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ASYMPTOTIC PROPERTIES OF GAUSSIAN RANDOM FIELDS

BY

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ABSTRACT. In this paper we study continuous mean zero Gaussian random fields $X(p)$ with an N-dimensional parameter and having a correlation function $\rho(p, q)$ for which $1 - \rho(p, q)$ is asymptotic to a regularly varying (at zero) function of the distance $\operatorname{dis}(p, q)$ with exponent $0 < \alpha \leq 2$. For such random fields, we obtain the asymptotic tail distribution of the maximum of $X(p)$ and an asymptotic almost sure property for $X(p)$ as $|p| \to \infty$. Both results generalize ones previously given by the authors for $N = 1$.

1. **Introduction.** In this paper, we are concerned with a real continuous Gaussian process with an N-dimensional parameter space. We denote such a random field by $X = \{X(p); p \in R^N\}$, and we assume without loss of generality that each $X(p)$ has mean zero and variance one. We also assume throughout this paper that X satisfies, for some positive constants C_1 and C_2,

$$(1.1) \qquad \begin{aligned} &(\mathrm{A}) \quad E((X(p) - X(q))^2) \leq 2C_2 |p - q|^\alpha H(|p - q|) \quad \text{and} \\ &(\mathrm{B}) \quad E((X(p) - X(q))^2) \geq 2C_1 |p - q|^\alpha H(|p - q|) \end{aligned}$$

for all p, q such that $|p - q| \leq$ some δ, where $0 < \alpha \leq 2$ and $H(s)$ is a slowly varying function (at zero). Here and throughout the paper, we define $|p| = (\Sigma_{i=1}^N p_i^2)^{1/2}$, where $p = (p_1, \cdots, p_N)$. For the preliminary proofs of §2, we further restrict X to be a stationary isotropic Gaussian random field satisfying

$$(1.2) \qquad \rho(p, q) = 1 - |p - q|^\alpha H(|p - q|) + o(|p - q|^\alpha H(|p - q|)) \quad \text{as } |p - q| \to 0,$$

where $\rho(p, q) = E(X(p)X(q))$ is the correlation function of X.

In §2, we give the asymptotic tail distribution of the maximum $Z(D) = \max_{p \in D} X(p)$, where D is an open bounded set with Lebesgue measure $\mu(D) = \mu(\bar{D})$. This is done first in Theorem 2.1 for stationary isotropic Gaussian random fields satisfying condition (1.2), then extended in the Corollary to Theorem 2.1 to obtain asymptotic bounds of the tail distribution of $Z(D)$ for Gaussian random fields satisfying condition (1.1) for all $p, q \in D$ with $|p - q| < \delta$. Lemma 2.3 and

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its corollary are useful discrete versions of Theorem 2.1 and its corollary. The results of this §2 generalize Theorem 2.1 in Qualls-Watanabe [4] and the results of Pickands [2] from continuous time Gaussian processes to Gaussian random fields.

In §3, we study the event

$$A_\phi = \{\exists r_0;\ X(p) \le \phi(|p|) \text{ for all } p \text{ satisfying } |p| \ge r_0\}$$

for arbitrary nondecreasing functions ϕ. In more descriptive language, the complement of A_ϕ occurs if $X(p)$ crosses the surface $\phi(|p|)$ infinitely often as $|p| \to \infty$. For a fairly wide class of Gaussian processes, we can expect that $P(A_\phi) = 0$ or 1. By using the results of §2, we give a criterion in terms of ϕ for deciding whether the probability of the event A_ϕ is 0 or 1. In order to obtain $P(A_\phi) = 0$, the only hypothesis necessary (beside the requirement on ϕ) is that part (A) of condition (1.1) holds for all p, q such that $|p|$, $|q| \ge$ some T_1 and $|p - q| <$ some δ_1. To obtain $P(A_\phi) = 1$, we require part (B) of condition (1.1) to hold for all p, q such that $|p|$, $|q| \ge$ some T_2 and $|p - q| <$ some δ_2, and we need a mixing condition

(1.3) $\quad$ (C) $\quad \rho(p, p + q) = O(|q|^{-\gamma})$ uniformly in p as $|q| \to \infty$,

for some $\gamma > 0$. This is an extension of the results in Qualls-Watanabe [4] to Gaussian random fields.

The proofs of these results of course bear similarities to those for the case $N = 1$. However, we take a somewhat different point of view in the present paper; and the details are considerably different for $N > 1$. A good source of general information about Gaussian processes and random fields is the book by Cramér and Leadbetter, *Stationary and related stochastic processes*, Wiley, New York, 1967.

2. The asymptotic distribution of the maximum.

We first list some definitions and properties of regular varying functions that will be required in the following. One general reference on regular variation is Feller [1].

Definition 2.1. A positive function $H(x)$ defined for $x > 0$ varies slowly at zero if for all $t > 0$

$$\lim_{x \to 0} \frac{H(tx)}{H(x)} = 1. \tag{2.1}$$

Definition 2.2. A positive function $Q(x)$ defined for $x > 0$ varies regularly at zero with exponent $\alpha \ge 0$ if for all $t > 0$

$$\lim_{x \to 0} \frac{Q(tx)}{Q(x)} = t^\alpha. \tag{2.2}$$

A function $Q(x)$ satisfies (2.2) if and only if $Q(x) = x^\alpha H(x)$, where $H(x)$ varies slowly. Let $Q(x)$ vary regularly with exponent $\alpha \ge 0$ and $H(x)$ vary slowly at zero. Then the following properties hold.

(2.3) The limits (2.1) and (2.2) converge uniformly in t on any compact subset of the half line $(0, \infty)$.

(2.4) For any $\epsilon > 0$, we have

$$\lim_{x \to 0} x^{-\epsilon} H(x) = \infty \quad \text{and} \quad \lim_{x \to 0} x^{\epsilon} H(x) = 0.$$

(2.5) If $H(x)$ is a slowly varying function at zero, then for any $\epsilon > 0$ and $t_0 > 1$, there exists a $\delta > 0$ such that $t^{-\epsilon} \leq H(tx)/H(x) \leq t^{\epsilon}$ for all $x > 0$ and all $t \geq t_0$ satisfying $tx < \delta$.

(2.6) Also for any $\epsilon > 0$ and positive $t_0 < 1$, there exists a $\delta > 0$ such that $t^{\epsilon} \leq H(tx)/H(x) \leq t^{-\epsilon}$ for all positive $t \leq t_0$ and all $x > 0$ satisfying $x < \delta$.

A Gaussian random field X will be called stationary or homogeneous if for each choice of points $p_1, p_2, \cdots, p_k$ in R^N the joint distribution of $X(p_1 + q), \cdots, X(p_k + q)$ does not depend on q in R^N. Also X will be called isotropic if the correlation function $\rho(p, q) = E(X(p)X(q))$ depends only on the distance $|p - q|$.

Define the function σ^2 by $\sigma^2(p, q) = E(|X(p) - X(q)|^2)$ and also let $\tilde{\sigma}^2(s) = 2|s|^{\alpha} H(s)$, where $H(\cdot)$ is a slowly varying function at zero and $0 < \alpha \leq 2$. When X is isotropic, $\sigma^2(p, q)$ can be written as $\sigma^2(|p - q|)$. We assume without loss of generality that $\tilde{\sigma}(\cdot)$ is monotone near the origin. Define

$$A_1(r) = \inf\{\sigma(p, q)/\tilde{\sigma}(|p - q|);\ 0 < |p - q| \leq r\},$$
$$A_2(r) = \sup\{\sigma(p, q)/\tilde{\sigma}(|p - q|);\ 0 < |p - q| \leq r\}.$$

For a space parameter of dimension $N > 1$, and X satisfying the condition (1.2), there is no loss in generality in taking $0 < A_1(r) \leq A_2(r) < \infty$ for all $r > 0$. For $N = 1$, there is very little loss in generality in assuming $A_1(r) > 0$ for all r of interest; the excluded periodic case is discussed in [3].

Theorem 2.1. *Let X be a stationary and isotropic, Gaussian random field satisfying condition* (1.2). *Let D be a bounded open set in R^N for which the Lebesgue measure $\mu(D) = \mu(\bar{D})$. Then for $Z(D) = \sup_{p \in D} X(p)$ we have*

$$\lim_{x \to \infty} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(\tilde{\sigma}^{-1}(1/x))^{-N}} = H_\alpha \tag{2.7}$$
$$\equiv \lim_{T \to \infty} T^{-N} \int_0^{\infty} e^{y} P\left(\sup_{0 \leq p_i \leq T} Y(p_1, \cdots, p_N) > y\right) dy$$

and $0 < H_\alpha < \infty$, where $\psi(x) = (2\pi)^{-1}x^{-1} \exp(-x^2/2)$, and $\{Y(p)\}$ is a nonhomogeneous Gaussian process with $Y(0, \cdots, 0) = 0$, $EY(p) = -|p|^{\alpha}/2$ and $\operatorname{Cov}(Y(p), Y(q)) = (|p|^{\alpha} + |q|^{\alpha} - |p - q|^{\alpha})/2.$

In order to prove this theorem, we first establish the following lemmas. Define $c(x) = (\tilde{\sigma}^{-1}(1/x))^{-1}$ for large x. Now by stationarity, we may assume $D \subset \Pi_{i=1}^{N} [0, t]$ for some $t > 0$.

Lemma 2.1. *Let X satisfy the conditions in Theorem 2.1. Then for any $a > 0$, we have*

$$\lim_{x\to\infty} \frac{P(Z(I_x[a\mathbf{n}/c(x)]) > x)}{\psi(x)} = H_\alpha(\mathbf{n}, a)$$

$$\equiv 1 + \int_0^\infty e^y P\Big(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} Y(a\mathbf{k}) > y\Big)\, dy < \infty,$$

where $\{Y(p)\}$ is described in Theorem 2.1, and

$$I_x[a\mathbf{n}/c(x)] = \{p \in R^N;\ p = a\mathbf{k}/c(x),\ \mathbf{k} = (k_1, \cdots, k_N),\ 0 \le k_i \le n,\ k_i \text{ integers}\}.$$

Proof.

$$P(Z(I_x[a\mathbf{n}/c(x)] > x) = P(X(0) > x) + P\Big(X(0) \le x, \max_{0\le k_i\le n;\ \mathbf{k}\ne 0} X(a\mathbf{k}/c(x)) > x\Big).$$

The second term is equal to

$$\int_{-\infty}^{x} P\Big(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} X(a\mathbf{k}/c(x)) > x \mid X(0) = u\Big)\phi(u)\,du,$$

where $\phi(u)$ is the standard normal density. Substituting $u = x - y/x$ and defining $Y_1(p) = x(X(p/c(x)) - x) + y$, the second term becomes

$$\psi(x)\int_0^\infty e^y P\Big(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} X(a\mathbf{k}/c(x)) > y \mid X(0) = x - y/x\Big) \exp(-y^2/(2x^2))\,dy$$

$$= \psi(x)\int_0^\infty e^y P\Big(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} Y_1(a\mathbf{k}) > y \mid X(0) = x - y/x\Big) \exp(-y^2/(2x^2))\,dy.$$

Now as in Lemma 2.2 of [4], we can obtain that

$$E(Y_1(p) \mid X(0) = x - y/x) = x\left\{\rho\left(\frac{1}{c(x)}|p|\right)\left(x - \frac{y}{x}\right) - x\right\} + y$$

$$= -x^2\left\{1 - \rho\left(\frac{1}{c(x)}|p|\right)\right\} + y\left\{1 - \rho\left(\frac{1}{c(x)}|p|\right)\right\}$$

$$= -x^2\tilde{\sigma}^2\left(\frac{1}{c(x)}\right)\frac{|p|^\alpha}{2} + o(1) = -\frac{|p|^\alpha}{2} + o(1) \quad \text{as } x \to \infty,$$

and

$$\operatorname{Cov}(Y_1(p), Y_1(q) \mid X(0) = x - y/x) = x^2\left\{\rho\left(\frac{1}{c(x)}\,|p-q|\right) - \rho\left(\frac{1}{c(x)}\,|p|\right)\rho\left(\frac{1}{c(x)}\,|q|\right)\right\}$$

$$= \frac{x^2}{2}\left\{-\tilde{\sigma}^2\left(\frac{1}{c(x)}\right)|p-q|^\alpha + \tilde{\sigma}^2\left(\frac{1}{c(x)}\right)|p|^\alpha + \tilde{\sigma}^2\left(\frac{1}{c(x)}\right)|q|^\alpha - \tilde{\sigma}^4\left(\frac{1}{c(x)}\right)\frac{|p|^\alpha|q|^\alpha}{2}\right\} + o(1)$$

$$= \tfrac{1}{2}\{-|p-q|^\alpha + |p|^\alpha + |q|^\alpha\} + o(1) \quad \text{as } x \to \infty.$$

Consequently,

$$P\left(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} Y_1(a\mathbf{k}) > y \mid X(0) = x - y/x\right) \to P\left(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} Y(a\mathbf{k}) > y\right) \text{ as } x \to \infty.$$

In order to see that the Lebesgue dominated convergence theorem completes the proof of Lemma 2.1, apply Boole's inequality to the left-hand side of the previous line to obtain a finite number of terms each dominated by an integrable function.

Corollary to Lemma 2.1. *Let X satisfy the same conditions as in Theorem 2.1, except that* $\lim_{s\to 0} \sigma^2(s)/\tilde{\sigma}^2(s) = C > 0$. *Then for any $a > 0$,*

$$\lim_{x\to\infty} \frac{P(Z(I_x[a\mathbf{n}/c(x)]) > x)}{\psi(x)} = H_\alpha(\mathbf{n}, C^{1/\alpha}a).$$

Proof. We have for the corresponding Y_1 in Lemma 2.1,

$$E(Y_1(p) \mid X(0) = x - y/x) \to C\,|p|^\alpha/2 \quad \text{as } x \to \infty$$

and

$$\operatorname{Cov}(Y_1(p), Y_1(q) \mid X(0) = x - y/x) \to C(|p|^\alpha + |q|^\alpha - |p-q|^\alpha)/2 \quad \text{as } x \to \infty.$$

Now, set $\tilde{Y}(t_1 \cdots t_N) = Y(C^{1/\alpha}t_1, \cdots, C^{1/\alpha}t_N)$, where Y is the process defined in Theorem 2.1. Then

$$P\left(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} Y_1(a\mathbf{k}) > y \mid X(0) = x - y/x\right) \to P\left(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} \tilde{Y}(a\mathbf{k}) > y\right)$$

$$= P\left(\max_{0\le k_i\le n;\ \mathbf{k}\ne 0} Y(C^{1/\alpha}a\mathbf{k}) > y\right) \quad \text{as } x \to \infty.$$

Lemma 2.2. *Let X satisfy the conditions in Theorem 2.1. For any $a > 0$,*

$$\varliminf_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)c(x)^N} \ge a^{-N}\left(1 - 2^N {\sum}^* (1 - \Phi(2^{-1}(|\mathbf{k}|a)^{\alpha/2}))\right),$$

where $D_{(x)} = \{p\colon p = a\mathbf{k}/c(x) \in D,\ k_i \text{ integers}\}$ and Σ^* denotes the N-fold sum over $0 \le k_i < \infty$ $(i = 1, \cdots, N)$ but $\mathbf{k} \ne \mathbf{0}$.

Proof. Denoting the event $B_{\mathbf{k}} = \{X(a\mathbf{k}/c(x)) > x\}$ and using stationarity we have

$$P(Z(D_{(x)}) > x) \geq P\left(\bigcup_{\mathbf{k}\in\tilde{\Lambda}} B_{\mathbf{k}}\right)$$
$$\geq \sum_{\mathbf{k}\in\tilde{\Lambda}}\cdots\sum P(B_{\mathbf{k}}) - \tfrac{1}{2}\sum\cdots\sum_{\mathbf{k}\neq l;\ \mathbf{k},l\in\tilde{\Lambda}} P(B_{\mathbf{k}} \cap B_l)$$
$$\geq \#(\tilde{\Lambda})\left(P(B_0) - 2^{N-1}\sum\cdots\sum_{0\leq k_i<m;\ \mathbf{k}\neq 0} P(B_0 \cap B_{\mathbf{k}})\right),$$

where $\tilde{\Lambda} = \{\mathbf{k} \in R^N : I_{a,x}(\mathbf{k}) \subset D\}$, $\#(\tilde{\Lambda})$ = the cardinal number of $\tilde{\Lambda}$,

$$I_{a,x}(\mathbf{k}) = \{p \in R^N : ak_i/c(x) \leq p_i \leq a(k_i + 1)/c(x),\ i = 1, \cdots, N\},$$

$m = [tc(x)/a] + 1$, and [ ] denotes the greatest integer function.

We recall the result of Pickands [2, Lemma 2.3],

$$P(B_0 \cap B_{\mathbf{k}}) \leq 2\psi(x)\{1 - \Phi(x(1-\rho)^{1/2}(1+\rho)^{-1/2})\},$$

where $\rho = \rho(a|\mathbf{k}|/c(x))$ is the correlation coefficient for $X(\mathbf{0})$ and $X(a\mathbf{k}/c(x))$ and Φ is the standard normal distribution function and $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2 + \cdots + k_N^2}$. Now we note that

$$\#(\Lambda)/\mu(D)(\tilde{\sigma}^{-1}(1/x))^{-N} = \mu\left(\bigcup_{\mathbf{k}\in\tilde{\Lambda}} I_{a,x}(\mathbf{k})\right)\Big/\mu(D)a^N \longrightarrow a^{-N} \quad \text{as } x \longrightarrow \infty,$$

since $\mu(\partial D) = 0$. Therefore we have

$$\varliminf_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \geq a^{-N}\left\{1 - 2^N \varlimsup_{x\to\infty} \sum\cdots\sum_{0\leq k_i\leq m;\ \mathbf{k}\neq 0} \{1 - \Phi(x(1-\rho)^{1/2}(1+\rho)^{-1/2})\}\right\}.$$

To study the lim sup of the above sum, we partition the sum into three parts according to (i) $|\mathbf{k}|a \leq t_0$, (ii) $|\mathbf{k}|a > t_0$, $|\mathbf{k}|a/c(x) < \delta$, and (iii) $|\mathbf{k}|a > t_0$, $|\mathbf{k}|a/c(x) \geq \delta$, where the values of $\delta > 0$ and $t_0 > 1$ will be selected later.

Since the number of terms in $\Sigma^{(i)}$ is finite, we have

$$\lim_{x\to\infty} \sum^{(i)} (1 - \Phi) = \sum^{(i)} \lim_{x\to\infty} (1 - \Phi).$$

We may ignore the third sum $\Sigma^{(iii)}$. For $|\mathbf{k}|a/c(x) \geq \delta$, there exists a constant κ such that $1 - \rho \geq \kappa > 0$, and

$$\sum^{(iii)}\left\{1 - \Phi\left(x\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\right)\right\} \leq \sum^{(iii)} (1 - \Phi(x(\kappa/2)^{1/2})) \leq m^N\psi(x(\kappa/2)^{1/2})$$
$$\leq (tc(x)/a)^N \exp(-\kappa x^2/2) \longrightarrow 0 \quad \text{as } x \longrightarrow \infty.$$

Now for Δ_1 sufficiently small, we have $A_1(\Delta_1) > 0$. Then for all the terms $\Sigma^{(ii)}$ when $|\mathbf{k}|a/c(x) < \Delta_1$, we have

$$x\left(\frac{1-\rho}{1+\rho}\right)^{1/2} \geq x\sigma(|\mathbf{k}|a/c(x)) = \frac{1}{2}\,\frac{\sigma(|\mathbf{k}|a/c(x))}{\tilde{\sigma}(1/c(x))}$$

$$\geq \frac{A_1(\Delta_1)}{2}\,\frac{\tilde{\sigma}(|\mathbf{k}|a/c(x))}{\tilde{\sigma}(1/c(x))} = \frac{A_1(\Delta_1)}{2}\,(|\mathbf{k}|a)^{\alpha/2}\left(\frac{H(|\mathbf{k}|a/c(x))}{H(1/c(x))}\right)^{1/2}.$$

By property (2.5), there exists a $\delta_\alpha > 0$ such that $H(|\mathbf{k}|a/c(x))/H(1/c(x)) \geq (|\mathbf{k}|a)^{-\alpha/4}$, provided $|\mathbf{k}|a/c(x) < \delta_\alpha$, and $|\mathbf{k}|a > t_0 > 1$. Now for the definition of $\Sigma^{(ii)}$, take $\delta = \min(\delta_\alpha, \Delta_1)$. Then

$$\inf_{T\leq x\leq\infty} x\left(\frac{1-\rho}{1+\rho}\right)^{1/2} \geq \frac{A_1(\delta)}{2}\,(|\mathbf{k}|a)^{\alpha/4},$$

provided $|\mathbf{k}|a/c(x) < \delta$, $|\mathbf{k}|a > t_0$ and T is large.

Finally, defining $a_{\mathbf{k}}(x) = \{1 - \Phi(x(1-\rho)^{1/2}(1+\rho)^{-1/2})\}$ for $|\mathbf{k}|a/c(x) < \delta$ and $a_{\mathbf{k}}(x) = \{1 - \Phi(2^{-1}A_1(\delta)(|\mathbf{k}|a)^{\alpha/4})\}$ for $|\mathbf{k}|a/c(x) \geq \delta$, we have

$$\sum{}^{*} \sup_{T\leq x<\infty} a_{\mathbf{k}}(x) \leq \sum{}^{*}\{1 - \Phi(2^{-1}A_1(\delta)(|\mathbf{k}|a)^{\alpha/4})\} < \infty.$$

Therefore, it follows that

$$\overline{\lim_{x\to\infty}} \sum\cdots\sum_{0\leq k_i<m;\ \mathbf{k}\neq 0} a_{\mathbf{k}}(x) \leq \sum{}^{*} \overline{\lim_{x\to\infty}}\, a_{\mathbf{k}}(x)$$

$$= \sum{}^{*}\{1 - \Phi(2^{-1}(|\mathbf{k}|a)^{\alpha/2})\} < \infty,$$

since

$$x(1-\rho)^{1/2}(1+\rho)^{-1/2} \sim \frac{x}{2}\cdot\sigma(|\mathbf{k}|a/c(x)) \sim \frac{1}{2}\,\frac{\tilde{\sigma}(|\mathbf{k}|a/c(x))}{\tilde{\sigma}(1/c(x))} \to \frac{1}{2}(|\mathbf{k}|a)^{\alpha/2} \quad \text{as } x\to\infty.$$

Lemma 2.2 now follows.

Corollary to Lemma 2.2. *For X satisfying the same conditions as in Theorem 2.1, except that $\lim_{s\to 0}\sigma^2(s)/\tilde{\sigma}^2(s) = C > 0$, we have*

$$\underline{\lim_{x\to\infty}} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(\tilde{\sigma}^{-1}(1/x))^{-N}} \geq a^{-N}\left(1 - 2^N \sum{}^{*}(1 - \Phi(2^{-1}(|\mathbf{k}|C^{1/\alpha}a)^{\alpha/2}))\right).$$

Proof. We have only to note that

$$x(1-\rho)^{1/2}(1+\rho)^{-1/2} \to \tfrac{1}{2}(|\mathbf{k}|a)^{\alpha/2}C^{1/2} \quad \text{as } x\to\infty.$$

Lemma 2.3. *Let X satisfy the conditions in Theorem 2.1. For $a > 0$,*

$$\lim_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} = \frac{H_\alpha(a)}{a^N}, \tag{2.8}$$

where $0 < H_\alpha(a) \equiv \lim_{n\to\infty}(H_\alpha(\mathbf{n}, a)/n^N) < \infty$.

Proof. Let $(\mathfrak{X}^+)^N = \{\mathbf{k} \in R^N: k_i$ are nonnegative integers, $i = 1, \cdots, N\}$. For $\mathbf{k} \in (\mathfrak{X}^+)^N$, let

$$B_{\mathbf{k}} = \{X(a\mathbf{k}/c(x)) > x\},$$

and also for arbitrary $n > 0$ and $l \in (\mathfrak{X}^+)^N$, let

$$A_l = \bigcup_{j_1=(l_1-1)n}^{l_1 n-1} \bigcup_{j_2=(l_2-1)n}^{l_2 n-1} \cdots \bigcup_{j_N=(l_N-1)n}^{l_N n-1} B_{\mathbf{j}}.$$

By stationarity, $P(A_l) = P(A_{1,\cdots,1})$ for all l with the $l_i \geq 1$.

$$P\left(\bigcup_{l\in\Lambda_1} A_l\right) \leq P(Z(D_{(x)}) > x) \leq P\left(\bigcup_{l\in\Lambda_2} A_l\right), \tag{2.9}$$

where

$$\Lambda_1 = \{l \in R^N: I_l^{(n)} \subset D\}, \qquad \Lambda_2 = \{l \in R^N: I_l^{(n)} \cap D \neq \emptyset\}$$

and

$$I_l^{(n)} = \{p \in R^N: a(l_i - 1)n/c(x) \leq p_i \leq a l_i n/c(x),\ i = 1, \cdots, N\}.$$

Consequently,

$$P(Z(D_{(x)}) > x) \leq \#(\Lambda_2)P(A_{1,\cdots,1}).$$

Therefore, using Lemma 2.1, we obtain

$$\lim_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \leq \overline{\lim_{x\to\infty}} \frac{\#(\Lambda_2)P(A_{1,\cdots,1})}{\mu(D)\psi(x)(c(x))^N} \frac{H_\alpha(\mathbf{n} - \mathbf{1}; a)}{a^N n^N} \tag{2.10}$$

On the other hand, (2.9) and stationarity imply

$$\begin{aligned} P(Z(D_{(x)}) > x) &\geq \sum_{l\in\Lambda_1}\cdots\sum P(A_l) - \frac{1}{2}\sum_{\mathbf{k}\neq l;\ \mathbf{k},l\in\Lambda_1}\cdots\sum P(A_{\mathbf{k}} \cap A_l) \\ &\geq \#(\Lambda_1)\left\{P(A_{1,\cdots,1}) - 2^{N-1}\sum_{0\leq j_\nu<n}\cdots\sum_{0\leq s_i<m;\ \text{some } s_i\geq n} P(B_{\mathbf{j}} \cap B_{\mathbf{s}})\right\}, \end{aligned} \tag{2.11}$$

where $m = \lfloor tc(x)/a \rfloor + 1$. We note here that

$$\#(\Lambda_1)/\mu(D)(c(x))^N \longrightarrow n^{-N}a^{-N} \quad \text{as } x \longrightarrow \infty.$$

Again using Lemma 2.3 in Pickands [2], we obtain

$$\begin{aligned} \underline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} &\geq n^{-N}a^{-N}\left\{H_\alpha(\mathbf{n} - \mathbf{1}, a) - 2^{N-1}\overline{\lim_{x\to\infty}}\sum\nolimits^{**}\frac{P(B_{\mathbf{j}} \cap B_{\mathbf{s}})}{\psi(x)}\right\} \\ &\geq n^{-N}a^{-N}\left\{H_\alpha(\mathbf{n} - \mathbf{1}, a) - 2^N\overline{\lim_{x\to\infty}}\sum\nolimits^{**}(1 - \Phi(x(1-\rho)^{1/2}(1+\rho)^{-1/2}))\right\}, \end{aligned}$$

where $\rho = \rho(a|\mathbf{j} - \mathbf{s}|/c(x))$ and

$$\Sigma^{**} = \sum_{j_1=0}^{n-1} \cdots \sum_{j_N=0}^{n-1} \left(\sum_{s_1=n}^{\infty} \sum_{s_2=0}^{\infty} \cdots \sum_{s_N=0}^{\infty} + \sum_{s_1=0}^{n-1} \sum_{s_2=n}^{\infty} \cdots \sum_{s_N=0}^{\infty} + \cdots + \sum_{s_1=0}^{n-1} \cdots \sum_{s_{N-1}=0}^{n-1} \sum_{s_N=n}^{\infty} \right).$$

In the study of $\overline{\lim}_{x\to\infty} \Sigma^{**}(1 - \Phi)$, we use the same device as in Lemma 2.2. So we have

$$\underline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \geq n^{-N}a^{-N}\left\{H_\alpha(\mathbf{n} - 1, a) - 2^N \sum^{**} (1 - \Phi(2^{-1}(|\mathbf{j} - \mathbf{s}|a)^{\alpha/2}))\right\}.$$

Since the left-hand side does not depend on n, we have

$$\underline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \geq a^{-N} \overline{\lim}_{n\to\infty} \frac{H_\alpha(\mathbf{n} - 1, a)}{n^N} - 2^N \underline{\lim}_{n\to\infty} \sum^{**} \frac{d_{\mathbf{j}-\mathbf{s}}}{n^N}. \tag{2.12}$$

Now Lemma 2.2 implies $\Sigma^{**}d_{\mathbf{j}-\mathbf{s}} < \infty$ and then Kronecker's lemma implies

$$\underline{\lim}_{n\to\infty} \sum^{**} d_{\mathbf{j}-\mathbf{s}}/n^N = 0.$$

Combining (2.10) and (2.12) yields

$$\frac{a^{-N} \overline{\lim}_{n\to\infty} H_\alpha(\mathbf{n} - 1, a)}{n^N} \leq \underline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N}$$
$$\leq \overline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \leq a^{-N} \underline{\lim}_{n\to\infty} \frac{H_\alpha(\mathbf{n} - 1, a)}{n^N}.$$

This establishes that $\lim_{n\to\infty} (H_\alpha(\mathbf{n}, a)/n^N) = H_\alpha(a)$ exists and that

$$\lim_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} = \frac{H_\alpha(a)}{a^N}.$$

Now (2.10) implies $H_\alpha(a) < \infty$. Since the right-hand side of (2.8) in Lemma 2.3 is positive for all sufficiently large a, we know $H_\alpha(a) > 0$ for all $a > a_0$, say. For arbitrary $a > 0$, we select an integer m such that $ma > a_0$. Then $H_\alpha(\mathbf{n}, am) \leq H_\alpha(\mathbf{n}m, a)$ implies $0 < H_\alpha(am) \leq m^N H_\alpha(a)$. So $H_\alpha(a) > 0$ for all $a > 0$.

Corollary to Lemma 2.3. *Let X be a continuous Gaussian random field with mean zero and variance one satisfying condition* (1.1) *for all $p, q \in D$ with $|p - q| < \delta$. Then*

$$\frac{H_\alpha(C_1^{*1/\alpha}a)}{a^N} \leq \underline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \leq \overline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \leq \frac{H_\alpha(C_2^{*1/\alpha}a)}{a^N} \tag{2.13}$$

for any C_1^*, C_2^* *such that* $0 < C_1^* < C_1 < C_2 < C_2^*$.

Proof. Define two stationary isotropic Gaussian processes $X^{(1)}, X^{(2)}$ having $\sigma_i^2(|p-q|) \equiv E(|X^{(i)}(p) - X^{(i)}(q)|^2) = 2C_i^*|p-q|^\alpha H(|p-q|) + o(|p-q|^\alpha H(|p-q|))$ as $|p-q| \to 0$. We use the label (i) in this proof to denote expressions involving the process $X^{(i)}$.

By hypothesis, we then have

$$\sigma_1^2(|p-q|) \le 2C_1\tilde{\sigma}^2(|p-q|) \le \sigma^2(p, q) \le 2C_2\tilde{\sigma}^2(|p-q|) \le \sigma_2^2(|p-q|)$$

for all $p, q \in I_l^{(n)}$ (when x is sufficiently large) and uniformly for $l \in \Lambda_2$. So, using Slepian's result [5], we have

$$P(A_l) \le P(A_l^{(2)}) \quad \text{for large } x,$$

where $A_{1,\ldots,1}^{(2)}$ is the event $A_{1,\ldots,1}$ defined in terms of the process $X^{(2)}$. Now

$$\text{(2.14)} \qquad \overline{\lim_{x\to\infty}} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N} \le \overline{\lim_{x\to\infty}} \frac{\#(\Lambda_2)P(A_l^{(2)})}{\mu(D)\psi(x)(c(x))^N} = \frac{H_\alpha(\mathbf{n}, C_2^{*1/\alpha}a)}{a^N n^N},$$

and also

$$P\left(\bigcup_{l\in\Lambda_1} A_l\right) \ge \#(\Lambda_1)P(A_l^{(1)}) - \frac{1}{2}\sum\cdots\sum_{\mathbf{k}\ne l} P(A_{\mathbf{k}} \cap A_l).$$

As in the proof of Lemma 2.3, the sum $\Sigma P(A_{\mathbf{k}} \cap A_l)$ can be bounded above by a sum $\Sigma P(B_{\mathbf{i}} \cap B_{\mathbf{j}})$. But

$$\begin{aligned} P(B_{\mathbf{i}} \cap B_{\mathbf{j}}) &= P(B_{\mathbf{i}}^{(1)}) + P(B_{\mathbf{j}}^{(1)}) - P(B_{\mathbf{i}} \cup B_{\mathbf{j}}) \\ &\le P(B_{\mathbf{i}}^{(1)}) + P(B_{\mathbf{j}}^{(1)}) - P(B_{\mathbf{i}}^{(1)} \cup B_{\mathbf{j}}^{(1)}) = P(B_{\mathbf{i}}^{(1)} \cap B_{\mathbf{j}}^{(1)}). \end{aligned}$$

Consequently, as in the proof of Lemma 2.3, the sum $\Sigma P(A_{\mathbf{k}} \cap A_l)$ can eventually be ignored (as $n \to \infty$). So

$$\text{(2.15)} \qquad \lim_{n\to\infty} \frac{H_\alpha(\mathbf{n}, C_1^{*1/\alpha}a)}{a^N n^N} \le \underline{\lim}_{x\to\infty} \frac{P(Z(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N}.$$

From (2.14) and (2.15) follows (2.13).

Lemma 2.4. *Under the same assumptions as in Theorem* 2.1, *we have, for any* $a > 0$, $2^{-\alpha/4} < b < 1$, *and* $\gamma > 0$,

$$\overline{\lim_{x\to\infty}} \frac{P(X(0) < x - \gamma/x,\ Z(I[a/c(x-\gamma/x)] > x)}{\psi(x)} \le M(a, \gamma)$$

where

$I[a/c(x-\gamma/x)] = \{p \in R^N : 0 \le p_i \le a/c(x-\gamma/x)\}$,

$$M(a, \gamma) = (\sqrt{N}a/2)^{\alpha/2} 2^N \sum_{k=0}^{\infty} 2^{k(N-\alpha/2)} R(\gamma(1-b)(2/a\sqrt{N})^{\alpha/2}(2^{\alpha/2}b)^k - 2^{-1}(a\sqrt{N}/2)^{\alpha/2} 2^{-\alpha k/2}),$$

and

$$R(x) = \int_x^\infty (1 - \Phi(s))\,ds.$$

Proof. We note that

$$\{X(\mathbf{0}) \le x - \gamma/x,\ Z(I[\mathbf{a}/c(x - \gamma/x)]) > x\} \subseteq \bigcup_{k=0}^{\infty} D_k,$$

where

$$D_k = \left\{ \max_{0 \le j_i \le 2^k - 1} X(a2^{-k}\mathbf{j}/c(x - \gamma/x)) \le x - \gamma b^k/x,\ \max_{0 \le j_i \le 2^{k+1}-1} X(a2^{-k-1}\mathbf{j}/c(x - \gamma/x)) > x - \gamma b^{k+1}/x \right\}.$$

Also, $D_k \subseteq \bigcup_{0 \le j_i \le 2^k - 1} E_{\mathbf{j},k}$ where

$$E_{\mathbf{j},k} = \left\{ X(a2^{-k}\mathbf{j}/c(x - \gamma/x)) \le x - \gamma b^k/x,\ \max_{2j_i \le l_i \le 2j_i + 1} X(a2^{-k}\mathbf{l}/c(x - \gamma/x)) > x - \gamma b^{k+1}/x \right\}.$$

By using [4, Lemma 2.6], we obtain

$$P(E_{\mathbf{j},k}) \le \sum_{\epsilon \ne 0} \rho^{-1}(1 - \rho^2)^{1/2}\psi(x)R(y),$$

where $\rho = \rho(a2^{-k-1}\sqrt{\epsilon_1^2 + \cdots + \epsilon_N^2}/c(x - \gamma/x))$, $\Sigma_{\epsilon \ne 0}$ is a sum over the set $\{\epsilon \ne \mathbf{0}\colon \epsilon_j = 0 \text{ or } 1\}$, and $y = y(x) = \gamma(1 - b)b^k\rho x^{-1}(1 - \rho^2)^{-1/2} - x(1 + p)^{-1}(1 - \rho^2)^{1/2}$. Consequently,

$$\overline{\lim_{x \to \infty}} \frac{P\{X(\mathbf{0}) \le x - \gamma/x,\ Z(I[\mathbf{a}/c(x - \gamma/x)]) > x\}}{\psi(x)}$$

$$\le \overline{\lim_{x \to \infty}} \sum_{k=0}^{\infty} \sum_{0 \le j_i \le 2^k - 1} P(E_{j,k})/\psi(x)$$

$$\le \overline{\lim_{x \to \infty}} \sum_{k=0}^{\infty} 2^{Nk} \sum_{\epsilon \ne 0} \rho^{-1}(1 - \rho^2)^{1/2}R(y(x)).$$

By using property (2.6) when k is sufficiently large, we can obtain the following estimates for large x,

$$x\rho^{-1}(1 - \rho^2)^{1/2} \le 2S^{-1}A_2 \cdot (\sqrt{N}a\,2^{-k-1})^{\alpha/4},$$

and

$$y = y(x) \ge 2^{-1}\{\gamma(1 - b)b^k S A_2^{-1}(2^{k+1}/a\sqrt{N})^{\alpha/4} - 4(1 + S)^{-1}A_2(2^{k+1}/a\sqrt{N})^{-\alpha/4}\},$$

where $S = \inf_{0 \le s \le a\sqrt{N}/c(x - \gamma/x)} \rho(s)$ and $A_2 = A_2(a\sqrt{N}/c(x - \gamma/x))$. From the above estimate, we can prove

$$\sum_{k=0}^{\infty} 2^{Nk} \sup_{T \le x < \infty} \sum_{\epsilon \ne 0} \rho^{-1}(1-\rho^2)^{1/2} R(y(x)) < \infty.$$

Therefore

$$\overline{\lim_{x\to\infty}} \frac{P(X(0) \le x - \gamma/x,\ \ Z(I[a/c(x-\gamma/x)]) > x)}{\psi(x)}$$

$$\le \sum_{k=0}^{\infty} \overline{\lim_{x\to\infty}}\, 2^{Nk} \sum_{\epsilon\ne 0} x\rho^{-1}(1-\rho^2)^{1/2} R(y(x))$$

$$= \sum_{k=0}^{\infty} 2^{Nk} \sum_{\epsilon \ne 0} (a|\epsilon|2^{-k-1})^{\alpha/2} R(\gamma(1-b)b^k(a|\epsilon|2^{-k-1})^{-\alpha/2} - 2^{-1}(a|\epsilon|2^{-k-1})^{\alpha/2})$$

$$< M(a,\gamma).$$

Proof of Theorem 2.1. Define

$$H_\alpha^+ = \overline{\lim_{x\to\infty}} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N}$$

and

$$H_\alpha^- = \underline{\lim_{x\to\infty}} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N}$$

Now, since $c(x-\gamma/x)/c(x) \to 1$ and $\psi(x-\gamma/x)/\psi(x) = e^\gamma$ as $x \to \infty$ for $\gamma > 0$, we see from Lemma 2.3 that

$$\lim_{x\to\infty} \frac{P(Z(D_{(x-\gamma/x)}) > x - \gamma/x)}{\mu(D)\psi(x)(c(x))^N} = e^\gamma \frac{H_\alpha(a)}{a^N}.$$

For $\gamma > 0$, we obtain

$$H_\alpha^- - e^\gamma \frac{H_\alpha(a)}{a^N} \le H_\alpha^+ - e^\gamma \frac{H_\alpha(a)}{a^N}$$

$$= \overline{\lim_{x\to\infty}} \frac{P(Z(D) > x) - P(Z(D_{(x-\gamma/x)}) > x - \gamma/x)}{\mu(D)\psi(x)(c(x))^N}$$

$$= \overline{\lim_{x\to\infty}} \frac{P(Z(D) > x,\ Z(D_{(x-\gamma/x)}) \le x - \gamma/x)}{\mu(D)\psi(x)(c(x))^N}$$

$$\le \overline{\lim_{x\to\infty}} \frac{\#(\Lambda_2) P(X(0) \le x - \gamma/x,\ Z(I[a/c(x-\gamma/x)]) > x)}{\mu(D)\psi(x)(c(x))^N}$$

$$\le \frac{1}{a^N} \overline{\lim_{x\to\infty}} \frac{P(X(0) \le x - \gamma/x,\ Z(I[a/c(x-\gamma/x)]) > x)}{\psi(x)}$$

$$\le \frac{M(a,\gamma)}{a^N}.$$

Therefore

$$(2.16)\qquad 0 \le H_\alpha^- - \frac{H_\alpha(a)}{a^N} \le H_\alpha^+ - \frac{H_\alpha(a)}{a^N} \le \frac{M(a,\gamma)}{a^N} + (e^\gamma - 1)\frac{H_\alpha(a)}{a^N}.$$

Since H_α^- and H_α^+ are finite, we can see that the lim sup and lim inf of $H_\alpha(a)/a^N$ as $a \to 0$ must be finite. Now choosing $\gamma = a^\beta$ with $0 < \beta < \alpha/2$ in (2.16), we have $M(a,\gamma)/a^N \to 0$ as $a \to 0$ and $H_\alpha \equiv H_\alpha^+ = H_\alpha^- = \lim_{a\to 0} H_\alpha(a)/a^N$.

In a manner similar to the proof of Theorem 2.1 in [4], we can show

$$H_\alpha = \lim_{T\to\infty} T^{-N}\left(1 + \int_0^\infty e^s P\left(\sup_{0\le t_i \le T} Y(\mathbf{t}) > s\right) ds\right).$$

Corollary to Theorem 2.1. *Let X be a continuous Gaussian random field with mean zero and variance one satisfying condition* (1.1) *for all $p, q \in D$ with $|p-q| < \delta$. Then*

$$(2.17)\qquad H_\alpha C_1^{N/\alpha} \le \underline{\lim}_{x\to\infty} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N} \le \overline{\lim}_{x\to\infty} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N} \le H_\alpha C_2^{N/\alpha}.$$

Proof. If X satisfies the conditions of the Corollary to Lemma 2.1, then note that

$$\lim_{x\to\infty} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N} = \lim_{a\to 0} \frac{H_\alpha(C^{1/\alpha}a)}{a^N} = C^{N/\alpha} H_\alpha.$$

Now let $D \subset \Pi_{i=1}^N [0, t]$, let l be an arbitrary number $< \delta/\sqrt{N}$, and let the rectangle $I_{\mathbf{k}}^l = \{p \in R^N : l \cdot k_j \le p_j \le l(k_j + 1)\}$. Let $K = \{\mathbf{k} : I_{\mathbf{k}}^l \cap D \ne \varnothing\}$. Then by the use of Slepian's theorem, we have

$$P(Z(D) > x) \le \sum_{\mathbf{k}\in K} P(Z(I_{\mathbf{k}}^l) > x) \le \#(K) P(Z^{(2)}(I_0^l) > x),$$

where $Z^{(2)}(A) = \max_{p\in A} X^{(2)}(p)$ and $X^{(2)}$ is a process defined in the proof of the Corollary to Lemma 2.3. Therefore

$$\overline{\lim}_{x\to\infty} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N} \le \frac{\#(K) l^N}{\mu(D)} \overline{\lim}_{x\to\infty} \frac{P(Z^{(2)}(I_0^l) > x)}{\mu(I_0^l)\psi(x)(c(x))^N}$$

$$= \frac{\#(K) l^N}{\mu(D)} C_2^{*N/\alpha} H_\alpha \to C_2^{*N/\alpha} H_\alpha \quad \text{as } l \to 0.$$

Since $C_2^* > C_2$ is arbitrary, we have the right-hand side of (2.17).

Now by the Corollary to Lemma 2.3,

$$\underline{\lim}_{x\to\infty} \frac{P(Z(D) > x)}{\mu(D)\psi(x)(c(x))^N} \ge \underline{\lim}_{x\to\infty} \frac{P(Z^{(1)}(D_{(x)}) > x)}{\mu(D)\psi(x)(c(x))^N}$$

$$= \frac{H_\alpha(C_1^{*1/\alpha} a)}{a^N} \to C_1^{*N/\alpha} H_\alpha \quad \text{as } a \to 0.$$

Since $C_1^* < C_1$ is arbitrary, we have the left-hand side of (2.17).

3. An asymptotic 0-1 behavior. In this section we will consider the ω-set

$$A_\phi = \{\exists r_0: X(p) \le \phi(|p|) \text{ for all } |p| \ge r_0\}$$

for an arbitrary nondecreasing function ϕ. Since X is a continuous Gaussian random field, A_ϕ is an event. In earlier papers [4] and [5], the authors give a criterion in terms of ϕ for deciding whether the $P(A_\phi)$ is 0 or 1 for processes with an $N = 1$ dimensional parameter space. Here this criterion is generalized to random fields defined on R^N with $N > 1$. The proofs are based on the corollaries to Lemma 2.3 and Theorem 2.1 in §2.

Theorem 3.1. *Suppose X is a continuous Gaussian random field with mean zero and variance one satisfying part* (A) *of condition* (1.1); *that is, the correlation function $\rho(p, q)$ satisfies*

$$\text{(A)} \qquad 1 - C_2|p - q|^\alpha H(|p - q|) \le \rho(p, q),$$

for all $|p - q| <$ some δ_1, $|p|$ and $|q| \ge$ some T_1, and some $C_2 > 0$. If for any positive nondecreasing function $\phi(r)$,

$$\int^\infty (\tilde{\sigma}^{-1}(1/\phi(r)))^{-N}(\phi(r))^{-1}\exp(-\tfrac{1}{2}\phi^2(r))r^{N-1}\,dr < \infty,$$

then

$$P(A_\phi) = 1.$$

Proof. Let $r_n = n\Delta$, where $\Delta > 0$ and $n = 1, 2, \cdots$. From (2.17) we have, for n sufficiently large,

$$P\left(\max_{r_n \le |p| \le r_{n+1}} X(p) \ge \phi(r_n)\right) \le \text{const}\ \mu(K_n)(\phi(r_n))^{-1}(\tilde{\sigma}^{-1}(1/\phi(r_n)))^{-N} \cdot \exp(-\tfrac{1}{2}\phi^2(r_n)),$$

where $K_n = \{p: r_n \le |p| \le r_{n+1}\}$. The convergence of the right-hand side of (2.17) can be shown to be uniform with respect to all the sets K_n; we would need

$$\#(\Lambda_2)/\mu(K_m)(c(x))^N \longrightarrow n^{-N}a^{-N}$$

uniformly for K_m as $x \longrightarrow \infty$. But

$$0 \le \frac{\#(\Lambda_2)}{\mu(K_m)}\left(\frac{an}{c(x)}\right)^N - 1 \le \frac{S_N(r_{m+1}^{N-1} + r_m^{N-1})\sqrt{N}\,(an/c(x))}{V_N(r_{m+1}^N - r_m^N)} \le \frac{N^{3/2}}{\Delta}\,\frac{an}{c(x)} \longrightarrow 0$$

uniformly in m, where the coefficients S_N, V_N satisfy $NV_N = S_N$. Therefore, we choose Δ small enough and then n_0 large enough that

$$\sum_{n=n_0}^{\infty} P\left(\max_{r_n \le |p| \le r_n+1} X(p) \ge \phi(r_n)\right)$$

$$\le \text{const} \sum_{n=n_0}^{\infty} \mu(K_n)(\phi(r_n))^{-1}(\tilde{\sigma}^{-1}(1/\phi(r_n)))^{-N}\exp(-\tfrac{1}{2}\phi^2(r_n))$$

$$\le \text{const} \int_{r \ge n_0\Delta} (\phi(r))^{-1}(\tilde{\sigma}^{-1}(1/\phi(r)))^{-N}\exp(-\tfrac{1}{2}\phi^2(r))r^{N-1}\,dr$$

$$< \infty.$$

So when we apply the Borel-Cantelli lemma, we obtain the conclusion of Theorem 3.1.

Theorem 3.2. *Suppose X is a continuous Gaussian random field with mean zero and variance one satisfying part* (B) *of condition* (1.1); *that is, there are positive constants C_1, δ_2, T_2 such that*

(B) $$\rho(p, q) \le 1 - C_1|p-q|^{\alpha}H(|p-q|)$$

for all p, q satisfying $0 < |p-q| < \delta_2$ and $|p|$, $|q| \ge T_2$.

Suppose X also satisfies condition (1.3); *that is, there is a $\gamma > 0$ such that*

(C) $$\rho(p, p+q) = O(|q|^{-\gamma}) \text{ uniformly in } p \text{ as } |q| \to \infty.$$

If, for any positive nondecreasing function $\phi(r)$,

$$I(\phi) = \int^{\infty} (\tilde{\sigma}^{-1}(1/\phi(r)))^{-N}(\phi(r))^{-1}\exp(-\tfrac{1}{2}\phi^2(r))r^{N-1}\,dr = \infty,$$

then $P(A_\phi) = 0$.

Before giving the proof we need the following lemma.

Lemma 3.1. *If Theorem 3.2 is true under the additional condition that, for large r, $2N \log r \le \phi^2(r) \le 3N \log r$, then it is true without this restriction.*

Proof. The proof of Lemma 3.1 is accomplished in the same way as for Lemma 3.1 in Qualls and Watanabe [4]. So we omit it.

Proof of Theorem 3.2. Let $r_n = 2n\Delta$, for large Δ, and let

$$D_n = \{p = (r, \phi_1, \cdots, \phi_{N-1}): r_n \le r \le r_n + \Delta,\ 0 \le \phi_i \le \pi,\ 0 \le \phi_{N-1} \le \epsilon\}$$

for $\epsilon > 0$. Let G_n be the set of lattice points in D_n obtained by dividing N-space into cubes with sides of length $1/l_n$. Now we take $l_n \sim (\tilde{\sigma}^{-1}(1/\phi(r_n + \Delta)))^{-1}$.

Now by applying the Corollary to Lemma 2.3, we can estimate the probabilities of the events $E_n = \{\max_{p \in G_n} X(p) \le \phi(r_n + \Delta)\}$ for large n. The convergence of the left-hand side of (2.13) can be shown to be uniform with respect to all the sets K_n; for example following inequality (2.11), we would need $\#(\Lambda_1)/\mu(K_n)(c(x))^N \to n^{-N}a^{-N}$ uniformly for K_m as $n \to \infty$ which can be done as in the proof of Theorem 3.1. Then

$$P(E_n^c) \geq \text{const}\ \mu(D_n)(\tilde{\sigma}^{-1}(1/\phi(r_n + \Delta)))^{-N}(\phi(r_n + \Delta))^{-1} \exp(-\tfrac{1}{2}\phi^2(r_n + \Delta))$$

implies

$$\sum_{n=n_0}^{\infty} P(E_n^c) \geq \text{const}\ I(\phi) = \infty.$$

Next we show $P(E_n^c \text{ i.o.}) = 1$, which would imply the conclusion of Theorem 3.2. Since $\Pi_{k=m}^{\infty} P(E_k) = 0$, we need to show

$$1 - P(E_n^c \text{ i.o.}) = \lim_{m\to\infty}\left(P\left(\bigcap_{k=m}^{\infty} E_k\right) - \prod_{k=m}^{\infty} P(E_k)\right) = 0.$$

Now, by Lemma 1.5 in Qualls and Watanabe [3], we estimate

$$A_{m,n} = \left|P\left(\bigcap_{k=m}^{n} E_k\right) - \prod_{k=m}^{n} P(E_k)\right|$$

by

$$A_{m,n} \leq \sum_{m\leq i<j\leq n} \sum_{\mu=0}^{K_i} \sum_{\nu=0}^{K_j} \rho \int_0^1 g(\phi(r_i + \Delta), \phi(r_j + \Delta), \lambda\rho)\, d\lambda,$$

where $\rho = \rho(p_{i,\mu}, p_{j,\nu})$, $K_n = \#(G_n) = \#\{p_{n,\nu};\ \nu = 1, 2, \cdots, K_n\}$, and $g(x, y, \rho) = (2\pi)^{-1}(1 - \rho^2)^{-1/2}\exp(-2^{-1}(1 - \rho^2)^{-1}(x^2 - 2\rho xy + y^2))$.

Because $|p_{i,\mu} - p_{j,\nu}| \geq \Delta$ and because of condition (C), Δ can be chosen large enough that, for some positive constant M,

$$|\rho(p_{i,\mu}, p_{j,\nu})| \leq M((j - i)\Delta)^{-\gamma}, \quad \text{for all } j > i \geq m,$$

and $|\rho| < \gamma/6N$. For m sufficiently large, Lemma 3.1 implies

$$u^2((2i + 1)\Delta) \equiv 2N \log((2i + 1)\Delta) \leq \phi^2((2i + 1)\Delta) \leq 3N \log((2i + 1)\Delta) \equiv v^2((2i + 1)\Delta)$$

for all $i \geq m$. Now

$$\begin{aligned} &g(\phi((2i + 1)\Delta), \phi((2j + 1)\Delta), \lambda\rho) \\ &\quad\leq (2\pi)^{-1}(1 - \rho^2)^{-1/2} \exp(-\tfrac{1}{2}(\phi^2((2i + 1)\Delta) - 2|\rho|\phi^2((2j + 1)\Delta) + \phi^2((2j + 1)\Delta))) \\ &\quad\leq (2\pi)^{-1}(1 - (\gamma/6N)^2)^{-1/2} \exp(-\tfrac{1}{2}(u^2((2i + 1)\Delta) + (1 - 2|\rho|)u^2((2j + 1)\Delta))) \\ &\quad\leq (2\pi)^{-1}(1 - (\gamma/6N)^2)^{-1/2} [1/(2i + 1)\Delta]^N [1/(2j + 1)\Delta]^{N(1-3|\rho|)}. \end{aligned}$$

Since $l_i \sim (\tilde{\sigma}^{-1}(1/\phi((2i + 1)\Delta)))^{-1}$, then

$$K_i \sim l_i^N \mu(D) \sim \text{const}(2i + 1)^{N-1}(\tilde{\sigma}^{-1}(1/\phi((2i + 1)\Delta)))^{-N} \quad \text{as } i \to \infty.$$

From Lemma 3.1 and property (2.4), we see that $K_i \leq \text{const}\ (2i + 1)^{N-1}(\log(2i + 1))^{N/(\alpha-\epsilon)}$ for large i and some $\epsilon > 0$ such that $\alpha - \epsilon > 0$. Consequently, we have, for large m,

$$A_{m,\infty} \le \text{const} \times \sum_{m \le i < j < \infty} \frac{(2i+1)^{N-1}(2j+1)^{N-1}(\log(2i+1))^{N/(\alpha-\epsilon)}(\log(2j+1))^{N/(\alpha-\epsilon)}}{(j-1)^{\gamma}}$$
$$\times [1/(2i+1)]^N[1/(2j+1)]^{N(1-3|\rho|)}$$

$$\le \text{const} \times \sum_{i=m}^{\infty}\sum_{k=1}^{\infty} \frac{(\log(2(k+i)+1))^{2N/(\alpha-\epsilon)}}{k^{\gamma}}\left(\frac{1}{2i+1}\right)\left(\frac{1}{2(k+i)+1}\right)^{1-3N|\rho|}$$

$$\le \text{const} \times \sum_{i=m}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{1-\delta+\gamma}\left(\frac{1}{2i+1}\right)^{1+\delta-\gamma/2}(\log(2(k+i)+1))^{2N/(\alpha-\epsilon)}.$$

Since $\log^{\beta}(x+y) \le \log^{\beta}2x + \log^{\beta}2y$ for any $\beta > 0$ and all $x, y \ge 1$, we see that this last sum is convergent if we choose δ so that $\gamma/2 < \delta < \gamma$. Therefore $\lim_{m\to\infty} A_{m,\infty} = 0$, which completes the proof.

REFERENCES

1. W. Feller, *An introduction to probability theory and its applications*. Vol. II, 2nd ed., Wiley, New York, 1966. MR 42 #5292.

2. J. Pickands III, *Upcrossing probabilities for stationary Gaussian processes*, Trans. Amer. Math. Soc. 145 (1969), 51–73. MR 40 #3606.

3. C. Qualls and H. Watanabe, *An asymptotic* 0-1 *behavior of Gaussian process*, Ann. Math. Statist. 42 (1971), 2029–2035.

4. ———, *Asymptotic properties of Gaussian processes*, Ann. Math. Statist. 43 (1972), 580–596.

5. D. Slepian, *The one-sided barrier problem for Gaussian noise*, Bell System Tech. J. 41 (1962), 463–501. MR 24 #A3017.

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