algebra which is a dense $*$-subalgebra of B. Let $P_1, \cdots, P_n$ be n distinct points of Ω. Let $B' = B_{P_1} \oplus \cdots \oplus B_{P_n}$ (the direct sum of the quotient algebras B_{P_k}, $1 \leq k \leq n$). B' is a dual B^*-algebra. Let $I = A \cap P_1 \cap \cdots \cap P_n$. The map π: $A/I \to B'$, defined for $a \in A$ by $\pi(a + I) = (a + P_1) \oplus \cdots \oplus (a + P_n)$, is a $*$-isomorphism of the locally B^*-equivalent Banach algebra A/I into B'. Let $b_1, \cdots, b_n$ be any n elements of B. Then as noted above, there exists $b \in B$ such that $b(P_k) = b + P_k = b_k + P_k$ for $1 \leq k \leq n$. Choose a sequence $\{a_j\} \subset A$ such that $|a_j - b| \to 0$. Then $\pi(a_j + I) \to (b_1 + P_1) \oplus \cdots \oplus (b_n + P_n)$ in B'. This proves that $A' = \pi(A/I)$ is a dense locally B^*-equivalent Banach $*$-subalgebra of the dual B^*-algebra B'. If $t = t^* \in A'$, then t has the same spectrum in A' and B'. Then by [1, Theorem 3.4], the spectrum of t in A' is at most countable. Therefore by [2, Theorem 4.1], $A' = B'$. It follows that if $b_1, \cdots, b_n$ are n elements of B, there exists $a \in A$ such that $a(P_k) = b_k + P_k$ for $1 \leq k \leq n$. Then A satisfies all the hypotheses of Theorem 1, so that $A = B$. We state this result as a proposition.

Proposition 6. *If B is a CCR algebra with Hausdorff structure space, then $A = B$.*

Lemma 7. *Let K be a closed B^*-equivalent $*$-ideal of A. Then there exists a closed B^*-equivalent $*$-ideal I of A such that $K \subset I$ and A/I has no nonzero closed B^*-equivalent $*$-ideals.*

Proof. Let $\mathcal{I}$ be the collection of all closed B^*-equivalent $*$-ideals of A that contain K. Then $\mathcal{I}$ is nonempty, and $\mathcal{I}$ is partially ordered by inclusion. Assume that $\{I_\lambda\}$, $\lambda \in \Lambda$, is a chain in $\mathcal{I}$. Then by Lemma 3 the $\|\cdot\|$-closure of $\bigcup_{\lambda \in \Lambda} I_\lambda$ is in $\mathcal{I}$. By Zorn's Lemma $\mathcal{I}$ contains a maximal element I. Assume that J is a closed B^*-equivalent $*$-ideal of A/I. Let π be the natural quotient map of A onto A/I. By [2, Lemma 2.4], $\pi^{-1}(J)$ is a closed B^*-equivalent $*$-ideal of A that contains I. Then by the maximality of I, $\pi^{-1}(J) = I$. This proves that J is the zero ideal in A/I.

Proposition 8. *If B is a GCR algebra, then $A = B$.*

Proof. By Lemma 7 A contains a closed B^*-equivalent $*$-ideal I such that A/I has no nonzero closed B^*-equivalent $*$-ideals. Then I is a closed $*$-ideal of B, and A/I is a dense locally B^*-equivalent $*$-subalgebra of B/I. By [4, Proposition 4.3.5], B/I is a GCR algebra. Let A_1 and B_1 denote the quotient algebras A/I and B/I, respectively. Assume $B_1 \neq \{0\}$ (that is, $B \neq I$). Then B_1 contains a nonzero closed *-ideal J which is a CCR algebra (see [4, p. 87]). Then J_1, the first nonzero ideal in the composition series described in [5, Theorem 6.2], is a closed *-ideal of B_1 which is a CCR algebra with Hausdorff structure space. Now $A_1 \cap J_1$ is dense in J_1 by Lemma 1. Then by Proposition 6, $J_1 \subset A_1$. This contradiction implies that $B = I$. Therefore $B = A$.

Now we prove the main result.

Theorem 2. *There exists a closed *-ideal I of A with the properties*:

(1) *I is B^*-equivalent.*

(2) *I contains the largest GCR ideal of B, and*

(3) *A/I has no nonzero closed B^*-equivalent ideals.*

Proof. Let J be the largest GCR ideal of B (see [5, Theorem 7.5]). By Lemma 1 $A \cap J$ is a dense *-subalgebra of J. Furthermore $A \cap J$ is locally B^*-equivalent. Then by Proposition 8, $J = A \cap J \subset A$. Then Lemma 7 implies the existence of a closed *-ideal I of A with properties (1)–(3).

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