INVERSE LIMITS ON GRAPHS AND MONTOTONE MAPPINGS

BY

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ABSTRACT. In 1935, Knaster gave an example of an irreducible continuum (i.e. compact connected metric space) K which can be mapped onto an arc so that each point-preimage is an arc. The continuum K is chainable (or arc-like). In this paper it is shown that every one-dimensional continuum M is a continuous image, with arcs as point-preimages, of some one-dimensional continuum M'. Moreover, if M is G-like, for some collection G of graphs, then M' can be chosen to be G-like. A corollary is that every chainable continuum is a continuous image, with arcs as point-inverses, of a chainable (and hence, by a theorem of Bing, planar) continuum. These investigations give rise to the study of certain special types of inverse limit sequences on graphs.

1. **Introduction.** If P is a proper subset of a continuum M, let $\mathfrak{S}(M - P)$ denote the collection to which e belongs if and only if e is the closure of a component of $M - P$. Then G is a *graph* if and only if G is a continuum (i.e. a compact connected metric space) and there exists a finite proper subset P of G (called a *partition* of G) such that $\mathfrak{S}(G - P)$ is a finite collection, each element of which is an arc with both endpoints in P, and no two elements of which have two points in common. (Hence every graph is homeomorphic to the space of a one-dimensional simplicial complex.) The elements of $\mathfrak{S}(G - P)$ are called *edges* of the partition P of G. Clearly, each edge of P has only its endpoints in P, and if two edges intersect, their intersection consists of a single common endpoint of both. A finite subset of G that contains a partition P of G is called a *refinement* of P, and is itself a partition of G.

If P_1 and P_2 are partitions of graphs G_1 and G_2 respectively, a map $f: G_1 \to G_2$ is *simplicial relative to* (P_1, P_2) if and only if $f(P_1) \subset P_2$, and the restriction of f to any edge of P_1 is either constant or a homeomorphism onto an edge of P_2. The map $f: G_1 \to G_2$ is *simplicial* if and only if it is simplicial relative to some partitions of G_1 and G_2.

A map is *light* if and only if each point-preimage is totally disconnected. It is easy to see that $f: G_1 \to G_2$ is light and simplicial relative to the partitions P_1 and P_2 of G_1 and G_2, respectively, if and only if the restriction of f to each

edge of P_1 is a homeomorphism onto an edge of P_2; moreover, if f is light and simplicial, then each point-preimage is in fact finite.

Denote by $d(x, y)$ the distance from the point x to the point y; by $D(H)$ the diameter of the point set H; by $\overline{H}$ its closure. If G is a collection of point sets, G^* denotes their union; if s is an arc with endpoints x and y, then $\mathrm{Seg}(s) = s - \{x, y\}$ is called the *segment* of s.

For an inverse limit sequence $\{X_i, f_i\}$, we require that f_i map X_{i+1} onto X_i, and denote by π_i the projection map from the limit onto X_i. As a matter of convention, we take metrics for the coordinate spaces so that $D(X_i) \leq 1$ for each i. The metric for the limit is defined by

$$d(x, y) = \sum_{i=1}^{\infty} d(\pi_i(x), \pi_i(y)) \cdot 2^{-i}.$$

If G is a collection of graphs, then a continuum M is said to be *G-like* if and only if, for each $\epsilon > 0$, there is an ϵ-map (i.e. point-preimages having diameter $< \epsilon$) from M onto an element of G. From [5], the G-like continua are precisely the continua homeomorphic to the limit of an inverse sequence with coordinate spaces in G.

The inverse limit sequence $\{G_i, f_i\}$ on graphs (i.e. G_i is a graph for each i) is said to be *light (or simplicial)* if and only if f_i is light (or simplicial) for each i. $\{G_i, P_i, f_i\}$ is called a *uniformly simplicial* inverse limit sequence on graphs if and only if, for each i, G_i is a graph, P_i is a partition of G_i, and f_i is a map from G_{i+1} onto G_i which is simplicial relative to (P_{i+1}, P_i).

In §2, we show that every one-dimensional continuum is homeomorphic to the limit of a light simplicial inverse limit sequence on graphs. This sequence need not be uniformly simplicial, however. In §3, we study uniformly simplicial inverse limits on graphs, characterizing their limits. Not every one-dimensional continuum is such a limit, since each such limit is a union of arcs, which disqualifies, for example, the pseudo-arc. On the other hand, several well-known continua are shown to be such limits, including the dyadic solenoid.

In §4, we apply these ideas to the problem concerning monotone mappings discussed in the abstract.

2. **Light simplicial inverse limits.**

Lemma 1. *Suppose x and y are points of the graph G, and a and b are endpoints of the arc s. Then there is a light simplicial map g from s onto G such that $g(a) = x$ and $g(b) = y$.*

Proof. Let P denote a partition of G. There exists a finite sequence $s(1)$, $s(2), \cdots, s(n)$ of arcs in G with x as one endpoint, each of which is a union of

edges of P, such that the other endpoint of $s(n)$ is y, and $\bigcup_{i=1}^{n} s(i) = G$. Let $w_0 = a$, w_1, w_2, $\cdots$, $w_{2n-1} = b$ denote a sequence of points in the indicated order along s. For each odd {even} positive integer $i < 2n$, let h_i denote a homeomorphism from the arc $[w_{i-1}, w_i]$ onto $s((i+1)/2)\{s(i/2)\}$ such that $h_i(w_{i-1}) = x$ $\{h_i(w_i) = x\}$. Define g such that if $c \in [w_{i-1}, w_i]$, then $g(c) = h_i(c)$. Clearly, g is both light and simplicial, since each h_i is, and $g(s) \supset \bigcup_{i=1}^{n} s(i) = G$.

Theorem 1. *Suppose* $\epsilon > 0$, *and* f *is a map from the graph* G *onto the graph* G'. *Then there is a light simplicial map* g *from* G *onto* G' *such that if* $x \in G$, *then* $d(f(x), g(x)) < \epsilon$.

Proof. Let P' denote a partition of G' such that each edge of P' has diameter $< \epsilon/4$. There exists a partition P of G and a map g' from G into G' such that (1) g' is simplicial relative to (P, P'), (2) if $x \in G$ and $f(x) \in P'$, then $g'(x) = f(x)$, and (3) if $x \in G$ and $f(x)$ lies in the interior of an edge s of P', then $g'(x) \in s$ [3, Theorem 7.3, p. 64]. From condition (2), $g'(P) = P'$, and from (3), if $x \in G$, $d(f(x), g'(x)) < \epsilon/4$. Still, $g'(G)$ need not be G', and g' need not be light.

Suppose s is an edge of G with endpoints a and b. Then either $g'(a) = g'(b)$, or $g'(a)$ and $g'(b)$ are the endpoints of the edge $g'(s)$ of G'. In any case, let U denote the union of all the edges of G' with either $g'(a)$ or $g'(b)$ as an endpoint. Since U is a graph, there is a light simplicial map g_s from s onto U such that $g_s(a) = g'(a)$ and $g_s(b) = g'(b)$, by Lemma 1. Clearly, $D(U) < 3 \cdot \epsilon/4$, so that if $x \in s$,

$$d[f(x), g_s(x)] \leq d[f(x), g'(x)] + d[g'(x), g_s(x)] < \epsilon/4 + 3 \cdot \epsilon/4 = \epsilon.$$

Hence, if we define $g(x)$, for each $x \in G$ to be $g_s(x)$, for any edge s of G that contains x, then $d(f(x), g(x)) < \epsilon$, and g is clearly both light and simplicial. Moreover, since $g'(P) = P'$, and for each edge s of G, $g(s)$ contains every edge of G' that intersects $g'(s)$, g throws G onto G'.

Theorem 2. *Every one-dimensional continuum,* M, *is homeomorphic to the limit of a light simplicial inverse limit sequence* $\{G_i, f_i\}$ *on graphs. Moreover, if* M *is* G-*like for some collection* G *of graphs, then* G_i *may be chosen from* G *for each* i.

Proof. It is well known that every one-dimensional continuum is homeomorphic to the limit of an inverse sequence on graphs [5]. Theorem 2 follows from this fact, Theorem 1, and Brown's approximation theorem for inverse limits [2, Theorem 3, p. 481].

3. Light uniformly simplicial inverse limits. While Theorem 2 shows that

every one-dimensional continuum can be obtained from a light simplicial inverse
limit sequence on graphs, the limits of light uniformly simplicial inverse sequences
are far more restricted. It is the purpose of this section to bring out some of their
properties in order to characterize them.

Theorem 3. *Suppose* (1) $\{G_i, P_i, g_i\}$ *is a light uniformly simplicial inverse
limit sequence,* (2) B *is the collection of all sequences* $\{s_i\}$ *where, for each i,
s_i is an edge of P_i and $s_i = g_i(s_{i+1})$,* (3) *for each* $\{s_i\} \in B$, $\lambda\{s_i\} =
\bigcap_{i=1}^{\infty} \pi_i^{-1}(s_i)$, *and* (4) A *is the collection of sets $\lambda\{s_i\}$ for all sequences $\{s_i\}$
in B. Then* (a) *each element of A is an arc,* (b) *for each i, the restriction of π_i
to any element of A is a homeomorphism onto an edge of G_i, and* (c) $M = A^*$.

Proof. Suppose $\alpha = \lambda\{s_i\} \in A$. Then α is the limit of the inverse sequence
$\{s_i, g_i \mid s_i\}$. Since f_i is light and simplicial for each i, and s_i is an edge of G_i,
$g_i \mid s_i$ is a homeomorphism. But if each of the bonding maps is a homeomorphism,
then so are the projection maps, $\pi_i \mid \alpha$, which proves (b). But then α is homeo-
morphic to $\pi_1(\alpha) = s_1$, an arc, which proves (a).

For (c), suppose $p(p_1, p_2, \cdots)$ is a point of M. If p_1 is not in P_1, then by
induction p_i is not in P_i for any i, and there is only one edge, s_i of G_i that
contains p_i. For each i, $g_i(s_{i+1})$ is some edge of G_i that contains $g_i(p_{i+1}) =
p_i$, but s_i is the only such edge, so $s_i = g_i(s_{i+1})$. Hence $\{s_i\} \in B$, and $p \in
\lambda\{s_i\} \in A$.

If, on the other hand, $p_1 \in P_1$, then, again by induction, $p_i \in P_i$ for each
i. Now, for each i, let H_i denote the collection of all edges of G_i with p_i as
an endpoint. Clearly if $s \in H_j$ and $i < j$, then $g_{ij}(s) \in H_i$. Hence some element
s_1 of the finite set H_1 is the image under g_{1j} of an element of H_j for infinitely
many (and hence clearly for all) positive integers $j > 1$. For each $j > 1$, let H_j^1
denote the set to which s belongs if and only if $s \in H_j$ and $g_{1j}(s) = s_1$. Continu-
ing by induction, construct sequences $\{s_i\}$ and $\{H_j^i\}$ ($i < j$) such that $s_i \in H_j^{i-1}$
and, for each $j > i$, $s \in H_j^i$ if and only if $s \in H_j^{i-1}$ and $g_{ij}(s) = s_i$ (the finiteness
of H_i^{i-1} assures the existence of an edge s_i which is the image under g_{ij} of
some element of H_j^{i-1}, for each $j > i$). Then $\{s_i\} \in B$, and $p \in \lambda\{s_i\} \in A$.

Theorem 4 is useful for generating examples of limits of light uniformly
simplicial inverse systems. We precede it with a lemma.

Lemma 2. *Suppose P and P_0 are partitions of the graphs G and G_0, respec-
tively, f is a light simplicial map from G onto G_0 relative to (P, P_0), Q_0 is a
refinement of P_0, and $Q = f^{-1}(Q_0)$. Then Q is a refinement of P, and f is
simplicial relative to (Q, Q_0).*

Proof. Since f throws G onto G_0, $f^{-1}(P_0) = P$, and since f is light and
simplicial, $f^{-1}(Q_0)$ is finite. Hence $Q = f^{-1}(Q_0) \supset f^{-1}(P_0) = P$, and Q is a

refinement of P. Moreover, each edge s of Q lies in an edge of P on which f is a homeomorphism. Hence $f(s)$ is an arc in G_0 with only its endpoints in Q_0, i.e. $f(s)$ is an edge of Q_0. So the restriction of f to any edge of Q is a homeomorphism onto an edge of Q_0 and f is simplicial relative to (Q, Q_0).

Theorem 4. *Suppose f is a map from the graph G onto itself, and P_1 is a partition of G such that $P_2 = f^{-1}(P_1)$ is a refinement of P_1, and f is light and simplicial relative to (P_2, P_1). Then $\{G_i, P_i, f_i\}$ is a light uniformly simplicial inverse sequence, where for each i, $G_i = G$, $f_i = f$, and $P_{i+1} = f^{-1}(P_i)$.*

Proof. By hypothesis, f_1 is simplicial relative to (P_2, P_1). Since P_2 is a refinement of P_1, $P_3 = f^{-1}(P_2)$ is a refinement of P_2 by Lemma 2, and $f_2 = f$ is simplicial relative to (P_3, P_2). The proof continues by induction.

Several well-known continua are limits of inverse sequences with a single bonding map that satisfies the hypothesis of Theorem 4. For example, if I denotes the unit interval, and $f: I \to I$ is defined by $f(x) = 2x$ if $0 \le x \le 1/2$ and $f(x) = (3/2) - x$ if $1/2 \le x \le 1$, then the limit of the inverse sequence $\{I, f\}$ (i.e. each coordinate space is I, and each bonding map is f) is the $\sin(1/x)$-continuum, and $P_1 = \{0, 1/2, 1\}$ satisfies the hypothesis of Theorem 4. If $g(x) = 2x$ ($0 \le x \le 1/2$) and $g(x) = 2 - 2x$ ($1/2 \le x \le 1$), then the limit of the inverse sequence $\{I, g\}$ is a well-known planar indecomposable continuum, and again $P_1 = \{0, 1/2, 1\}$ satisfies the theorem. Finally, if C denotes the unit circle in the complex plane, and $s(z) = z^2$ for each complex number z in C, then $\lim\{C, s\}$ is the dyadic solenoid, and $P_1 = \{1, -1\}$ satisfies the theorem. Hence each of these continua is a limit of a light uniformly simplicial inverse sequence. We are now ready to characterize all the limits of such sequences.

Theorem 5. *In order that the continuum M be homeomorphic to the limit of a light uniformly simplicial inverse sequence on graphs, it is both necessary and sufficient that there exist a map π from M onto $[0, 1]$ such that (1) $V = \pi^{-1}(\{0, 1\})$ is totally disconnected, and (2) if $\alpha \in \mathscr{E}(M - V)$, then $\pi \mid \alpha$ is a homeomorphism onto $[0, 1]$.*

Proof of necessity. We assume that M is homeomorphic to such a sequence $\{G_i', P_i', f_i\}$. Our first object is to modify the inverse sequence so that $[0, 1]$ is the first coordinate space. Then the first projection mapping will satisfy the requirements for π.

For each edge e of P_2', let x_e denote a particular nonendpoint of e, and let X denote the union of the points x_e for all edges e of P_2', and $P_2 = X \cup P_2'$. Then P_2 is a refinement of P_2'. Let $G_1 = [0, 1]$, $G_2 = G_2'$, and $P_1 = \{0, 1\}$. Define $\phi: P_2 \to P_1$ such that $\phi(P_2') = 0$ and $\phi(X) = 1$. Then ϕ is easily extended to a light simplicial map g_1 from G_2 onto G_1 relative to (P_2, P_1).

For each $n > 1$, let $G_{n+1} = G'_{n+1}$, and $P_{n+1} = f_n^{-1}(P_n)$. Then by Lemma 2, P_{n+1} is a refinement of P'_{n+1}, and f_n is simplicial relative to (P_{n+1}, P_n). So if $f_n = g_n$ for each $n > 1$, then $\{G_i, P_i, g_i\}$ is a light uniformly simplicial inverse system with limit M' homeomorphic to M, since only the first coordinate space and bonding map have been altered.

We now show that the first projection map $\pi_1: M' \to G_1 = [0, 1]$ has the required properties relative to M'. First, g_i is light for each i, so π_1 is light, and $\pi_1^{-1}(0)$ and $\pi_1^{-1}(1)$ are totally disconnected. For condition (2), we begin by letting A denote the collection as defined in Theorem 3 relative to $\{G_i, P_i, g_i\}$, and $\alpha = \lambda\{s_i\}$ and $\alpha' = \lambda\{s'_i\}$ denote two elements of A. Hence, letting $V = \pi_1^{-1}(\{0, 1\})$, both $\mathrm{Seg}(\alpha)$ and $\mathrm{Seg}(\alpha')$ lie in $M' - V$. Let n denote the first positive integer i such that $s_i \neq s'_i$. Then $\mathrm{Seg}(s_i)$ and $G_i - s_i$ are mutually exclusive open subsets of G_i, and $\mathrm{Seg}(s'_i) \subset G_i - s_i$. Hence $\pi_i^{-1}(\mathrm{Seg}(s_i))$ and $\pi_i^{-1}(G_i - s_i)$ are mutually exclusive open subsets of $M' - V$, containing $\mathrm{Seg}(\alpha)$ and $\mathrm{Seg}(\alpha')$ respectively. We conclude that if α and α' are elements of A, then $\mathrm{Seg}(\alpha)$ and $\mathrm{Seg}(\alpha')$ lie in different components of $M' - V$. Hence $\mathrm{Seg}(\alpha)$ is itself a component of $M' - V$, for every point of $(M' - V) - \mathrm{Seg}(\alpha)$ lies in the segment of some other arc of A. Hence the components of $M' - V$ are precisely the segments of arcs of A, and the elements of $\mathfrak{S}(M' - V)$ are these arcs. The restriction of π_1 to each of these arcs is now a homeomorphism onto $[0, 1]$ by Theorem 3.

Proof of sufficiency. Suppose π is a map from M onto $[0, 1]$ satisfying the conditions of the theorem. Let $A = \pi^{-1}(0)$ and $B = \pi^{-1}(1)$ and note that it follows from the conditions of the theorem that if x is a point of $M - V$, then α_x, the closure of the component of $M - V$ that contains x, is an arc with one endpoint in A and the other in B. We first consider the case in which it is true that

(*Assumption* Z) If x and y lie in $M - V$, and both endpoints of α_x are endpoints of α_y, then $\alpha_x = \alpha_y$.

Now, since $A \subset V$, A is totally disconnected; so there is a sequence $\{A_i\}$ such that, for each i, (1) A_i is a finite collection of mutually exclusive closed subsets of A, each of diameter $< 2^{-i}$, (2) $A_i^* = A$, and (3) each element of A_{i+1} lies in some element of A_i. There is a sequence $\{B_i\}$ for B with similar properties.

Suppose i is a positive integer. Let ϕ_i denote a one-to-one map from the collection $A_i \cup B_i$ onto a finite subset V_i of E^3 of diameter 1, such that no three points of V_i are collinear, and no four are coplanar. Let $e_i: M \to E^3$ denote the map such that (1) if $a \in A_i$, $e_i(a) = \phi_i(a)$, (2) if $b \in B_i$, $e_i(b) = \phi_i(b)$, and (3) if $x \in \mathfrak{S}(M - V)$ and α_x has endpoints in $a \in A_i$ and $b \in B_i$, respectively, then $e_i(x)$ is the point between $\phi_i(a)$ and $\phi_i(b)$ given by $t\phi_i(b) + (1 - t)\phi_i(a)$, where $t = \pi(x) \in [0, 1]$. Let $G_i = e_i(M)$, a graph in E^3.

Now, if i is a positive integer and $a \in A_{i+1}$ [or $b \in B_{i+1}$], then there is only one element $v(a)$ [or $v(b)$] of A_i [or B_i], that contains a [or b]. Hence there is a linear map g_i from G_{i+1} onto G_i such that if $x \in M - V$, and α_x has endpoints in $a \in A_{i+1}$ and $b \in B_{i+1}$ respectively, and $y \in G_{i+1}$ is the point $e_{i+1}(x) = t\phi_{i+1}(b) + (1 - t)\phi_{i+1}(a)$, where $t = \pi(x)$, then $g_i(y)$ is the point $e_i(x) = t\phi_i[v(b)] + (1 - t)\phi_i[v(a)]$ of G_i. So, $e_i = g_i e_{i+1}$, and it is not difficult to see that $\{G_i, V_i, g_i\}$ is a light uniformly simplicial inverse sequence on graphs.

We now show that M and M', the limit of the inverse sequence $\{G_i, V_i, f_i\}$ are homeomorphic. For each $x \in M$, let $h(x) = (e_1(x), e_2(x), \cdots)$. Then for each i, $e_i(x) \in G_i$, and $e_i(x) = g_i e_{i+1}(x)$, so that $h(x) \in M'$.

If $x \in M$, and O is an open set in M' containing $h(x)$, then there is a positive integer i and an open set R in G_i containing $\pi_i h(x) = e_i(x)$ such that $\pi_i^{-1}(R) \subset O$. Let $O' = e_i^{-1}(R)$. Then O' is open in M and contains x, and

$$h(O') \subset \pi_i^{-1}[e_i(O')] = \pi_i^{-1}[R] \subset O.$$

Hence h is continuous.

That $h(M) = M'$ follows from the fact that for each i, $\pi_i[h(M)] = e_i(M) = G_i$, and no closed proper subset of M' projects onto every coordinate space.

Finally, h is reversible, for suppose x and y are distinct points of M. If $x \in A$, there is a positive integer i such that $x \in a \in A_i$, but $y \notin a$. So $e_i(x) = \phi_i(a) \neq e_i(y)$, and $h(x) \neq h(y)$. The argument is similar if either x or y lies in $A \cup B$. So suppose neither x nor y lies in V. If $\alpha_x = \alpha_y$, then $e_1(x) \neq e_1(y)$ since $e_1 | \alpha_x$ is a homeomorphism, and again, $h(x) \neq h(y)$. If $\alpha_x \neq \alpha_y$, let $c_x \in A$ [or $c_y \in B$] and $d_x \in B$ [$d_y \in B$] denote the endpoints of α_x [α_y]. By Assumption Z, either $c_x \neq c_y$ or $d_x \neq d_y$; suppose $c_x \neq c_y$. Then for some i, $c_x \in a_x \in A_i$ and $c_y \in a_y \in A_i$, and $a_x \neq a_y$. Hence

$$e_i(c_x) = \emptyset_i(a_x) \neq \emptyset_i(a_y) = e_i(c_y).$$

But $e_i(x)$ lies between $e_i(c_x)$ and $e_i(d_x)$ and $e_i(y)$ lies between $e_i(c_y)$ and $e_i(d_y)$. Since no three of the points of e_i $(\{c_x, c_y, d_x, d_y\})$ are collinear, and no four coplanar, $e_i(x) \neq e_i(y)$, and so $h(x) \neq h(y)$.

This completes the argument if Assumption Z is satisfied. If it is not satisfied, define $g: I \to I$ such that $g(x) = 2x$ $(0 \leq x \leq 1/2)$ and $g(x) = 2 - 2x$ $(1/2 \leq x \leq 1)$. Let $\pi' = g\pi$, $V' = \pi'^{-1}(\{0, 1\})$ and note that π' and V' satisfy the conditions required of π and V in the statement of the theorem. For each $x \in M - V'$, let α_x' denote the element of $\mathcal{E}(M - V')$ that contains x. It is easily verified that if $y \in M - V$, and $u = \pi^{-1}(3/4) \cap \alpha_y$, and $v = \pi^{-1}(1/4) \cap \alpha_y$, then $\alpha_y = \alpha_u' \cup \alpha_v'$, $\alpha_u' \cap \alpha_v' = \pi^{-1}(1/2) \cap \alpha_y$, and if x and y are points of $M - V$ and $\alpha_x \neq \alpha_y$, then $\pi^{-1}(1/2) \cap \alpha_x \neq \pi^{-1}(1/2) \cap \alpha_y$. It follows quickly then that if x

and y lie in $M - V'$, and both endpoints of α_x' are endpoints of α_y', then $\alpha_x' = \alpha_y'$. That is, Assumption Z is satisfied for π' and V'. The proof is now similar to the first case, using π' and V' in place of π and V.

Note that it follows from this characterization that each arc-component of the limit of a light uniformly simplicial inverse sequence is nondegenerate. This fact can be used to generate a number of continua which are not such limits.

4. **An application to the theory of monotone mappings.** The object of this section is to use the ideas developed in the previous two sections to prove the following theorem.

Theorem 6. *If M is a one-dimensional continuum, then there exists a one-dimensional continuum M' and a map θ from M' onto M under which each point-preimage is an arc. Moreover, if M is G-like for some collection G of graphs, then M' can be chosen to be G-like.*

Proof. We take M to be the limit of a light simplicial inverse sequence $\{G_i, f_i\}$ as in Theorem 2, picking G_i from G, for each i, if M is G-like. To avoid later confusion, we denote the projection map from M onto G_i by π_i'. Using Lemma 2, we construct sequences $\{P_i\}$ and $\{P_i'\}$ such that for each i,

(1) P_i' is a refinement of the partition P_i of G_i,

(2) f_i is simplicial relative to (P_{i+1}, P_i'),

(3) if s is an edge of P_i, then

    (a) $P_i' \cap s$ consists of an even number ≥ 8 of points,

    (b) at most one endpoint of s is of order > 2 in G_i,

    (c) no point of P_i is adjacent to both endpoints of s, and

    (d) if $j < i$, then $D[f_{ji}(s)] < 2^{j-i}$.

For each $i > 0$ and $n > 0$, let $C_i(n)$ denote the collection of all subsets L of G_i such that if x and y are points of L, then there is a sequence $s_1, \cdots, s_m$, where $m \leq n$, of edges of P_i such that $x \in s_1$, $y \in s_m$, and if $|j - k| = 1$, then s_j and s_k are adjacent. Suppose $c \in C_i(n)$. Then if $j < i$, it follows from property (3d) above that $D[f_{ji}(c)] \leq n \cdot 2^{j-i}$. Hence, if $c' = (\pi_i')^{-1}[c]$,

$$D[c'] \leq \sum_{j=1}^{\infty} D[\pi_j'(c')] \cdot 2^{-j} \leq \sum_{j=1}^{j} D[f_{ji}(c)] \cdot 2^{-j} + \sum_{j=i+1}^{\infty} (1) \cdot 2^{-j}$$

$$\leq \sum_{j=1}^{i} [(n \cdot 2^{j-i}) \cdot 2^{-j}] + 2^{-i}$$

$$= \sum_{j=1}^{i} (n \cdot 2^{-i}) + 2^{-i} = in2^{-i} + 2^{-i} = 2^{-i}(in + 1).$$

We have shown that

(A^*) if $c \in C_i(n)$, then $D[(\pi_i^l)^{-1}[c]] \leq 2^{-i}(in + 1)$.

For each i, if $p \in P_i$ and p is of order > 2 in G_i, let $z(p)$ denote a definite point of P_i adjacent to p. We now define a map $\phi_i : P_i' \to P_i$. Suppose s is any edge of P_i, and let $p_1, \cdots, p_n$ denote the points of $P_i' \cap s$, in order along s, with p_1 the point of order > 2 in G_i, if there is one. Then

(1) if $1 \leq j < n$ and j is odd, $\phi_i(p_j) = p_1$,

(2) if $2 < j \leq n$ and j is even, $\phi_i(p_j) = p_n$,

(3) $\phi_i(p_2) = z(p_1)$ if the order of p_1 in $G_i > 2$, and

(4) $\phi_i(p_2) = p_n$ if the order of $p_1 \leq 2$.

Roughly, the points $p_1, \cdots, p_n$ are thrown atlternately to p_1 and p_n, except that if the order of $p_1 > 2$, then p_2 is thrown to $z(p_1)$. Define ϕ_i in this way on all the edges of p_i, and extend ϕ_i in the obvious manner to a light simplicial map e_i from G_i onto G_i relative to (P_i', P_i).

If s is an edge of P_i [or P_i'] let $S(s)$ [$S'(s)$] denote the union of all the edges of P_i [P_i'] that either are s, or are adjacent to s, and let $T(s)$ [$T'(s)$] denote the collection of all edges of P_i [P_i'] that are either in $S(s)$ [$S'(s)$] or adjacent to an edge in $S(s)$ [$S'(s)$]. If s' is an edge of P_i', let $E(s')$ denote the edge of P_i that contains it. Then it follows from the properties of e_i, by a straightforward consideration of a number of different cases, that, for each i,

(B^*) if s' is an edge of P_i', then $e_i[T'(s')]$ lies in the union of $E(s')$ and *one* edge of P_i adjacent to $E(s')$, and

(C^*) if s' is an edge of P_i', then $S'(s') \subset S[e_i(s')]$.

For each positive integer i, let $g_i = e_i f_i$. Note that g_i is light and simplicial relative to (P_{i+1}, P_i). Also, if s is an edge of P_{i+1}, and $s' = f_i(s)$ (so that s' is an edge of P_i'), then $f_i(S(s)) \subset S'(s')$ and $f_i(T(s)) \subset T'(s')$, since f_i is simplicial. Hence,

$$g_i[T(s)] = e_i f_i[T(s)] \subset e_i[T'(s')].$$

It now follows from (B^*) that

(D^*) if s is an edge of P_{i+1}, then $g_i[T(s)]$ lies in the union of two adjacent edges of P_i.

Also, from (C^*),

$$f_i(S(s)) \subset S'(s') \subset S[e_i(s')] = S[e_i(f_i(s))] = S[g_i(s)],$$

so that

(E^*) if s is an edge of P_{i+1}, then $f_i[S(s)] \subset S[g_i(s)]$.

We are now ready to define M' as the limit of the light uniformly simplicial inverse sequence $\{G_i, P_i, g_i\}$. For each i, let π_i^g denote the projection from M' onto G_i, and let A denote the collection of arcs in M' as in Theorem 3. Suppose

$\alpha = \lambda\{s_i\} \in A$. For each i, let $M_i(\alpha) = (\pi_i^f)^{-1}[S(s_i)] \subset M$. Since $S(s_i) \in C_i(3)$, it follows from (A^*) that $D[M_i(\alpha)] \le 2^{-i}(3i + 1)$, so that $\lim_{i \to \infty} D[M_i(\alpha)] = 0$. Also, from (E^*), $f_i[S(s_{i+1})] \subset S[g_i(s_{i+1})] = S(s_i)$, so

$$M_{i+1}(\alpha) = (\pi_{i+1}^f)^{-1}[S(s_{i+1})] \subset (\pi_i^f)^{-1}[f_i[S(s_{i+1})]] \subset (\pi_i^f)^{-1}[S(s_i)] = M_i(\alpha).$$

Since M is compact, these two facts imply that $\bigcap_{i=1}^{\infty} M_i(\alpha)$ consists of a single point, which we denote by $\theta'(\alpha)$.

We now establish the first of two statements about θ'.

(F^*) If $\alpha = \lambda\{s_i\} \in A$ and $\alpha' = \lambda\{s_i'\} \in A$, then $\theta'(\alpha) = \theta'(\alpha')$ if and only if, for each i, either $s_i = s_i'$ or s_i and s_i' are adjacent edges of P_i.

Suppose the latter is true. Then for each i, $S(s_i)$ intersects $S(s_i')$ and so $M_i(\alpha)$ intersects $M_i(\alpha')$, and $\lim_{i \to \infty} D[M_i(\alpha) \cup M_i(\alpha')] = 0$. But

$$\theta'(\alpha) \cup \theta'(\alpha') \subset \bigcap_{i=1}^{\infty} [M_i(\alpha) \cup M_i(\alpha')],$$

which has diameter 0, so $\theta'(\alpha) = \theta'(\alpha')$. On the other hand, if $\theta'(\alpha) = \theta'(\alpha')$, then $\pi_i^f \theta'(\alpha) \in S(s_i) \cap S(s_i')$ for each i. Suppose i is any positive integer. Then, since $S(s_{i+1})$ intersects $S(s_{i+1}')$, there is an edge s of P_{i+1} such that both s_{i+1} and s_{i+1}' lie in $T(s)$. Hence $s_i \cup s_i' = g_i(s_{i+1} \cup s_{i+1}') \subset g_i[T(s)]$ which, by (D^*), lies in the union of two adjacent edges of G_i, and each of the edges s_i and s_i' must be one of these edges, which completes the proof of (F^*).

(G^*) If p is a point of two elements, α and α' of A, then $\theta'(\alpha) = \theta'(\alpha')$.

This is an immediate consequence of (F^*) since, for each i, the edge $s_i = \pi_i^g(\alpha)$ and the edge $s_i' = \pi_i^g(\alpha')$ have $\pi_i^g(p)$ in common.

We now define the function $\theta: M' \to M$, making use of (G^*). If $p \in M'$, and $\alpha = \lambda\{s_i\}$ is any element of A that contains p, then $\theta(p) = \theta'(\alpha) = \bigcap_{i=1}^{\infty} (\pi_i^f)^{-1}[S(s_i)]$. We conclude by showing that θ satisfies the requirements of the theorem.

First, θ is continuous, for suppose $\epsilon > 0$, and $p(p_1, p_2, \cdots) \in M'$. Then p belongs to some element $\alpha = \lambda\{s_i\}$ of A. For each i, the interior, $S^\circ(s_i)$, of $S(s_i)$ is an open set in G_i that contains p_i. Hence $O_i = (\pi_i^g)^{-1}[S^\circ(s_i)]$ is open in M', and contains p. Suppose $p' \in O_i$. Then p' lies in some element $\alpha' = \lambda\{s_i'\}$ of A. Since $\pi_i^g(p') \in s_i' \cap S^\circ(s_i)$, s_i' must lie in $S(s_i)$, and $S(s_i') \subset T(s_i)$. So, by the definition of θ,

$$\theta(p') \subset (\pi_i^f)^{-1}[S(s')] \subset (\pi_i^f)^{-1}[T(s_i)].$$

Hence $\theta(O_i) \subset (\pi_i^f)^{-1}[T(s_i)]$. But $T(s_i) \in C_i(5)$, so by (A^*), $D[\theta(O_i)] \le 2^{-i}(5i + 1) < \epsilon$, for sufficiently large i, and θ is continuous.

To show that $\theta(M') = M$, let $p(p_1, p_2, \cdots) \in M$, and for each i, let s_i denote an edge of G_i containing p_i, and α_i denote an element of A such that

$\pi_i^g(\alpha_i) = s_i$. Then $\theta(\alpha_i) \subset (\pi_i')^{-1}[S(s_i)]$, which contains p and is of diameter $\leq 2^{-i}(3i + 1)$, since $S(s_i) \in C_i(3)$. Hence the compact set $\theta(M')$ is dense in M, and $\theta(M') = M$.

Finally, suppose $p(p_1, p_2, \cdots) \in M$. We show that $\theta^{-1}(p)$ is an arc. Let A_0 denote the collection of all arcs of A that are mapped to p by θ. Then $A_0^* = \theta^{-1}(p)$. Now, A_0 cannot contain three arcs, $\alpha_j = \lambda\{s_i^j\}$ $(j = 1, 2, 3)$ of A, for suppose it does. There is a positive integer i such that if $k \geq i$, then s_k^1, s_k^2, and s_k^3 are all distinct. Then by (F^*), each two of the edges s_{i+1}^1, s_{i+1}^2, and s_{i+1}^3 are adjacent. Hence

$$\bigcup_{j=1}^3 s_i^j \subset g_i \left[\bigcup_{j=1}^3 s_{i+1}^j \right] \subset g_i[T(s_{i+1}^1)],$$

which, by (D^*), lies in the union of two adjacent edges of G_i. But this contradicts the fact that s_i^1, s_i^2, and s_i^3 are all distinct edges of G_i. So either A_0 contains only one arc of A, or only two. In the first case, $\theta^{-1}(p)$ is that arc. In the second, suppose $\alpha = \lambda\{s_i\} \in A$ and $\alpha' = \lambda\{s_i'\}$ are those arcs. There is a positive integer n such that $s_n \neq s_n'$. By (F^*), s_n and s_n' are adjacent edges of G_n, and by Theorem 3, $\pi_i^g \mid \alpha$ and $\pi_i^g \mid \alpha'$ are homeomorphisms onto s_n and s_n', respectively. Hence α and α' have only an endpoint in common and $\theta^{-1}(p) = \alpha \cup \alpha'$ is an arc.

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