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ON A THEOREM OF CHERN

BY

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To Professor Chern on his sixtieth birthday

ABSTRACT. A new proof is given for Chern's theorem showing that the Laplace operator for differential forms commutes with decomposition of forms associated with G-structures admitting a suitable connection. An analogous result is proved for symmetric tensor fields, and an application is made to determine all harmonic symmetric fields on a compact space of constant negative curvature. Vector-valued forms are also discussed.

1. **Introduction.** At the root of the Hodge decomposition of harmonic forms on a Kähler manifold stands the fact that the Laplace operator Δ commutes with the operator $L: \phi \longrightarrow \phi \wedge \Omega$, where Ω is the fundamental 2-form of the Kähler structure, cf. [6], [11].

In [5] Chern showed that a similar phenomenon holds under broad conditions for general G-structures. Here we present a different proof of Chern's theorem, in which the complicated curvature calculations of [5] are replaced by considerations which seem much simpler.

§§2 and 3 give brief discussion of G-structures and connections; in §4 a general Laplace operator is defined for vector bundles, and in §5 we show that it commutes with certain types of bundle homomorphisms. Our proof of Chern's theorem, based on simple curvature expressions, is given in §6. Some applications to well-known facts about the Laplace operator on a Kähler manifold are mentioned in §7.

In §8 we consider *symmetric* covariant tensor fields. A Laplace operator Δ_s is defined in a straightforward manner, and it is shown that an analogue of Chern's theorem holds for Δ_s. Applications are made in §9 to two operators P, Q for symmetric fields. These operators commute with Δ_s. The harmonic fields for compact spaces of constant curvature are discussed in some detail.

Finally, in §10, vector-valued forms are considered briefly. Laplace operators are defined in a natural manner, but the appropriate analogue of Chern's

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theorem no longer holds in general. The failure of the analogue is expressed by a certain operator which is exhibited explicitly.

2. **Vector bundles.** Let E denote an r-dimensional vector bundle over a differentiable manifold M (all our ingredients will be assumed C^∞). A *frame* (U, e) of E consists of an open $U \subset M$ and an r-tuple $e = (e_1, \cdots, e_r)$ of C^∞ sections over U which span the fibre E_a at each point a of U. If (V, e') is another such frame, then in $U \cap V$ we shall have $e' = eA$ $(e'_\alpha = e_\beta A^\beta_\alpha)$, where A is a matrix-valued function on $U \cap V$, called the *transition function* for the pair of frames.

G being a Lie group with a faithful representation ρ on $\mathbf{R}^r$, a (G, ρ)-structure on E (we shall say simply G-structure) consists of a covering $\mathfrak{U}$ of M by frames of E such that the transition function of any two overlapping frames in $\mathfrak{U}$ has values in the image $\rho(G)$ in $\mathrm{GL}(\mathbf{R}^r)$. A frame of E will be called a G-frame if $\mathfrak{U}$, when augmented by the adjunction of this frame, still defines a G-structure. Observe that a G-structure on E induces a G-structure on the dual bundle E^*, corresponding to the representation ${}^t\rho^{-1}$.

Let E, $\tilde{E}$ be two bundles over M with G-structures associated with representations ρ, $\tilde{\rho}$. We shall call the G-structures *coherent* if there is given a one-to-one correspondence between G-frames of E and $\tilde{E}$ (over the same open sets) such that for any two pairs of corresponding G-frames, say (U, e), $(U, \tilde{e})$ and (V, e'), $(V, \tilde{e}')$, the transition matrices from e to e', resp. $\tilde{e}$ to $\tilde{e}'$, at each point of $U \cap V$ correspond to the same element of G in the representations ρ, $\tilde{\rho}$.

If that is so, then the frames $(U, e \otimes \tilde{e})$ of $E \otimes \tilde{E}$ define a G-structure on $E \otimes \tilde{E}$, and in a natural way it is coherent with the given G-structures on E, $\tilde{E}$; it is associated with the representation $\rho \otimes \tilde{\rho}$ of G. In a similar way we can define G-structures on exterior powers $\bigwedge^p E$, etc. (cf. §6).

3. **Connections.** Let $\Gamma(E)$ denote the module of C^∞ sections of the vector bundle E over M; and let $T^* = T^*(M)$ denote the cotangent bundle of M. A connection D on E is an $\mathbf{R}$-linear mapping $D\colon \Gamma(E) \to \Gamma(E \otimes T^*)$ which is local (i.e. diminishes supports) and satisfies $D(fs) = s \cdot df + f \cdot Ds$ for any C^∞ function f and any section $s \in \Gamma(E)$; cf. [4, §2]. (We omit the symbol $\otimes$ when convenient.)

If (U, e) is a frame of E, then D gives on U a *connection matrix* $L = (L^\alpha_\beta)$ of 1-forms on U defined by $De_\beta = e_\alpha L^\alpha_\beta$. The *curvature* matrix of the connection D relative to the frame is the matrix of 2-forms $K = dL + L \wedge L$ $(K^\alpha_\beta = dL^\alpha_\beta + L^\alpha_\gamma \wedge L^\gamma_\beta)$.

Now let E have a G-structure. We call D a *G-connection* if the matrix L has values in the Lie algebra $\mathfrak{g}$ of $\rho(G)$ whenever the frame (U, e) is a G-frame.

If that is so, then the curvature matrix K also has values in $\mathfrak{g}$. Indeed, if ξ and η are tangent vector fields near $a \in U$, then $2dL(\xi, \eta) = \xi \cdot L(\eta) - \eta \cdot L(\xi) - L([\xi, \eta])$, and this lies in $\mathfrak{g}$ simply because $\mathfrak{g}$ is a vector space. And $2L \wedge L(\xi, \eta) = [L(\xi), L(\eta)]$, which is in $\mathfrak{g}$ because $\mathfrak{g}$ is closed under the bracket operation (cf. [3] and [7, Chapter 9] for Kähler connections).

Let E, $\tilde{E}$ be bundles over M with coherent G-structures, as defined above. Let D, $\tilde{D}$ denote G-connections on E resp. $\tilde{E}$. We shall call the connections *coherent* if for any pair of corresponding G-frames of E, $\tilde{E}$ the connection matrices have values which correspond under the isomorphism of $\rho(G)$ and $\tilde{\rho}(G)$ given by $\tilde{\rho} \circ \rho^{-1}$. If that is so, then it is easy to verify that the connection $\bar{D}$ on $E \otimes E$ defined by $\bar{D}(s \otimes \tilde{s}) = (Ds) \otimes \tilde{s} + s \otimes \tilde{D}\tilde{s}$ (cf. [4, p. 80]) is a G-connection for the induced G-structure on $E \otimes \tilde{E}$. (In the term $(Ds) \otimes \tilde{s}$ it is understood that the factor in T^* is to be put on the right.)

4. **Laplace operator.** We now suppose that E is equipped with a connection D and that M is equipped with a Riemannian metric $g = g_{ij}dx^i dx^j$. With D and g we now associate a "Laplace operator" $\mathfrak{L}$ on $\Gamma(E)$ as follows: If $s \in \Gamma(E)$, then $Ds \in \Gamma(E \otimes T^*)$. Let D_0 denote the Riemannian connection (i.e. on T^*). As indicated above, we obtain a connection $\bar{D}$ on $E \otimes T^*$ by the rule $\bar{D}(s\omega) = (Ds)\omega + s(D_0\omega)$ for any 1-form ω. We define

$$\mathfrak{L}s = -\operatorname{trace} \bar{D}Ds$$

for $s \in \Gamma(E)$. Relative to a frame (U, e) of E and local coordinates (x^i) in U, $\bar{D}Ds$ has an expression $s^{\alpha}_{ij} e_{\alpha} \otimes dx^i \otimes dx^j$, and then $\mathfrak{L}s = -(g^{ij}s^{\alpha}_{ij})e_{\alpha}$, where the g^{ij} are the components of the contravariant metric tensor.

5. **G-homomorphisms.** Let E, $\tilde{E}$ be vector bundles over M of dimension r, $\tilde{r}$, with coherent G-structures associated with representations ρ, $\tilde{\rho}$ of G, and with coherent G-connections D, $\tilde{D}$. Let $F = (f^{\alpha}_{\beta})$ be an $\tilde{r} \times r$ matrix such that $\tilde{\rho}(u)F = F\rho(u)$ for all $u \in G$. From this matrix we obtain a bundle homomorphism, also called F, from E to $\tilde{E}$. Namely let (U, e), resp. $(U, \tilde{e})$, be corresponding G-frames of E, $\tilde{E}$. We map $F: e_{\beta} \to \tilde{e}_{\alpha} f^{\alpha}_{\beta}$, obtaining a homomorphism over U. Evidently it does not depend upon the choice of corresponding G-frames, and thus we obtain a global homomorphism $F: E \to \tilde{E}$.

Again let T^* carry a Riemannian metric g, as in the preceding paragraph. With g and D, resp. $\tilde{D}$, we associate Laplace operators $\mathfrak{L}$, resp. $\tilde{\mathfrak{L}}$, on $\Gamma(E)$, resp. $\Gamma(\tilde{E})$. By a direct computation we find that $F\mathfrak{L} = \tilde{\mathfrak{L}}F$. Apart from the essential assumption of coherence, the chief point involved here is that the bundle homomorphism F, relative to corresponding G-frames, has constant coefficients. The calculation presents no difficulty and we therefore omit it here.

6. **Differential forms.** We now turn to the bundles $E^p = \bigwedge^p T^*$ of p-forms M. We suppose that $T^* = E^1$ has a given G-structure such that $\rho(G) \subset O_n$, ρ denoting a faithful representation of G on $\mathbf{R}^n$, where $n = \dim M$. If we take the G-frames as orthonormal frames, then of course we obtain a Riemannian metric g on M. If D is a G-connection on T^*, and if D has no torsion, then by a classical theorem D coincides with the Riemannian (i.e. Levi-Civita) connection.

By the general rules alluded to in §2, the G-structure on E^1 induces a G-structure on each E^p. If (U, θ) is a frame of E^1, then it gives rise to a frame $(U, \theta^{(p)})$ of E^p, where $\theta^{(p)} = (\theta^{i_1 \cdots i_p})_{i_1 < \cdots < i_p}$ and $\theta^{i_1 \cdots i_p} = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}$. If (U, θ) is a G-frame, then $(U, \theta^{(p)})$ is a G-frame of E^p.

The connection D defines a connection, also called D, on each E^p. For example, for $p = 2$, $D(\omega \wedge \phi) = (D\omega) \wedge \phi + \omega \wedge (D\phi)$, being induced by the tensor product (it is understood here that the 1-form arising from D is to be written on the right). If D is a G-connection on E^1, then the extension D is a G-connection on E^p, as is plain.

The Laplace operator Δ on p-forms has the expression

$$\Delta = \mathfrak{L} + \mathfrak{R}^{(p)} + \mathfrak{S}^{(p)},$$

where $\mathfrak{R}^{(p)}$, $\mathfrak{S}^{(p)}$ are endomorphisms of E^p defined by the Riemannian curvature [9, §26]. Namely, relative to a frame (U, θ) for T^*,

$$\mathfrak{R}^{(p)}(\theta^{i_1 \cdots i_p}) = -\sum_{\nu=1}^{p} \theta^{i_1 \cdots h_\nu \cdots i_p} R_h^{i_\nu}, \qquad p \geq 1,$$

and

$$\mathfrak{S}^{(p)}(\theta^{i_1 \cdots i_p}) = -\sum_{\mu<\nu} \theta^{i_1 \cdots j_\mu \cdots h_\nu \cdots i_p} R_{jh}^{i_\mu i_\nu}, \qquad p \geq 2,$$

where $R_{jh}^{kl} = g^{lq} R_{qjh}^k$ is the Riemann curvature operator and $R_h^k = R_{jh}^{kj}$ is the Ricci tensor (our notation differs slightly from that of de Rham, *loc. cit.*). In the formula j_μ, resp. h_ν, signify that j, h appear in the μth, resp. νth, place. Of course, we have $\mathfrak{S}^{(p)} = 0$ for $p = 0, 1$; and $\mathfrak{R}^{(0)} = 0$.

We now assume that the Riemannian connection D is a G-connection, and that the frame (U, θ) is a G-frame. For each (k, l), the matrix $(i, j) \mapsto R_{jkl}^i$ is in the Lie algebra $\mathfrak{g}$ of $\rho(G)$, as explained in §3. Let $B_\lambda = (b_{j\lambda}^i)$, $\lambda = 1, \cdots, t$, be a basis for $\mathfrak{g}$. Then we may write $R_{jkl}^i = \Sigma_{\lambda=1}^t a_\lambda(k, l) b_{j\lambda}^i$ for certain a_λ. Recalling that $g_{ij} = \delta_{ij}$, and using the symmetry properties of the curvature tensor, we obtain the expression $R_{jkl}^i = R_{kl}^{ij} = \Sigma_{\lambda,\mu} a_{\lambda\mu} b_{j\lambda}^i b_{l\mu}^k$, where $a_{\lambda\mu} = a_{\mu\lambda}$. From the vanishing of the torsion tensor we have $R_{jkl}^i + R_{ljk}^i + R_{klj}^i = 0$, from which it follows that the preceding expression can also be written

$$R^{ij}_{kl} = \sum a_{\lambda\mu}(b^i_{k\lambda} b^j_{l\mu} - b^i_{l\lambda} b^j_{k\mu}),$$

and, for the Ricci tensor

$$R^i_h = \sum a_{\lambda\mu} b^i_{j\lambda} b^j_{h\mu}$$

(recall that the B_λ are skew-symmetric, since $\rho(G) \subset O_n$).

Now consider a homomorphism F from the typical fibre $\bigwedge^p \mathbf{R}^n$ of E^p to the typical fibre $\bigwedge^q \mathbf{R}^n$ of E^q. Let us assume that F is equivariant with respect to the representations $\bigwedge^p \rho$, resp. $\bigwedge^q \rho$. Then (see §5 above) it is quickly seen that F induces a bundle homomorphism $F: E^p \to E^q$ and an associated homomorphism $F: \Gamma(E^p) \to \Gamma(E^q)$. Namely, relative to the canonical bases of the typical fibres, F is given by an alternating tensor with certain components $f^{i_1 \cdots i_p}_{j_1 \cdots j_q}$. For our G-frame, the bundle map is then given (over U) by

$$\theta^{i_1 \cdots i_p} \mapsto f^{i_1 \cdots i_p}_{j_1 \cdots j_q} \theta^{j_1 \cdots j_q}.$$

If G is connected, as we henceforth assume, the equivariance condition for F is equivalent to the infinitesimal condition

$$\sum_{\nu=1}^{p} f^{i_1 \cdots h_\nu \cdots i_p}_{j_1 \cdots j_q} b^{i_\nu}_h = \sum_{\nu=1}^{q} f^{i_1 \cdots i_p}_{j_1 \cdots h_\nu \cdots j_q} b^h_{j_\nu}$$

for every $B = (b^i_j)$ in the Lie algebra $\mathfrak{g}$. Let $C = (c^i_j)$ be another element of $\mathfrak{g}$. In the equation above we change j_μ to k, multiply by $c^k_{j_\mu}$ and sum, obtaining

$$\sum_{\mu,\nu} f^{i_1 \cdots h_\nu \cdots i_p}_{j_1 \cdots k_\mu \cdots j_q} b^{i_\nu}_h c^k_{j_\mu} = \sum_\nu f^{i_1 \cdots i_p}_{j_1 \cdots k_\nu \cdots j_q} b^h_{j_\nu} c^k_h$$
$$+ \sum_{\mu\ \nu} f^{i_1 \cdots i_p}_{j_1 \cdots h_\mu \cdots k_\nu \cdots j_q} (b^h_{j_\mu} c^k_{j_\nu} - b^h_{j_\nu} c^k_{j_\mu}).$$

Now in the equivariance equation above change $b^{i_\nu}_h$ to $c^{i_\mu}_k$, multiply the new equation by $b^{i_\nu}_h$ and sum, obtaining

$$\sum_{\mu,\nu} f^{i_1 \cdots h_\nu \cdots i_p}_{j_1 \cdots k_\mu \cdots j_q} b^{i_\nu}_h c^k_{j_\mu} = \sum_\nu f^{i_1 \cdots k_\nu \cdots i_p}_{j_1 \cdots j_q} b^{i_\nu}_h c^h_k$$
$$+ \sum_{\mu<\nu} f^{i_1 \cdots h_\mu \cdots k_\nu \cdots i_p}_{j_1 \cdots j_q} (b^{i_\mu}_h c^{i_\nu}_k - b^{i_\nu}_h c^{i_\mu}_k).$$

Thus the right-hand members of the last two equations are equal. From our formulas above for R^{ij}_{kl} and R^k_h, it follows at once that $F(\mathfrak{R}^{(p)} + \mathfrak{S}^{(p)}) = (\mathfrak{R}^{(q)} + \mathfrak{S}^{(q)})F$. Indeed one has merely to replace B, resp. C, by B_λ, resp. B_μ, then multiplying by $a_{\lambda\mu}$ and summing.

Finally we observe that the differential operator $\mathfrak{L}$ commutes with F, as indicated in §5. Thus we have proved Chern's theorem:

Let $F: E^p \to E^q$ be an equivariant homomorphism of the bundles of differential forms, in the sense described above. If the structure group G of T^* is connected, and if the connection of the G-metric is a G-connection, then the Laplacian Δ commutes with F.

7. **Kähler manifolds.** We consider exterior forms on a Hermitian manifold M. The exterior derivative d and the codifferential δ split into sums $d = d' + d''$, resp. $\delta = \delta' + \delta''$, of bidegrees $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ respectively (cf. [11]). Now $\Delta = d\delta + \delta d = (d'\delta' + \delta' d') + (d''\delta'' + \delta'' d'') + (d'\delta'' + \delta'' d') + (d''\delta' + \delta' d'')$. The last two terms are of bidegree $(1, -1)$, resp. $(-1, 1)$. If the Hermitian metric is in fact a Kähler metric, then the metric connection is compatible with the complex structure, and the Laplace operator on p-forms is compatible with the decomposition of forms according to complex type. Hence the last two terms in the expression above for Δ are zero. Call the first two terms Δ', resp. Δ''. Thus our equation is $\Delta = \Delta' + \Delta''$. Since δ' is the formal adjoint of d' and δ'' is the formal adjoint of d'', the operators Δ' and Δ'' are real. By degree considerations we have $\overline{\Delta'\phi} = \Delta''\overline{\phi}$ (bar denotes complex conjugate); there follows $\Delta = 2\Delta' = 2\Delta''$ (see [11, Chapter II, Théorème 2]).

If $\Omega = (i/2) g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ denotes the fundamental 2-form of the Kähler structure on M, then we obtain a bundle homomorphism $L: E^p \to E^{p+2}$ by $L: \phi \to \phi \wedge \Omega$. This operator satisfies the requirements of the theorem of §6 and therefore commutes with Δ. This fact leads to Hodge's canonical decomposition of a harmonic form into a sum of harmonic forms $L^r\phi_r$, where the ϕ_r are *primitive*. See [11, Chapter 2, Theorem 2] and [6, §43]. In a remarkable observation, Serre [10, Chapter IV, §4] has shown that Hodge's decomposition theorem can be connected with the representation theory of $\mathfrak{sl}_2$ (cf. §9 below).

8. **Symmetric fields.** M^n again being a Riemannian manifold, we now consider the bundles S^p of covariant symmetric tensors of rank p. For brevity we shall call sections of S^p *p-fields*. The symmetrization operator, call it s_p, gives a projection of the full covariant tensor bundle $T^{0,p}(M)$ onto S^p.

As before, we assume that $T^*(M) = S^1$ has a G-structure which is a reduction of the O_n-structure given by the metric. Then on each S^p we obtain a G-structure by tensor products. If (U, θ) is a frame for S^1, $\theta = (\theta^1, \cdots, \theta^n)$, then the sections $\theta_s^{i_1 \cdots i_p} = s_p(\theta^{i_1} \otimes \cdots \otimes \theta^{i_p})$, $i_1 \leq \cdots \leq i_p$, form a frame for S^p over U. If (U, θ) is a G-frame, then this associated frame of S^p is a G-frame. We install upon S^p the fibre metric for which those sections are orthonormal at each point of U. We write $\langle \phi, \phi' \rangle_a$ for the inner product of two elements of the fibre S_a^p.

For a p-field ϕ we define a derivative by $D_s\phi = (p+1)_{s_{p+1}}(D\phi)$, where D

denotes covariant differentiation relative to the Riemannian connection. Relative to a frame the components are given by the formula

$$(D_s\phi)_{i_1\cdots i_p} = \sum_{\nu=1}^{p+1} \phi_{i_1\cdots\hat{\nu}\cdots i_{p+1},\, i_\nu}.$$

For p-fields ϕ, ϕ' we introduce the inner product

$$(\phi, \phi') = \int_M (\phi, \phi')_a \, dV_a,$$

where $dV_a = \sqrt{g}\ |\theta^1 \wedge \cdots \wedge \theta^n|$ is the volume element at the point a.

The formal adjoint D_s^* of D_s is the operator characterized by the equation $(D_s\phi, \psi) = (\phi, D_s^*\psi)$ for an arbitrary p-field ϕ and an arbitrary $(p+1)$-field ψ whose common support is compact. It is easily shown that for a p-field ϕ the components of $D_s^*\phi$ are given by the formula

$$(D_s^*\phi)_{i_1\cdots i_{p-1}} = -g^{jk}\phi_{i_1\cdots i_{p-1}jk}.$$

Of course we define $D_s^* = 0$ for 0-fields.

In analogy with the theory of harmonic integrals, we introduce the Laplace operator $\Delta_s = D_s^*D_s - D_sD_s^*$. It can be shown that for p-fields we have $\Delta_s = \mathfrak{L} + \mathfrak{R}'^{(p)} + \mathfrak{S}'^{(p)}$, with the following formulas for the operation on the frame elements introduced above:

$$\mathfrak{L}(\theta_s^{i_1\cdots i_p}) = -g^{jk}\theta_s^{i_1\cdots i_p}{}_{,jk},$$

$$\mathfrak{R}'^{(p)}(\theta_s^{i_1\cdots i_p}) = \sum_{\nu=1}^{p} \theta_s^{i_1\cdots h_\nu\cdots i_p} R_h^{i_\nu}, \qquad p \geq 1,$$

$$\mathfrak{S}'^{(p)}(\theta_s^{i_1\cdots i_p}) = \sum_{\mu<\nu} \theta_s^{i_1\cdots j_\mu\cdots h_\nu\cdots i_p} \overline{R}_{jh}^{i_\mu i_\nu}, \qquad p \geq 2.$$

Again j_μ, h_ν indicate the positions of the indices j, h; and we have $\mathfrak{R}'^{(0)} = 0$ and $\mathfrak{S}'^{(p)} = 0$ for $p = 0, 1$.

Here we have the new curvature operator $\overline{R}_{kl}^{jh} = g^{hq}R_{kql}^j + g^{jq}R_{kql}^h$. Let us assume again that the frame (U, θ) is a G-frame. Referring back to our expression for the curvature in §6, we obtain for $\overline{R}$ the expression

$$\overline{R}_{kl}^{jh} = \sum a_{\lambda\mu}(b_{k\lambda}^j b_{l\mu}^h + b_{k\lambda}^h b_{l\mu}^j).$$

Observe that $R_l^j = \overline{R}_{hl}^{jh}$, as follows from the skew-symmetry of the matrices $(b_{j\lambda}^i)$.

Let F' denote an equivariant homomorphism, relative to the G-action, from the typical fibre of S^p to the typical fibre of S^q. Then F' induces a bundle homomorphism $F'\colon S^p \to S^q$ (see §5). For the G-frame (U, θ), F' is given by a symmetric tensor:

$$F': \theta_s^{i_1\cdots i_p} \mapsto f'^{i_1\cdots i_p}_{j_1\cdots j_q}\,\theta_s^{j_1\cdots j_q};$$

the components f' are constant and independent of the choice of G-frame, and they satisfy symmetry relations similar to those given for F in §6. From the similarity of the expressions for Δ, resp. Δ_s, it is clear that, with only minor modifications, the argument of §6 can be applied to Δ_s, and we obtain the following theorem:

Let $F': S^p \to S^q$ be an equivariant bundle mapping, as described above. If the structure group G is connected, and if the Riemannian connection is a G-connection, then Δ_s commutes with F'.

9. **Harmonic fields.** We shall continue with some observations concerning *harmonic* p-fields, i.e. fields ϕ satisfying $\Delta_s\phi = 0$. To simplify statements, we shall assume that M is compact.

The operator Δ_s is related to a variational problem, as follows: Define the "energy" of a p-field ϕ by $E(\phi) = \frac{1}{2}(D_s\phi, D_s\phi) - \frac{1}{2}(D_s^*\phi, D_s^*\phi) = \frac{1}{2}(\phi, \Delta_s\phi)$. For a variation $\delta\phi$ of ϕ we obtain $\delta E(\phi) = (\delta\phi, \Delta_s\phi)$. Hence $\Delta_s\phi = 0$ is the condition for a free extremal of E. If we impose the constraint $(\phi, \phi) = 1$, say, then according to the theory of Lagrange multipliers, the extremal condition becomes $\Delta_s\phi = c\phi$ ($c = \text{const}$). Of course the energy functional $E(\phi)$ is not always positive.

Consider now a 1-field ϕ on M, i.e. a Pfaffian form. If $D_s\phi = 0$, then necessarily $D_s^*\phi = 0$, as is readily seen, and so $\Delta_s\phi = 0$. Now $D_s\phi = 0$ is none other than Killing's equation, and so Killing fields are harmonic fields (see [2]). A p-field whose covariant derivative vanishes is clearly harmonic, the metric tensor g being the most important example.

In general, geometric interpretations of the spaces of harmonic p-fields are not known. Presently we shall take up in detail the case of constant curvature K. Then the curvature operator has components $\bar{R}^{jb}_{kl} = K(2g^{jb}g_{kl} - \delta^j_k\delta^b_l - \delta^j_l\delta^b_k)$, and the Ricci tensor is $R^j_l = K(1-n)\delta^j_l$. The operators $\mathcal{R}'^{(p)}$ and $\mathcal{S}'^{(p)}$ then take on a particularly simple form.

Before imposing the curvature restriction, however, we shall develop a very useful formalism involving the metric tensor.

Let us denote the symmetric product by $\vee$. Thus e.g. for 1-fields ξ, η we have $\xi \vee \eta = \frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi)$. Define operators P, Q for p-fields by

$$P\phi = -\tfrac{1}{2}(p+1)(p+2)\cdot\phi \vee g;$$

$$Q = -(p+1)^{-1}(p+2)^{-1}P^* \quad \text{for } p \geq 2,$$

$$Q = 0 \quad \text{for } p = 0, 1.$$

Here P^* denotes the adjoint of P, and g is the metric tensor. For components we have the formulas

$$(P\phi)_{i_1\cdots i_{p+2}} = -\sum_{\mu<\nu}\phi_{i_1\cdots\hat{\mu}\cdots\hat{\nu}\cdots i_{p+2}}\cdot g_{i_\mu i_\nu}, \qquad p \geq 2;$$

$$(Q\phi)_{i_1\cdots i_{p-2}} = \tfrac{1}{2}\phi_{i_1\cdots i_{p-2}jk}\cdot g^{jk}, \qquad p \geq 2.$$

We have the commutation relations $D_sP - PD_s = 0$, $D_sQ - QD_s = D_s^*$, and $PQ - QP = n/2 + p$ (for p-fields). These hold for $p \geq 0$. Write W for the latter commutator. Then $[P, Q] = W$, $[W, P] = 2P$, $[W, Q] = -2Q$. Therefore the typical fibre of the bundle $S = \Sigma_{p=0}^{\infty} S^p$ becomes an $\mathfrak{sl}_2(\mathbf{R})$-module (cf. Serre [10, Chapitre 4]).

It is readily verified that P and Q are equivariant bundle maps for the O_n-structure of T^*, in the sense understood in §8. We may therefore apply the theorem of §8 to conclude that Δ_s *commutes with* P *and* Q. This may also be seen from the commutation rules above. The situation is in striking analogy with the Kähler case, but does not lead to parallel conclusions because of the infinite dimensionality of our representation of $\mathfrak{sl}_2$.

We have subbundles Ker $Q \subset$ Ker $Q^2 \subset$ Ker $Q^3 \subset$ etc., from which we obtain the orthogonal decomposition $S^p = S_0^p + \cdots + S_{\langle p\rangle}^p$ ($\langle p\rangle = p/2$ for even p and $(p-1)/2$ for odd p), where S_k^p is the orthogonal complement of the bundle $S^p \cap$ Ker Q^k in $S^p \cap$ Ker Q^{k+1}. The subspace $S_{0,a}^p$ of the fibre S_a^p is irreducible for the orthogonal group of T_a^* (see [12, Chapter 5, §7]). The subspaces $S_{k,a}^p$ are also irreducible, as will presently be evident.

The operators PQ and QP are selfadjoint in each fibre S_a^p. The commutator $[P, Q^k]$ is a multiple of Q^{k-1} on p-fields, from which it is easily seen that Q maps S_k^p into S_{k-1}^{p-2} $(k \geq 1)$, and that each subbundle S_k^p is stable for PQ and QP. On S_k^p we have

$$PQ|S_k^p = -k(n/2 + p - k - 1), \qquad QP|S_k^p = -(k+1)(n/2 + p - k).$$

This is easily established by induction on k. For let $\phi \in S_k^p$, $k \geq 1$, and write $PQ\phi = \psi$. Then $QPQ\phi = Q\psi$; and $Q\phi$, $Q\psi$ are in S_{k-1}^{p-2}. By induction assumption, $QPQ\phi = cQ\phi$, for a certain factor c, whence $Q(c\phi - \psi) = 0$. But $c\phi - \psi$ is in S_k^p and thus is orthogonal to Ker Q, whence $c\phi = \psi$, and so $PQ\phi = c\phi$. The value of c is quickly found inductively with use of the relation $QP = PQ - (n/2 + p)$.

As just pointed out, Q maps S_k^p injectively into S_{k-1}^{p-2} for $k > 0$. We see now that P is injective. Indeed, if $\phi \in S_k^p$ and $P\phi = 0$, then $QP\phi = 0$, and so $PQ\phi = (n/2 + p)\phi$. Comparing with the result above for $PQ \mid S_k^p$, we obtain $\phi = 0$.

The harmonic elements in $\Gamma(S^p)$ form a finite-dimensional vector space H^p, for compact M, as follows from standard elliptic theory applied to the strongly elliptic system $\Delta_s\phi = 0$; and the equation $\Delta_s\phi = \psi$ for C^∞ fields has a solution if and only if ψ is orthogonal to all elements of H^p. See for example [1], [8]. Since Δ_s commutes with P and Q, the space H^p splits into an orthogonal sum $H_0^p + \cdots + H_{\langle p\rangle}^p$, with $H_k^p \subset \Gamma(S_k^p)$.

For a section ϕ of S_0^p we have $D_s^*\phi = -QD_s\phi$, and so ϕ is harmonic if $D_s\phi = 0$. We observe that the energy functional $E(\phi)$ is ≥ 0 for $p = 1$ and $n \leq 4$. In these special cases it follows that a global harmonic 1-field ϕ satisfies $D_s\phi = D_s^*\phi = 0$ on a compact manifold.

By linear algebra, P maps *onto* the orthogonal complement of Ker $P^* =$ Ker Q, and therefore we find that P maps S_k^p isomorphically upon S_{k+1}^{p+2} $(k = 0, 1, \cdots, \langle p\rangle)$. As P commutes with Δ_s, P maps H_k^p isomorphically onto H_{k+1}^{p+2}. Thus for harmonic fields it suffices to consider the bundles S_0^p. Since P commutes with D_s, if $\phi \in \Gamma(S_k^p)$ satisfies $D_s\phi = 0$, then by reducing via P to $k = 0$, we see that $\Delta_s\phi = 0$, and so $D_sD_s^*\phi = 0$. For the compact manifold M it follows that $D_s^*\phi = 0$ also.

Let $\phi \in \Gamma(S_0^p)$ and $\psi \in \Gamma(S_0^q)$. If ϕ, ψ satisfy the orthogonality relation expressed in terms of components by

$$\phi_{i_1\cdots i_{p-1}k}\psi_{j_1\cdots j_{q-1}l}g^{kl} = 0,$$

then it is not hard to see that $\phi \vee \psi$ is a section of S_0^{p+q}. Since, moreover,

$$D_s(\phi \vee \psi) = \frac{p+q+1}{p+1}(D_s\phi) \vee \psi + \frac{p+q+1}{q+1}\phi \vee (D_s\psi),$$

it follows that $D_s(\phi \vee \psi) = 0$, hence $\phi \vee \psi$ harmonic, whenever $D_s\phi = D_s\psi = 0$. For example, if $\phi^1, \cdots, \phi^p$ are orthonormal covariant Killing fields, then $\phi^1 \vee \cdots \vee \phi^p$ is a harmonic section of S_0^p.

Let us note in passing that the fibre dimension of S^p is

$$d_p = \sum_{m=1}^{\min(n,\,p)} \binom{n}{m}\binom{p-1}{m-1}.$$

With the foregoing observations concerning $P: S^p \to S^{p+2}$ in mind, we see that S_0^p has fibre dimension $d_p - d_{p-2}$ $(p \geq 2)$. For example, for $n = 2$ this dimension is 2; and for $n = 3$ it is $2p + 1$.

For $n = 2$ the typical fibres of S_0^p $(p \geq 2)$ are irreducible O_2-modules (of dimension 2), hence are isomorphic. This means that for any orthonormal basis b of the typical fibre of S_0^p, there is an orthonormal basis b' of the typical fibre of S_0^q $(p, q \geq 2)$ for which the O_2-action is represented by the same matrices. Clearly b' is uniquely determined by b, and we may conclude that the bundles

S_0^p $(p \geq 2)$ are all equivariantly isomorphic. Hence for $n = 2$ only the bundles S_0^1, S_0^2 are of importance in the present connection.

Let us now turn to the case of constant curvature, for which the tensor $\bar{R}$ has the form given earlier. It is readily shown that the following formulas result for a p-field ϕ:

$$R'^{(p)}\phi = -pK(n-1)\phi, \qquad S'^{(p)}\phi = -p(p-1)K\phi - 4KPQ\phi.$$

Thus for a section ϕ of S_0^p the harmonic equation reduces to $\mathfrak{L}\phi = pK(n+p-2)\phi$

Suppose that $K \leq 0$. Then for a harmonic section ϕ of S_0^p $(p > 0)$, $\mathfrak{L}\phi = -r\phi$, with $r \geq 0$. Since $\mathfrak{L}$ is a positive operator, and $-r$ is a negative operator, there follows $(\mathfrak{L}\phi, \phi) = r(\phi, \phi) = 0$. If $K < 0$, we find that $\phi = 0$. Hence, for a compact manifold of constant negative curvature the only harmonic nonzero p-fields are those of the form $g \vee \cdots \vee g \times \text{const}$.

If $K = 0$, then we have just $(\mathfrak{L}\phi, \phi) = 0$. But $(\mathfrak{L}\phi, \phi) = (D\phi, D\phi)$, and so $D\phi = 0$. Thus $(\mathfrak{L}\phi, \phi) = 0$, and in particular $\mathfrak{L}\phi = 0$ if and only if ϕ is a uniform field (i.e. has covariant derivative zero). (The argument here is substantially that of Bochner—cf. [2]).

10. **Vector-valued forms.** Computations similar to the foregoing can be made for other tensor bundles over M^n. Here we shall take up rather summarily the bundles of vector-valued p-forms $\hat{E}^p = E^p \otimes T$ and of vector-valued p-fields $\hat{S}^p = S^p \otimes T$, where T denotes the tangent bundle of M. These bundles are of interest in connection with deformation theory.

We again suppose that there is given a G-structure on T^* which is a reduction of a smooth O_n-structure. If $(U, (\theta^i))$ is a frame of T^*, and if $(U, (\theta_i))$ is the dual frame of T, then the system $(U, (\theta_k^{i_1 \cdots i_p}))_{i_1 < \cdots < i_p}$ is a frame of $\hat{E}^p$ over U, where $\theta_k^{i_1 \cdots i_p} = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \otimes \theta_k$. If $(U, (\theta^i))$ is a G-frame, then we declare this associated frame of $\hat{E}^p$ to be a G-frame; and in this manner we obtain a G-structure on $\hat{E}^p$.

Further we introduce on $\hat{E}_a^p$ the fibre metric $\langle\ ,\ \rangle_a$ for which these elements form an orthonormal basis. In particular, we have once more the Riemannian metric $g = g_{ij}\theta^i\theta^j$ on T, and $g_{ij} = \delta_{ij}$ for a G-frame. By means of the metrics we obtain a global inner product

$$(\phi, \phi') = \int_M \langle \phi, \phi' \rangle_a \, dV_a,$$

as in §6.

For a section ϕ of $\hat{E}^p$ we define a derived form $d\phi = (p+1)A_{p+1}D\phi$, where D is the covariant derivative and A_{p+1} is the alternation operator on the "covariant indices". If ϕ has components $\phi^k_{i_1 \ldots i_p}$ relative to a frame (U, θ) of T^*, then $d\phi$ has components

$$(d\phi)^k_{i_1\cdots i_{p+1}} = \sum (-1)^{\nu-1}\phi^k_{i_1\cdots\hat{\nu}\cdots i_{p+1},\, i_\nu}.$$

The formal adjoint d^* of d is given, for a section ϕ of $\hat{E}^p$, by

$$(d^*\phi)^k_{i_1\cdots i_{p-1}} = -\phi^k_{bi_1\cdots i_{p-1},\, l}g^{bl},$$

where as usual the g^{bl} are the components of the metric tensor g relative to the frame in use. For $p = 0$ we define $d^*\phi = 0$.

The corresponding Laplace operator $\Delta = dd^* + d^*d$ for sections of $\hat{E}^p$ has the expression $\Delta = \mathfrak{L} + \mathfrak{R}^{(p)} + \mathfrak{S}^{(p)} + \mathfrak{T}^{(p)}$, where $\mathfrak{L}$ is as in §4 and $\mathfrak{R}^{(p)}$, $\mathfrak{S}^{(p)}$ are quite as in §6. $\mathfrak{T}^{(p)}$ is the operator given on the frame $(U, (\theta^{i_1\cdots i_p}_k))_{i_1<\cdots<i_p}$ by

$$\mathfrak{T}^{(p)}\theta^{i_1\cdots i_p}_k = \sum_{\nu=1}^{p} \theta^{i_1\cdots h_\nu\cdots i_p}_l \hat{R}^{li_\nu}_{kh} \qquad (p > 0),$$

where $\hat{R}^{li}_{kh} = g^{im}R^l_{kmh}$.

Now let F be an equivariant linear mapping $\bigwedge^p \mathbf{R}^n \otimes \mathbf{R}^n \to \bigwedge^q \mathbf{R}^n \otimes \mathbf{R}^n$. Relative to the canonical bases F is given by a system of components $f^{i_1\cdots i_p l}_{j_1\cdots j_q k}$ which are skew-symmetric in the i's, resp. the j's. As in §6, F gives a bundle homomorphism $F: \hat{E}^p \to \hat{E}^q$ which, for the G-frames defined above, is given by

$$F: \theta^{i_1\cdots i_p}_k \mapsto f^{i_1\cdots i_p l}_{j_1\cdots j_q k}\theta^{j_1\cdots j_q}_l.$$

If G is connected, then the equivariance condition reduces to the following relation, for all elements (b^i_j) in the Lie algebra of G:

$$\sum_{\nu=1}^{q} f^{i_1\cdots i_p l}_{j_1\cdots h_\nu\cdots j_q k} b^h_{j_\nu} - f^{i_1\cdots i_p m}_{j_1\cdots j_q k} b^l_m = \sum_{\nu=1}^{p} f^{i_1\cdots h_\nu\cdots i_p}_{j_1\cdots j_q k} b^{i_\nu}_h - f^{i_1\cdots i_p l}_{j_1\cdots j_q m} b^m_k.$$

By operating upon this identity as in §6, we can determine the commutator $\Delta F - F\Delta$. It is found that this operator is expressed in terms of components by

$$\sum_{\nu=1}^{q} f^{i_1\cdots i_p}_{j_1\cdots h_\nu\cdots j_q m}\hat{R}^{hm}_{j_\nu k} - \sum_{\nu=1}^{p} f^{i_1\cdots h_\nu\cdots i_p m}_{j_1\cdots j_q k}\hat{R}^{i_\nu l}_{hm}.$$

For certain F this commutator vanishes. For example, if our G-structure is a Kähler structure with fundamental 2-form ω, and if F denotes the operation $F\phi = \omega \wedge \phi$, then it can be shown by direct computation that $\Delta F = F\Delta$. Thus we can conclude that a Hodge decomposition holds for harmonic vector-valued forms on a Kähler manifold.

The situation is analogous for vector-valued p-fields, i.e. sections of the bundle $\hat{S}^p$. A Laplace operator Δ_s is defined as in §8. For an equivariant bundle map $F: \hat{S}^p \to \hat{S}^q$ of the sort considered above (but with symmetric components $f^{i_1 \cdots i_p l}_{j_1 \cdots j_q k}$ in the i's, resp. j's, rather than the skew-symmetric ones pertaining to the $\hat{E}^p$), we find that $\Delta_s F - F\Delta_s$ is given by the same expression as that just written for the case of exterior forms.

We can define operators P, Q for sections of $\hat{S}^p$, just as in §9. For these operators the commutator vanishes, as is fairly easily seen.

REFERENCES

1. P. Bidal and G. de Rham, *Les formes différentielles harmoniques*, Comment. Math. Helv. 19 (1946), 1–49. MR 8, 93.

2. S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. 52 (1946), 776–797. MR 8, 230.

3. ——, *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. 53 (1947), 179–195. MR 8, 490.

4. R. Bott and S.-S. Chern, *Hermitian vector bundles and the equidistribution of their holomorphic sections*, Acta Math. 114 (1965), 71–112. MR 32, #3070.

5. S. S. Chern, *On a generalization of Kähler geometry. Algebraic geometry and topology* (A Sympos. in Honor of S. Lefschetz), Princeton Univ. Press, Princeton, N.J., 1957, pp. 103–121. MR 19, 314.

6. W. V. D. Hodge, *The theory and application of harmonic integrals*, Cambridge Univ. Press, Cambridge, 1941. MR 2, 296.

7. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. Vol. II, Interscience Tracts in Pure and Appl. Math., no. 15, vol. II, Interscience, New York, 1969. MR 38 #6501.

8. R. Palais et al., *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, no. 57, Princeton Univ. Press, Princeton, N.J., 1965. MR 33 #6649.

9. G. de Rham, *Variétés différentiables*, Actualités Sci. Indust., no. 1222, Hermann, Paris, 1955. MR 16, 957.

10. J.-P. Serre, *Algèbres de Lie semi-simples complexes*, Benjamin, New York, 1966. MR 35 #6721.

11. A. Weil, *Variétés Kählériennes*, Hermann, Paris, 1958.

12. H. Weyl, *The classical groups*, Princeton Univ. Press, Princeton, N. J., 1939. MR 1, 42.

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