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k-CONGRUENCE ORDERS FOR E_k

BY

GRATTAN P. MURPHY

ABSTRACT. This paper generalizes the notion of congruence order for metric spaces to k-metric (k-dimensional metric) spaces. The k-congruence order of E_k with respect to the class of oriented semi k-metric spaces is determined. An example shows that this result is sharp.

Introduction. The congruence order and 'best' congruence indices of classes of semi-metric spaces is well known [1, pp. 93–118]. This paper deals with the analogous problem for oriented semi k-metric spaces with respect to E_k (considered as an oriented k-metric space).

A *k-metric space* consists of a set S together with a real-valued function d_k defined for $k+1$-tuples of S and satisfying

(1) if a_1, a_2 are 2 distinct points of S, there exist points $a_3, \cdots, a_{k+1}$ such that $d_k(a_1, \cdots, a_k, a_{k+1}) \neq 0$, and

(2) for each $k+2$ points of S there exist $k+2$ points of E_{k+1}, the $k+1$-dimensional Euclidean space, and a 1-1 mapping between the two $k+2$-tuples such that the values of d_k and the k-dimensional hypervolume (unsigned) are the same for corresponding $k+1$-tuples.

This reduces to a metric space for $k = 1$, $k = 2$ gives a generalized area and other values give generalized 'volume' spaces. The Euclidean space E_n is a k-metric space for every $k \leq n$ if we take the k-dimensional volume for the k-metric.

A 1-1 onto mapping between S and S', subsets of k-metric spaces, is called a k-congruence if it preserves the k-metric. S and S' are then said to be k-congruent and we write $S \approx S'$. With this definition condition (2) above could be changed to: each $k+2$ points are k-congruent with $k+2$ points of E_{k+1}.

If condition (2) is reduced to the requirement that every $k+1$-tuple be k-congruent with a $k+1$-tuple of E_k, the resulting space is called a *semi k-metric space*. The spaces are said to be *oriented* if each ordered $k+1$-tuple is attached a sign according to some rule. The usual orientation for E_k is given by the sign of the determinant which gives the hypervolume of each $k+1$-tuple. If a

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k-congruence between two sets in oriented k-metric spaces either preserves all orientations or reverses all orientations, it is said to be positive (denoted by k_+-*congruence* and $\equiv$).

The results mentioned earlier for metric spaces depended heavily on the fact that an independent $n+1$-tuple in E_n is a complete metric basis for E_n (i.e., any point of E_n can be uniquely determined by giving its distances from points in an independent $n+1$-tuple and congruences between subsets of E_n can be extended to motions if the subsets contain independent $n+1$-tuples). Considering E_k as a k-metric space the corresponding statement is true for an independent $k+1$-tuple only if the k-congruences are all positive. If a, b, c and d are the vertices of a rectangle in E_2, then a, b, c, d are 2-congruent with a, b, d, c, but the correspondence cannot be extended to all of E_2 (where would the point of intersection of the diagonals go?). The difficulty in this example lies in the fact that the 2-congruence between bcd and bdc is not positive, while that between abc and abd is. The correspondence between abc and abd can be extended positively to $abcd$ by mapping d into the point which is the reflection of c through d. The latter 2-congruence can be extended to a 2-*motion* of E_2 (a k-motion of a k-metric space M is a k-congruence of M with itself).

Bases in E_k.

Definition. If each point of a k-metric (k_+-metric) space M is uniquely determined when the values of the k-metric (k_+-metric) for each ordered $k+1$-tuple containing that point and k points of some subset B are given, then B is called a *k-metric* (*k_+-metric*) *basis* for M.

Definition. A k-metric (k_+-metric) basis B is said to be *complete* if, whenever $B \approx B^*$ ($B \equiv B^*$) and $B^* \subset S^* \subset M$, there exists a subset S of M containing B and k-congruent (k_+-congruent) with S^* and this correspondence is an extension of that between B and B^*.

Defintion. A $k+1$-tuple in E_k is called *independent* if it is not contained in a lower dimensional subspace.

If $(x_{i1}, \cdots, x_{ik})$, $i = 0, 1, \cdots, k$, are the rectangular representations of $k+1$ points of E_k, then V_k, the k_+-metric (signed k-dimesional volume) of the ordered $k+1$-tuple, is given by

$$(k!)\, V_k(x_0, \cdots, x_k) = \begin{vmatrix} x_{01} & x_{02} & \cdots & x_{0k} & 1 \\ x_{11} & & & & 1 \\ \vdots & & & & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kk} & 1 \end{vmatrix} .$$

In order to facilitate working with these determinants, we will adopt the convention that if x is a point in E_k, x' is the point in E_{k+1} whose first k coordinates are the same as x and whose $k+1$st coordinate is 1. The value of the k_+-metric above would then be $(1/k!)\det(x_0', x_1', \cdots, x_k')$.

Theorem 1. *If $p_0, p_1, \cdots, p_k$ and $q_0, q_1, \cdots, q_k$ are two independent $k+1$-tuples in E_k with the same k-dimensional volume and x is a point of E_k, then there is one and only one y in E_k such that $p_0, p_1, \cdots, p_k, x \equiv q_0, q_1, \cdots, q_k, y$.*

Proof. Without loss of generality we may assume the p's to be in an E_k such that $p_0 = (0, 0, \cdots, 0)$, $p_i = (a_{1i}, a_{2i}, \cdots, a_{ii}, 0, \cdots, 0)$ and the q's in an E_k with $q_0 = (0, \cdots, 0)$, $q_i = (b_{1i}, \cdots, b_{ii}, 0, \cdots, 0)$. For definiteness we assume that $p_0, \cdots, p_k$ and $q_0, \cdots, q_k$ have the same orientation.

Let x and y be given by $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$. The k_+-congruence requires that the $k+1$ equations

$$\begin{aligned}(3)\qquad &\det(p_0', \cdots, p_{i-1}', x', p_{i+1}', \cdots, p_k') \\ &= \det(q_0', \cdots, q_{i-1}', y', q_{i+1}', \cdots, q_k'), \qquad i = 0, \cdots, k,\end{aligned}$$

be satisfied. The theorem will be proved by showing that there is a unique solution to the system of equations obtained by taking $i \geq 1$ and that it satisfies the equation for $i = 0$. By expanding the determinants for $i \geq 1$ we get k linear equations in the k unknowns y_i, $i = 1, \cdots, k$. The ith equation so obtained is of the form

$$(4)\qquad \sum_{j=i}^{k} C_j y_j = \sum_{j=i}^{k} D_j x_j, \quad \text{where } C_i = \frac{\prod_{j=1}^{k} b_{jj}}{b_{ii}}$$

and is, therefore, not zero. The coefficient vectors in the system are then in echelon form. It follows that there is a unique solution y. In order to show that the solution obtained is 'compatible' with the equation for $i = 0$, we examine the determinants a little more closely. From

$$\det(p_0', \cdots, p_{i-1}', x', \cdots, p_k') = \det(q_0', \cdots, q_{i-1}', y', \cdots, q_k')$$

follows

$$(5)\qquad \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ a_{i-1\,1} & & \cdots & 0 \\ x_1 & & \cdots & x_k \\ \vdots & & & \vdots \\ a_{k1} & & \cdots & a_{kk} \end{vmatrix} = \begin{vmatrix} b_{11} & 0 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ b_{i-1\,1} & & \cdots & 0 \\ y_1 & & \cdots & y_k \\ \vdots & & & \vdots \\ b_{k1} & & \cdots & b_{kk} \end{vmatrix}$$

If the x's and y's are moved to the top row and the preceding rows down one, the determinants are the same as the minors of the elements in the $i+1$st row and last column of the determinants for $i = 0$, $\det(x', p_1', \cdots, p_k')$ and $\det(y', p_1', \cdots, p_k')$. The minors of the elements in the 1st row and last column are equal because they are the volumes of the p's and the q's. Pairwise equality of these minors implies equality of the determinants when we recall that the elements in the last column are all 1's.

Corollary 1.1. *An independent $k+1$-tuple in E_k is a k_+-metric basis for E_k.*

Proof. Let $p_i = q_i$ for $i = 0, \cdots, k$.

Corollary 1.2. *There is one and only one k_+-motion of E_k that takes $p_0, \cdots, p_k$ onto $q_0, \cdots, q_k$, where p_i and q_i are the points in Theorem* 1.

Proof. If there is a motion it must take each point onto that point determined in the proof of the theorem, so it is unique. We must show that the mapping which takes each x onto the y such that $p_0, \cdots, p_k, x \equiv q_0, \cdots, q_k, y$ is a k_+-motion. That the mapping is onto E_k follows from the fact that the row space in the system of equations is of dimension k. It remains to show that the k_+-metric is preserved. Let $x^0, x^1, \cdots, x^k$ be $k+1$ points of E_k that map onto $y^0, y^1, \cdots, y^k$, respectively. From (4) we see that $y_i^r = \Sigma_{j=1}^k B_j x_j^r$ and these are the elements in the ith column of $\det(y^{0'}, \cdots, y^{k'})$. By elementary column operations the elements in the ith column may be reduced to $B_i x_i^r$. It follows that $\det(y^{0'}, \cdots, y^{k'}) = A \det(x^{0'}, \cdots, x^{k'})$ for some constant A. From $\det(p_0', \cdots, p_k') = \det(q_0', \cdots, q_k')$, we have $A = 1$ and the mapping is a k_+-motion.

Corollary 1.3. *An independent $k+1$-tuple in E_k is a complete k_+-metric basis for E_k.*

Proof. Let I and I^* be independent $k+1$-tuples in E_k, $I^* \subseteq S^* \subseteq E_k$ for some S^* and $I \equiv I^*$. Since $I \equiv I^*$, there is just one k_+-motion ϕ which will take I onto its image I^*. The proof is completed by letting S be the set of preimages of S^* under ϕ.

The need for orientation can be seen by noting that without it, equations (3) would have to be $\det(p_0', \cdots) = \pm \det(q_0', \cdots)$ and there would not be unique solutions for all x.

The analogue in an arbitrary k-metric space of an independent $k+1$-tuple in E_k is a $k+1$-tuple on which the k-metric function is not zero. Such a $k+1$-tuple is called *nontrivial*.

k_+-congruence order of E_k.

Definition. A semi k-metric space is said to be *k-congruently imbeddable* (*k_+-congruently imbeddable*) in a semi k-metric space M if S is k-congruent (k_+-congruent) with a subset of M.

Definition. A semi k-metric space M has *k-congruence indices* (*k_+-congruence indices*) (n, q) with respect to a class $\{S\}$ of spaces provided any space S of $\{S\}$, containing more than $n + q$ pairwise distinct points, is k-congruently imbeddable (k_+-congruently imbeddable) in M whenever each n of its points has that property.

Definition. If a space M has k-congruence indices (k_+-congruence indices) $(n, 0)$ with respect to a class $\{S\}$ of spaces, M is said to have *k-congruence order* (*k_+-congruence order*) n with respect to that class.

Two $k + 1$-tuples in a semi k-metric space are said to *k-touch* if they have k points in common.

Definition. Two nontrivial $k + 1$-tuples, P and Q, of points in a semi k-metric space are *chain connected* if there exists a finite sequence X_i $(i = 0, \cdots, n)$ of nontrivial $k + 1$-tuples of points of the semi k-metric space with each X_i k-touching X_{i+1}, $X_0 = P$ and $X_n = Q$.

A semi k-metric space has *property* C_k if each pair of its nontrivial $k + 1$-tuples is chain connected.

Lemma 1. *E_n has property C_k for each $k \leq n$.*

Proof. The theorem is obvious for $k = 1$. We assume it is true for $k - 1$ and proceed by induction. It is sufficient to show that if $x_0, \cdots, x_i, y_0, \cdots, y_j$ $(i + j = k)$ are $k + 2$ points in E_n such that the first $k + 1$ of them are independent and that the y's are independent, then for a suitable ordering of the x's, $x_1, \cdots, x_i, y_0, \cdots, y_j$ is an independent $k + 1$-tuple.

Assume $x_1, \cdots, x_i, y_0, \cdots, y_j$ is not an independent $k + 1$-tuple. The points satisfy the hypothesis of the lemma for $k - 1$, so there is an x which may be deleted to leave an independent k-tuple. Let it be x_1. Then $x_2, \cdots, x_i, y_0, \cdots, y_j$ generate an E'_{k-1}. Since adding x_1 does not yield an independent $k + 1$-tuple, x_1 must be in E'_{k-1}. If $x_0, x_2, \cdots, x_i, y_0, \cdots, y_j$ is independent, the theorem is proved; if not, then x_0 is in E'_{k-1}. But x_0 and x_1 both in E'_{k-1} contradicts the independence of $x_0, \cdots, x_i, y_0, \cdots, y_{j-1}$.

Let $\mathfrak{K}$ be the class of oriented semi k-metric spaces which have property C_k. We may now state the following:

Theorem 2. *E_k has k_+-congruence order $k + 3$ with respect to the class $\mathfrak{K}$.*

Proof. Let $p_0, p_1, \cdots, p_k$ be a nontrivial $k + 1$-tuple of a semi k_+-metric space M in $\mathfrak{K}$, every $k + 3$ points of which are k_+-congruent with $k + 3$ points of E_k. Let $p_0', p_1', \cdots, p_k'$ be a $k+1$-tuple of E_k with $p_0, p_1, \cdots, p_k \equiv p_0', p_1', \cdots, p_k'$. If x is a point in M, then there exist points $p_0'', p_1'', \cdots, p_k'', x''$ in E_k such that $p_0, p_1, \cdots, p_k, x \equiv p_0'', p_1'', \cdots, p_k'', x''$. Since $V_k(p_0'', \cdots, p_k'') = V_k(p_0', \cdots, p_k')$, we know that there is a k_+-motion of E_k, call it g, with $g(p_i'') = p_i'$ $(i = 0, 1, \cdots, k)$. Let x' denote $g(x'')$. By repeating the procedure for each x in M we define a mapping of M into a subset of E_k. In order to show that the mapping is a k_+-congruence we let $X_0, X_1, \cdots, X_n$ be a chain connecting $P = [p_0, \cdots, p_k]$ and the nontrivial $k+1$-tuple $[x_0, \cdots, x_k]$, then show that the mapping may be carried across this chain. Let $X_1 = [u, p_1, \cdots, p_k]$. For each x in M, there exist points $u^*, p_0^*, \cdots, p_k^*, x^*$ in E_k such that $u, p_0, \cdots, p_k, x \equiv u^*, p_0^*, \cdots, p_k^*, x^*$. Since $V_k(u^*, p_1^*, \cdots, p_k^*) = V_k(u', p_1', \cdots, p_k')$, there exist p_0^{**} and x^{**} with $u', p_1', \cdots, p_k', p_0^{**}, x^{**} \equiv u^*, p_1^*, \cdots, p_k^*, p_0^*, x^*$, by Corollary 1.3. From this follows $u, p_1, \cdots, p_k, p_0 \equiv u', p_1', \cdots, p_k', p^{**}$, which, together with $u, p_1, \cdots, p_k, p_0 \equiv u', p_1', \cdots, p_k', p_0'$ and transitivity of k_+-congruence, yields $p_0 = p_0^{**}$ by Theorem 1. That $x' = x^{**}$ will then follow from $p_0', \cdots, p_k', x' \equiv p_0', \cdots, p_k', x^{**}$ in a similar manner. Thus the mapping determined by the nontrivial $k+1$-tuple $u, p_1, \cdots, p_k$ coincides with a mapping determined by $p_0, \cdots, p_k$. The same argument may be employed to establish that if the mapping determined by X_{i-1} takes each x onto x', then the one determined by X_i may be made to do that also. By induction, the mapping can be carried across to X_n, from which we have $V_k(x_0', \cdots, x_k') = d_k(x_0, \cdots, x_k)$, the value of the k_+-metric on X_n.

We still need to show that the mapping is a k_+-congruence for trivial $k+1$-tuples. We assume that $x_0', \cdots, x_k'$ is nontrivial and $x_0, \cdots, x_k$ is trivial, for otherwise the theorem follows.

By Lemma 1 there is a chain connecting $p_0', \cdots, p_k'$ and $x_0', \cdots, x_1'$. We note that this chain contains only points from the two $k+1$-tuples. This chain and the proof above may be used to constuct a chain in M because the k_+-congruence of each X_i with its counterpart X_i' in E_k was established as soon as the mapping was extended to X_{i-1}. Consequently, the nontriviality of X_i' establishes that of the X_i and the chain in M is just the 'parallel' to the one in E_k.

Since $X_i \equiv X_i'$ for each X_i' in the chain and $V_k(x_0, \cdots, x_k) \neq 0$, then $d_k(x_0, \cdots, x_k) \neq 0$, a contradiction.

In the last part of the proof of the theorem we did not need the fact that M was chain connected. It is natural to ask if the theorem is true for a wider class of spaces than $\mathfrak{K}$. It turns out that, if k is less than 4, then we need not assume that M is chain connected, and may simply drop that requirement. However, for $k > 3$, we must assume that more points are k_+-congruently contained in E_k in order to 'bridge' nontrivial $k+1$-tuples, if we do not have chain connectedness. The following theorem, together with the fact that $k + 3 \geq 2k$ if $k \leq 3$, indicates the reason for this behavior.

Theorem 3. *If every $2k$ points of a k-metric space M are k-congruently contained in E_k, then M is chain connected.*

Proof. Let $x_0, \cdots, x_k$ and $p_0, \cdots, p_k$ be two nontrivial $k+1$-tuples with no points in common. The imbeddability of the $k+2$-tuple $p_0, \cdots, p_k, x_0$ in E_{k+1} implies that, for some j, $p_0, \cdots, p_{j-1}, x_0, p_{j+1}, \cdots, p_k$ is nontrivial. We suppose, without loss of generality, that the $k+1$-tuple $p_0, \cdots, p_{k-1}, x_0$ is nontrivial. If there exist i, j, $i > 0$ and $j < k$, such that $x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k, p_j$ is nontrivial, then the union of the $k+1$-tuples $x_0, \cdots, x_k, p_j$ and $p_0, \cdots, p_{k-1}, x_0$ contains $2k$ points. Since the $2k$ points are k-congruently contained in E_k, an application of Lemma 1 completes the proof in that case. If no such i and j exist, then $d_k(p_j, x_1, \cdots, x_k) = d_k(x_0, \cdots, x_k)$ for every $j < k$ so the $k+1$-tuple $p_j, x_1, \cdots, x_k$ is nontrivial. One of the $k+1$-tuples $p_0, \cdots, p_{j-1}, p_{j+1}, \cdots, p_k, x_1$ is nontrivial for $j \leq k$. We may combine that $k+1$-tuple with $p_n, x_1, \cdots, x_k$ $(n \neq j)$ to get $2k$ points and then proceed as above.

An easy corollary is

Corollary 3.1. *The k_+-congruence order of E_k with respect to the class $\mathcal{S}$ of all oriented semi k-metric spaces is* $\max\{k+3, 2k\}$.

We note that for $k = 1, 2$ or 3, Corollary 3.1 is a sharper result than Theorem 2.

Examples. The following examples show that the previous results are sharp.

Example 1 is a 4-metric space, every 7 points of which are 4-congruently contained in E_4, but which is not chain connected. Therefore $2k$ may not be replaced by $2k-1$ in Theorem 3.

k_+-congruence order m is the same as k_+-congruence indices $(m, 0)$. E_1 has congruence order 4, but also has congruence indices (3, 1), see [1, p. 118]. Example 2 shows that, for $k \geq 2$, E_k does not have k_+-congruence indices $(k+2, n)$ for any positive integer n. Therefore, for $k \geq 2$, Theorem 2 and Corollary 3.1 are sharp whether they are stated in terms of k_+-congruence order or k_+-congruence indices.

Example 1. Let $S = \{x_0, x_1, x_2, x_3, x_4, p_0, p_1, p_2, p_3, p_4\}$ and define $d_4(X) = 0$ if X is any 5-tuple containing 3 p's and 2 x's or 3 x's and 2 p's; $d_4(X) = 1$ for all other 5-tuples of distinct points of S. Let Y be a subset of S containing 7 points. Since d_4 is symmetric with respect to p's and x's and independent of indices, we may assume $Y = \{p_0, p_1, p_2, p_3, x_0, x_1, x_2\}$ or $\{p_0, p_1, p_2, p_3, p_4, x_0, x_1\}$. In the first case let $f(p_0) = (0, 0, 0, 0)$, $f(p_1) = (1, 0, 0, 0)$, $f(p_2) = (0, 2, 0, 0)$, $f(p_3) = (0, 0, 3, 0)$, $f(x_0) = (0, 0, 0, 4) = f(x_1) = f(x_2)$. The second case is the same, except $f(p_4) = (0, 0, 0, 4)$ and $f(x_0) = (1, -2, 3, -4) = f(x_1)$. In both cases f is a 4-congruence.

Example 2. We construct the example first for k even.

Let k be an even positive integer. Let $M_k = (S, d_k)$, where $S = \{a_i\}$ is any countable set and d_k is defined as follows:

$$d_k(a_{i_0}, a_{i_1}, \cdots, a_{i_k}) = \frac{1}{k!} \quad \text{if } \{i_j\} \text{ is increasing,}$$

$$d_k(\pi(a_{i_0}, \cdots, a_{i_k})) = -d_k(a_{i_0}, \cdots, a_{i_k}) \quad \text{if } \pi \text{ is a transposition, and}$$

$$d_k(a_{i_0}, a_{i_0}, a_{i_1}, \cdots) = 0.$$

For any increasing indices $i_0, i_1, \cdots, i_{k+1}$ define $f(a_{i_j}) = P_j$ where the P_j's are points in E_k with $P_0 = (0, 0, \cdots, 0)$, $P_1 = (1, 0, \cdots, 0)$, $P_2 = (0, 1, 0, \cdots, 0) \cdots$, $P_k = (0, 0, \cdots, 0, 1)$ and $P_{k+1} = (1, -1, 1, -1, \cdots, 1, -1, 1, 1)$. Verification that f is a k_+-congruence mapping the $k+2$-tuple $\{a_{i_j}\}$ onto E_k is straightforward and may be done by induction.

Let $M_{k+1} = (S^*, d_{k+1})$ where $S^* = S \cup \{x\}$ ($x \neq a_i$ for any i) and

$$d_{k+1}(x, a_{i_0}, a_{i_1}, \cdots, a_{i_k}) = d_k(a_{i_0}, \cdots, a_{i_k}),$$

$$d_{k+1}(a_{i_0}, a_{i_1}, \cdots, a_{i_{k+1}}) = 0.$$

Each $k+3$-tuple of M_{k+1} may be imbedded in E_{k+1} by mapping the a_i's into a k-flat in the manner described above and by mapping x to an appropriate point outside of the k-flat.

That the spaces M_k and M_{k+1} are not congruently contained in E_k and E_{k+1} follows from the fact that each contains an independent 'tuple' which would uniquely determine the image of every other point. In fact, all other points would have to map onto the same point in E_k or E_{k+1}.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MAINE, ORONO, MAINE 04473