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ON THE PRINCIPAL SERIES OF Gl_n OVER p-ADIC FIELDS

BY

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ABSTRACT. The entire principal series of $G = Gl_n(F)$, for a p-adic field F, is analyzed after the manner of the analysis of Bruhat and Satake for the spherical principal series. If K is the group of integral matrices in $Gl_n(F)$, then a "principal series" of representations of K is defined. It is shown that precisely one of these occurs, and only once, in a given principal series representation of G. Further, the spherical function algebras attached to these representations of K are all shown to be abelian, and their explicit spectral decomposition is accomplished using the principal series of G. Computation of the Plancherel measure is reduced to MacDonald's computation for the spherical principal series, as is computation of the spherical functions themselves.

Here we seek to illustrate, in the simplest case, the principal series of Gl_n, some facts which should be general features of harmonic analysis on semisimple p-adic groups; that is, they should hold to a large extent both for other series of representations of Gl_n, and for other semisimple groups. The main point I wish to clarify is the relation between harmonic analysis on Gl_n itself and on certain compact subgroups.

We will begin with a review of the construction and basic facts of irreducibility and equivalence of the principal series. Then we will define the principal series for the standard maximal compact subgroup, and show the relation between the two by the study of the associated algebras of spherical functions.

We will need notation. Let F be a p-adic field, R its ring of integers, π a prime element. Let $\overline{F} = R/\pi R$ be the residue class field, of order $q = p^a$.

Write $G = Gl_n(F)$. B is the Borel subgroup of upper triangular matrices, N the group of unipotent upper triangular matrices, A the diagonal matrices, W the Weyl group of permutation matrices, P any parabolic subgroup containing B, N_P the unipotent radical of P, N_P' the opposite group to N_P, and M_P the standard (block diagonal) levi component of P.

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Let $G_0 = K = Gl_n(R)$ be the standard maximal compact subgroup of G. Write B_0, N_0, A_0, etc., for $B \cap K$, $N \cap K$, $A \cap K$, etc.

Let $K_i = 1 + \pi^i M_n(R)$ for $i \geq 1$. Then $\overline{G} = K/K_1 = Gl_n(\overline{F})$. Write $\overline{B}$, $\overline{N}$, $\overline{A}$, etc., for $B_0/B_0 \cap K_1$, $N_0/N_0 \cap K_1$, etc. Write $\widetilde{B}$, $\widetilde{N}$, $\widetilde{A}$, etc., for the inverse image of $\overline{B}$, $\overline{N}$, $\overline{A}$, etc., in K. Note $\widetilde{N}$ is a maximal pro-p subgroup of G.

As is well known, we have the decompositions $G = KB$, $B = A \cdot N$, with N normal in B. N is also the derived subgroup of B. Somewhat more precisely, if D denotes the group of diagonal matrices whose entries are powers of π, then $A = D \cdot A_0$, and $G = KDN$, and any $g \in G$ may be written $g = kdn$, with $d \in D$ and unique, and k determined up to right multiplication by $n \in N_0$.

Let dg, dk, db, da, dn denote left Haar measures on G, K, B, A, and N respectively. We suppose the total measures of K, A_0, B_0, N_0 are all 1. Define a function δ on D by $\delta(d) = q^{2\rho(d)}$, where

$$2\rho(d) = \sum_{i>j} m_i - m_j \quad \text{and} \quad d = \operatorname{diag}(\pi^{m_i}) = \begin{pmatrix} \pi^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_n} \end{pmatrix}.$$

Note δ is multiplicative. Extend it to a multiplicative function on A by letting it be 1 on A_0. Then $d(ana^{-1}) = \delta(a)dn$, and with the above normalizations $db = dadn$ and $dg = dk\,\delta(a)db$. That is, $\int_B f(b)\,db = \int_{A\times N} f(an)\,da\,dn$, and $\int_G f(g)\,dg = \int_{K\times A\times N} f(kan)\,\delta(a)\,dk\,da\,dn$.

From the formula, it follows that, on the space of continuous functions on G such that $f(gan) = f(g)\delta^{-1}(a)$, the integral $\int_K f(k)\,dk$ defines a functional invariant under left translations.

The principal series may be defined as the collection of (unitary) representations of G induced from linear characters on B. To be explicit, let ψ be a unitary character of A. Define ψ on B by $\psi(an) = \psi(a)$. This yields all linear characters of B. Consider the continuous functions f on G such that $f(gan) = f(g)\psi(a)\delta(a)^{-1/2}$. G acts on the left on this space: $L_g(f)(h) = f(g^{-1}h)$. According to the remark above the bilinear form $\langle f_1, f_2\rangle = \int_K f_1(k)\overline{f}_2(k)\,dk$ is invariant under the action of G. (Note the f_i are determined by their restrictions to K.) Completing in the norm attached to this inner product, we get a unitary representation $U(\psi)$—the principal series representation attached to ψ. As ψ varies over $\hat{A}$ (the Pontryagin dual of A) we get the principal series. (One may also consider ψ nonunitary. Then one gets the "analytically continued principal series", a series of nonunitary representations. Of these $U(\psi)$ and $U(\psi^{-1})$ are contragredient to one another.)

The basic facts on irreducibility and equivalence of the principal series are summarized by the following theorem.

Theorem A. *If ψ is nondegenerate, that is, stabilized by no element of the Weyl group acting on $\hat{A}$, then $U(\psi)$ is irreducible. If ψ_1, ψ_2 belong to the same* *orbit of the Weyl group, and* $U(\psi_1)$, $U(\psi_2)$ *are irreducible, then they are equivalent. If* ψ_1, ψ_2 *are in different orbits* $U(\psi_1)$, $U(\psi_2)$ *are disjoint.*

About the proof, see [1].

Since $A = D \times A_0$, $\hat{A} = \hat{D} \times \hat{A}_0$. $\hat{D}$ is a torus, and $\hat{A}_0$ is a discrete torsion group. Thus the cosets of $\hat{D}$, the sets of ψ which restrict to a given fixed ψ_0 on A_0, are the connected components of $\hat{A}$, and $\hat{A}$ is the discrete union of them. Thus the principal series breaks up naturally into subseries $\mathrm{P.\,S.}(\psi_0) = \{U(\psi), \psi|_{A_0} = \psi_0\}$. We will call $\mathrm{P.\,S.}(\psi_0)$ the principal series attached to ψ_0.

Now we wish also to attach to ψ_0 a representation of K, and we will refer to the collection of representations of K so constructed as the principal series for K. (Actually, there are several ways to realize these representations. We want to define them in such a way as to emphasize the similarity and relation to the principal series of G. It is convenient to do this only for certain ψ_0; however, these ψ_0 do represent every W-orbit in $\hat{A}_0$, and so Theorem A shows they are enough.)

Let $S = R^{\times}$ denote the multiplicative group of units of R. Define $S_i \subseteq S$ for $i \geq 1$ by $S_i = 1 + \pi^i R$. For $\phi \in \hat{S}$, define the conductor of ϕ to be the largest S_i contained in $\ker \phi$. Put $|\phi|_{\hat{S}} = q^i$ (and $|1|_{\hat{S}} = 0$). Then $d_{\hat{S}}(\phi_1, \phi_2) = |\phi_1\phi_2^{-1}|_{\hat{S}}$ is a symmetric, invariant ultrametric distance on $\hat{S}$.

We have $A_0 \simeq S^n$ in the obvious way, so a character ψ_0 of A_0 is specified by an n-tuple of characters $(\phi_1, \cdots, \phi_n)$ of S. The action of the Weyl group in these coordinates is simply given by permuting the ϕ_j's. Hence, modulo conjugation by W, we may assume the ϕ_j's are arranged in some specified order. The order in which we are interested is that of the following lemma.

Lemma 1. *Let X be an ultrametric space, with metric $d(\ ,\ )$, and let $Y \subseteq X$ finite subset. Then the elements of Y may be arranged in an order $y_1, \cdots, y_n$ such that if $i < j < k$, then $d(y_i, y_j) \leq d(y_i, y_k) \geq d(y_j, y_k)$.*

Proof. Choose, among the elements of Y, two whose mutual distance is maximum. Call them a, b. Define $Z_1 = \{y \in Y, d(a, y) < d(a, b)\}$ and $Z_2 = Y - Z_1 = \{y \in Y, d(a, y) = d(a, b)\}$. Then if $x_i \in Z_i$, $d(x_1, x_2) = d(a, b)$. For, if $d(x_1, x_2) < d(a, b)$, then $d(a, x_2) \leq \max(d(a, x_1), d(x_1, x_2)) < d(a, b)$, and $x_2 \in Z_1$. Since Z_1, Z_2 are nonempty, we may by induction assume they may be ordered according to the lemma. Then juxtaposing the two orderings orders Y in the proper way.

Suppose now the ϕ_i are arranged according to Lemma 1. Then in particular, if $\phi_i = \phi_k$, $\phi_j = \phi_k$ for $i \leq j \leq k$. Put $d_{ij} = d_{\hat{S}}(\phi_i, \phi_j)$. Then $d_{ik} = \max(d_{ij}, d_{jk})$ for $i \leq j \leq k$. Let $|\ |_F$ be the usual ultrametric absolute value on F. Consider the set Y of matrices $M = \{m_{ij}\}$ such that $|m_{ij}| \leq 1$, and $|m_{ij}|d_{ij} \leq 1$ for $i > j$. Then if $M, N \in Y$, we have $MN = O$, with $o_{ik} = \sum_{l=1}^{n} m_{il} n_{lk}$. Obviously $|o_{ij}| \leq 1$ for all i, j. Moreover, if $i > j$, and $l \leq j$, $|m_{il} n_{lj}| \leq |d_{il}|^{-1} \leq |d_{ij}|^{-1}$; if $j < l \leq i$, then

$|m_{il}n_{lj}| \leq \min(d_{il}^{-1}, d_{lj}^{-1}) = d_{ij}^{-1}$; if $i < l$, then $|m_{il}n_{lj}| \leq |n_{lj}| \leq d_{lj}^{-1} \leq d_{ij}^{-1}$. Hence $|o_{ij}|d_{ij} \leq 1$ also, and Y is closed under multiplication. Put $H = H_{\psi_0} = K \cap Y$. H is clearly an open subgroup of K, containing B_0.

Now consider the subset Z of Y consisting of matrices M such that $m_{ij} = 0$ if $i > j$ and $d_{ij} > 0$. Then M is closed under multiplication also, and $M \cap H = P_0 = P \cap K$ for some parabolic subgroup P of G. (We note that, in the framework of B-N pairs, P is the parabolic subgroup associated to the isotropy group in W of ψ_0. In particular, if ψ_0 is nondegenerate, i.e., if $\phi_i \neq \phi_j$ for $i \neq j$, then $P = B$.)

Next, let J be the subgroup of H consisting of matrices of the form $1 + N$, where $n_{ij} = 0$ unless $i > j$ and $d_{ij} > 0$. Clearly $P_0 \cap J = \{1\}$, and the decomposition $H = JP_0$, that is, every $h \in H$ may be uniquely written $h = jp$, $j \in J$, $p \in P_0$, is a special case of the following lemma, which is proved by a very simple approximation argument.

Lemma 2. *Suppose* $\tilde{P} \supseteq H \supseteq P_0$ *for some open compact subgroup* H, *and some* $P \supseteq B$. *Then if* $J = H \cap N_P'$, *we have* $H = JP_0$, *and also, since* $P_0 = (M_P)_0(N_P)_0$, $H = J(M_P)_0(N_P)_0$. *Any permutation of the factors is permissible.*

By construction, it is clear that ψ_0 defines a character on P_0. I claim that in fact ψ_0 extends to a character of H, trivial on J. To show this suppose $P = P^1 \subseteq P^2 \subseteq \cdots \subseteq P^l = G$ is the sequence of parabolics defined as follows. Let $i_1, \cdots, i_l$ be determined by the criteria: $\phi_{i_j} \neq \phi_{i_k}$ if $j \neq k$, but $\phi_j = \phi_{i_k}$ if $i_{k-1} < j \leq i_k$. Then P^k is the parabolic consisting of elements $M = (m_{ij})$ with $m_{ij} = 0$ if $i > i_k$, $i > j$ and $d_{ij} > 0$; $P^{l-1} = Q$ is a maximal parabolic, and we may assume by induction that ψ_0 may be extended to $Q \cap H$. Put $H \cap M_Q = Y$, $H \cap N_Q = Z$, and $H \cap N_Q' = J \cap N_Q = X$. Then $H = XYZ$ and Y normalizes X and Z, and $YZ = Q \cap H$, and X and Z are abelian. Extend ψ_0 to H by the formula $\psi_0(xyz) = \psi_0(yz)$. We must verify that ψ_0 is still multiplicative. We have $\psi_0(x_1y_1z_1x_2y_2z_2) = \psi_0(x_1y_1x_2y_1^{-1}y_1x_2^{-1}z_1x_2y_2z_2) = \psi_0(y_1x_2^{-1}z_1x_2y_2)$. Now if $x_2 = 1 + a$, then $x_2^{-1} = 1 - a$. Also, let $z_1 = 1 + b$. Then $x_2^{-1}z_1x_2 = (1 - a)(1 + b)(1 + a) = 1 + b + [b, a] - aba$. Put $\alpha = -aba$, $\beta = [b, a]$, $\gamma = b$. Then $x_2^{-1}z_1x_2 = 1 + \alpha + \beta + \gamma = (1 + \alpha(1 + \beta)^{-1})(1 + \beta - \alpha(1 + \beta)^{-1}\gamma)(1 + (1 + \beta)^{-1}\gamma)$, the factors being in X, Y, Z respectively. Call the middle one y_3. Then $\psi_0(y_1x_2^{-1}z_1x_2y_2) = \psi_0(y_1y_3y_2) = \psi_0(y_1)\psi_0(y_2)\psi_0(y_3) = \psi_0(x_1y_1z_1)\psi_0(x_2y_2z_2)\psi_0(y_3)$. Thus we must show $\psi_0(y_3) = 1$. Multiplying ψ_0 by a character of K if necessary, which will not affect our arguments, allows us to assume $\phi_n = \phi_{i_l} = 1$. Now $y_3 = (1 + \beta)(1 - (1 + \beta)^{-1}\alpha(1 + \beta)^{-1}\gamma)$, and a brief calculation shows, if $(1 + \beta)^{-1}\alpha(1 + \beta)^{-1}\gamma = \{a_{ij}\}$ then $a_{ij} = 0$ unless $i > i_{l-1}$, $j > i_{l-1}$. Hence, by our normalization of ϕ_n, $\psi_0(y_3) = \psi_0(1 + \beta)$. Now $\beta = ba - ab$, and by the same reasoning as just above $\psi_0(y_3) = \psi_0(1 + ba)$. Write $ba = \{c_{ij}\}$. Then direct computation shows $|c_{ij}| < d_{nj}^{-1}$. From this, it is a simple matter to see that

$1 + ba = n_1' dn_2$, with $n_1' \in N_0'$, $d \in A_0$, $n_2 \in N_0$, and $d = \operatorname{diag}(d_1 \cdots d_n)$, with $|d_j - 1| < d_{nj}^{-1}$. Then $\psi_0(y_3) = \psi_0(d)$. But from the definition of the d_{nj}'s, and since $\phi_n = 1$, we finally see $\psi_0(d) = 1$.

Let now $V(\psi_0)$ be the representation of K induced from ψ_0 on H. We call the collection of these representations the principal series of K.

Theorem 1. *$V(\psi_0)$ is irreducible. For $U(\psi) \in P.S.(\psi_0)$, $Y(\psi_0)$, the restriction of $U(\psi)$ to K, contains $V(\psi_0)$ exactly once. If ψ_0' is not in the W-orbit of ψ_0, then $U(\psi')$ restricted to K does not contain $V(\psi_0)$. If $V(\psi_0)$, $V(\psi_0')$ are defined, then they are equivalent if and only if ψ_0 and ψ_0' belong to the same W-orbit.*

Proof. It is easily seen from the definition of $U(\psi)$, that the restriction of $U(\psi)$ to K is just the representation of K induced from ψ_0 on B_0. Clearly then $V(\psi_0)$ is a subrepresentation of $Y(\psi_0)$. Thus, if we can show the intertwining number of $Y(\psi_0)$ and $V(\psi_0)$ is one, then $V(\psi_0)$ is a fortiori irreducible, as well as being contained in $Y(\psi_0)$ only once. Similarly, if $V(\psi_0)$ is not contained in $Y(\psi_0')$, $V(\psi_0)$ must be different from $V(\psi_0')$. Also noting that it is obvious that if ψ_0, ψ_0' are in the same W-orbit, then $V(\psi_0)$ and $V(\psi_0')$ intertwine at least once, we see that to prove all statements, we need only compute the intertwining number of $V(\psi_0)$ and $Y(\psi_0')$.

For $k \in K$, let $\operatorname{Ad}(k)(H) = kHk^{-1}$, and $\operatorname{Ad}^*(k)(\psi_0)(khk^{-1}) = \psi_0(h)$. It is well known that the intertwining number of $V(\psi_0)$ and $Y(\psi_0')$ is given by the number of (B_0, H) double cosets B_0kH, such that ψ_0' and $\operatorname{Ad}^*k(\psi_0)$ agree on $B_0 \cap kHk^{-1}$. From [3], we know that, if $\{w_i\}$ are a set of coset representatives of $W \bmod W_P$, where W_P is the isotropy group of ψ_0, then $K = \bigcup_i B_0 w_i \tilde{P}$. Hence, if $\{x_j\}$ are a set of coset representatives for J in $N_P' \cap K_1$ we certainly have $K = \bigcup_{i,j} B_0 w_i x_j H$.

Write $\psi_0 = (\phi_1, \cdots, \phi_n)$ and $\psi_0' = (\phi_1', \cdots, \phi_n')$, and define $d_{A_0}(\psi_0, \psi_0') = \max d_{\mathfrak{S}}(\phi_i, \phi_i')$. Since $Y(\psi_0')$ is, by Theorem A, a function only of the W-orbit of ψ_0', we may choose ψ_0' in that orbit so that $d_{A_0}(\psi_0, \psi_0') \le d_{A_0}(\psi_0, \operatorname{Ad}^*w(\psi_0'))$ for any $w \in W$.

Since $x_j \in K_1$, and $K_a \subseteq H$, where $q^a = d_{1n} = \max d_{ij}$, we see that $A_0 \cap K_{a-1} \in B_0 \cap kHk^{-1}$ for any $k = w_i x_j$. Hence, if B_0kH is to support an intertwining operator ψ_0' and $\operatorname{Ad}^*w(\psi_0)$ must agree on $A_0 \cap K_{a-1}$. By our choice of ψ_0', it follows that in particular ψ_0 and ψ_0' agree on $A_0 \cap K_{a-1}$.

Now if θ is any linear character of K, then clearly, the intertwining number of $V(\psi_0)$ and $Y(\psi_0')$ is the same as that of $V(\psi_0 \cdot \theta)$ and $Y(\psi_0' \cdot \theta)$. Moreover, $d_{A_0}(\psi_0, \psi_0') \le d_{A_0}(\psi_0, \operatorname{Ad}^*w(\psi_0'))$ if and only if the same holds for $\psi_0 \cdot \theta$ and $\psi_0' \cdot \theta$. Therefore, we may normalize ψ_0 so that, for example, $\phi_1 = 1$, without loss of generality. From now on, therefore, we take $\phi_1 = 1$.

Now let $j_0 = \max\{j: d_{1j} < d_{1n}\}$. Then, for $j \le j_0$, ϕ_j is trivial on S_{a-1}, because of our normalization, while ϕ_j is nontrivial on S_{a-1} for all $j > j_0$. It follows from our previous observations that if $B_0 w_i x_j H$ supports an intertwining operator, then w_i cannot interchange ϕ_l and ϕ_m if $l \leq j_0 < m$.

Let P_1 be the (maximal) parabolic consisting of elements $M = \{m_{ij}\}$ such that $m_{ij} = 0$ if $j \leq j_0 < i$. We have seen just above that $B_0 w_i x_j H$ can support an intertwining operator only if $w_i \in W_{P_1}$. Now we will show that, unless x_j can also be chosen to be in P_1, then $B_0 w_i x_j H$ cannot support an intertwining operator. Then the problem will be reduced to two problems of lower rank, and by an obvious induction, the result is proved.

Write $x_j = n_2 n_1$, with $n_1 \in P_1$ and $n_2 \in N'_{P_1}$. Then x_j is determined modulo J, so it can be taken in P_1 if and only if $n_2 \in J$ if and only if $n_2 \in K_a$. If $n_2 \notin K_a$, write $n_2 = 1 + T$, $T = \{t_{ij}\}$. Put $b = \min_{i,j} \operatorname{ord}_F(t_{ij})$, where ord_F is the usual valuation on F, and choose l, m such that $\operatorname{ord}_F(t_{lm}) = b$, but $\operatorname{ord}_F(t_{ij}) > b$ if either $i > l$, $j = m$ or $i = l$, $j < m$. Let $E_{ml} = \{\delta_{im}\delta_{jl}\}$ be the (m, l)th matrix unit. Note that $b < a$. Put $z = 1 + \pi^{a-b-1} r E_{ml}$, $r \in R$. Then $z \in (N_{P_1})_0$.

Now n_1 normalizes $(N_{P_1})_0$. Therefore we can find $z_1 \in (N_{P_1})_0$ such that $n_1 z_1 n_1^{-1} = z$. Then $x_j z_1 x_j^{-1} = n_2 z n_2^{-1} = z + \pi^{a-b-1} r([T, E_{ml}] - TE_{ml}T) = z(1 + z^{-1}(\pi^{a-b-1} r([T, E_{ml}] - TE_{ml}T))) = z\alpha$. From the properties of T listed above, it follows that α is in $(B_0 \cap K_{a-1}) \cdot K_a$ and that it has diagonal entries in S_{a-1} at the lth and mth diagonal places. Since x_j normalizes $(N_0 \cap K_{a-1}) \cdot K_a$, we can alter z_1 by some y in this group, so that $x_j z_1 y x_j^{-1} = \delta z$, where δ is a diagonal matrix with entries in S_{a-1} at the lth and mth spots and 1's elsewhere. Moreover, by varying r in the definition of z, we may make the lth entry of δ arbitrary in S_{a-1}. Thus, in particular, by definition of P_1, and since $w_i = w \in W^{\sim}_{P_1}$, we have $S_{a-1} \subseteq \ker \phi_{w(m)}$ but $S_{a-1} \not\subseteq \ker \phi_{w(l)}$. Thus, we may assume that $\operatorname{Ad}^* w^{-1}(\psi_0)(\delta) \neq 1$.

Again, since $w_i \in W^{\sim}_{P_1}$, we have $\operatorname{Ad} w_i x_j (z_1 y) = b \in B_0$. And since ψ_0 and ψ'_0 agree on $A_0 \cap K_{a-1}$, we have $\psi'_0(b) = \operatorname{Ad}^* w^{-1}(\psi_0)(\operatorname{Ad} x_j(z_1 y)) = \operatorname{Ad}^* w^{-1}(\psi_0)(\delta z) \neq 1$. On the other hand, we clearly have $\operatorname{Ad}^* w_i x_j \psi_0(b) = \psi_0(z_1 y) = 1$, since z_1 and y are each in $\ker \psi_0$ (for different reasons). Thus ψ'_0 and $\operatorname{Ad}^*\psi_0$ do not agree on $B_0 \cap \operatorname{Ad} w_i x_j(H)$ unless w_i and x_j can be taken in P_1, and Theorem 1 is proved.

Now let $\mathcal{S}(\psi_0)$ denote the space of compactly supported functions f on G such that $f(h_1 g h_2) = \psi_0(h_1) f(g) \psi_0(h_2)$, for $h_1, h_2 \in H$. $\mathcal{S}(\psi_0)$ is an algebra under convolution, and the function which is ψ_0 on H and 0 elsewhere is the identity of $\mathcal{S}(\psi_0)$ (if Haar measure is renormalized so H has measure 1). We refer to $\mathcal{S}(\psi_0)$ as the ψ_0-spherical function algebra. Clearly each (H, H) double coset in G supports at most one element of $\mathcal{S}(\psi_0)$ up to scalar multiples. We want to determine the (H, H) double cosets which actually do support nonzero elements of $\mathcal{S}(\psi_0)$.

Let P again be the parabolic subgroup associated to ψ_0. Let $\overline{W}$ be the affine Weyl group, which in our case is the semidirect product $W \times_s D$. Then, again by [3], if $\{\overline{w}_i\}$ are representatives of the (W_P, W_P) double cosets in $\overline{W}$, then also $G = \bigcup_i \tilde{P}\, \overline{w}_i \tilde{P}$, the union being disjoint. Hence, $\{x_j\}$ again being a set of representatives for J in $N_P' \cap K_1$, we see $G = \bigcup_{i,j,k} Hx_j^{-1}\overline{w}_i x_k H$.

It is clear that, for any $d \in D$, ψ_0 and $\mathrm{Ad}^* d(\psi_0)$ agree on $H \cap dHd^{-1}$. Therefore all double cosets HdH support nontrivial elements of $\mathcal{S}(\psi_0)$.

Lemma 3. *The cosets HdH are precisely the (H, H) double cosets which support nontrivial elements of $\mathcal{S}(\psi_0)$.*

Proof. This calculation is very similar in spirit and detail to the calculation of Theorem 1. First, we note that $A_0 \cap K_{a-1}$ is contained in $(x_j H x_j^{-1}) \cap \overline{w}_i x_k H x_k^{-1} \overline{w}_i^{-1}$. Furthermore, conjugation by x_j, x_k and the diagonal part of $\overline{w}_i$ does not change the values of ψ_0 on $A_0 \cap K_{a-1}$. Therefore, if P_1 is defined as before, we must once again have $\overline{w}_i \in P_1$. Now put $x_j = n_1 n_2$, $x_k = m_1 m_2$, with $n_1, m_1 \in (N_{P_1}')_0$, and $n_2, m_2 \in P_1$. We see that if we can show if $Hx_j^{-1}\overline{w}_i x_k H$ supports an intertwining operator, then $n_1^{-1} w_i m_1 \in J_1 \overline{w}_i J_1$, where $J_1 = N_{P_1}' \cap J$, then we will be done, by the same sort of induction argument as in Theorem 1.

There are essentially two cases. First, suppose $n_1 = 1$. This covers the case when, although $n_1 \neq 1$, $n_1^{-1} w_i m_1 \in J_1 \overline{w}_i (N_{P_1}' \cap K_1)$. By symmetry, this also covers the case when $m_1 = 1$. Write $m_1 = 1 + T$, with $T = \{t_{ij}\}$. Let $E_{kl} = \{\delta_{ik}\delta_{jl}\}$ be the (k, l)th matrix coefficient. Then $m_1 = \Pi_{i,j} (1 + t_{ij}E_{ij})$ (order is immaterial). Since m_1 is determined only up to an element of $(\overline{w}_i^{-1} J_1 \overline{w}_i) \cdot J_1 = X$, we may and do assume that $t_{ij} = 0$ unless $1 + t_{ij}E_{ij} \notin X$. Now choose (k, l) such that $|t_{kl}|_F \geq |t_{ij}|_F$ for all i, j, and $|t_{kl}|_F > |t_{ij}|_F$ if $i < k$, $j = l$, or $i = k$, $j > l$. We have seen in Theorem 1 that we may find $z \in H$, such that $z' = x_k z x_k^{-1} = 1 + rE_{lk} + rt_{kl}(E_{kk} - E_{ll})$, with $\mathrm{ord}_F(r) + \mathrm{ord}_F(t_{kl}) = a - 1$. Moreover, $z \in \ker \psi_0$, but $z' \notin \ker \psi_0$ for appropriate r. Now we see that if $\overline{w}_i z' \overline{w}_i^{-1} \in H$, then it still does not belong to $\ker \psi_0$ for appropriate r. But if $\overline{w}_i E_{kl} \overline{w}_i^{-1} = sE_{k'l'}$, then $\overline{w}_i E_{lk} \overline{w}_i^{-1} = s^{-1} E_{l'k'}$, since $E_{lk}E_{kl} = E_{ll}$. But we have $\mathrm{ord}_F(t_{kl}) \geq 1$, but $\overline{w}_i(1 + t_{kl}E_{kl})\overline{w}_i^{-1} \notin J_1$. Hence $\mathrm{ord}_F s + \mathrm{ord}_F(t_{kl}) < a$. Thus $\mathrm{ord}_F(s^{-1}r) = \mathrm{ord}_F(r) - \mathrm{ord}_F(s) = a - 1 - \mathrm{ord}_F(s) - \mathrm{ord}_F(t_{kl}) > -1$. Thus $\overline{w}_i z' \overline{w}_i^{-1} \in H$, and this case is proved.

Next we assume we cannot take m_1 to be 1. Again writing $m_1 = 1 + T = \Pi_{i,j}(1 + t_{ij}E_{ij})$, suppose $\overline{w}_i(1 + t_{kl}E_{kl})\overline{w}_i^{-1} \in (N_{P_1})' \cap K_1$. Then $n_1^{-1}\overline{w}_i m_1 = n_1'^{-1}\overline{w}_i m_1'$, where $m_1' = (1 - t_{kl}E_{kl})m_1$ and $n_1' = n_1 \overline{w}_i (1 - t_{kl}E_{kl})\overline{w}_i^{-1}$. Thus we may and do assume that for all $t_{ij} \neq 0$ appearing in the expansion of m_1 as a product, $\overline{w}_i(1 + t_{ij}E_{ij})\overline{w}_i^{-1} \notin K_1$. This being so, pick (k, l) by the same criteria as before, and again pick z, so that $z' = 1 + rE_{lk} + rt_{kl}(E_{kk} - E_{ll})$, and

$z \in \ker \psi_0$, but $z' \notin \ker \psi_0$. Now, by our normalization of m_1, we have, if $\overline{w}_i E_{kl} \overline{w}_i^{-1} = sE_{k'l'}$, $\operatorname{ord}_F(t_{ij}) + \operatorname{ord}_F(s) \leq 0$. Thus $\operatorname{ord}_F(rs^{-1}) = a - 1 - (\operatorname{ord}_F(t_{ij}) + \operatorname{ord}_F(s)) \geq a - 1$. Therefore $\overline{w}_i z' \overline{w}_i^{-1} \in K_{a-1} \cap H$, but $\overline{w}_i z' \overline{w}_i^{-1} \notin \ker \psi_0$ for appropriate r. Thus we see $x_j^{-1} \overline{w}_i z' \overline{w}_i^{-1} x_j \in H$, and still is not in $\ker \psi_0$, since $\mathrm{Ad}^* x_j$ leaves ψ_0 invariant on $K_{a-1} \cap H$. Lemma 3 is now complete.

We are now prepared to do the harmonic analysis of $\mathcal{S}(\psi_0)$. In this, we follow the lines of investigation of Bruhat [2] and Satake [5]. We begin by recalling some of the basic notions about spherical functions.

Anticipating the result that $\mathcal{S}(\psi_0)$ is abelian, we define a ψ_0-spherical function on G to be a function Σ such that $\Sigma(h_1 g h_2) = \psi_0(h_1)\Sigma(g)\psi_0(h_2)$ for $h_i \in H$, and such that, for all $f \in \mathcal{S}(\psi_0)$, $f * \Sigma = \mu(f)\Sigma$, where $*$ denotes convolution, and $\mu(f) \in \mathbf{C}$. Clearly $f \longrightarrow \mu(f)$ defines an algebra homomorphism from $\mathcal{S}(\psi_0)$ to $\mathbf{C}$.

The best way to find spherical functions is via representation theory. Let U be a representation of G whose restriction to H contains ψ_0 exactly once. If v is a vector in the representation space which transforms under H according to ψ_0, and l is a linear functional such that $l(v) = 1$ and l vanishes on all other isotopic components of H, then $\Sigma(g) = l(U(g)(v))$ defines a ψ_0-spherical function on G, such that the associated linear functional μ is given by $U(f)(v) = \mu(f)(v)$.

Thus in particular, for any $U(\psi) \in \mathrm{P.S.}(\psi_0)$, we may by Theorem 1 associate a spherical function Σ_ψ. By Lemma 3, the support of Σ_ψ is contained in HDH. On this set, it may be realized as follows. There is a unique function σ_ψ such that, for $b = an \in B$, $h \in H$, we have $\sigma_\psi(han) = \psi_0(h)\psi(a)\delta(a)^{-1/2}$. (This formula clearly determines σ_ψ on HB, while Theorem 1 essentially shows σ_ψ must vanish off HB.) There is a well-known integral formula expressing Σ_ψ in terms of ψ. Recall that the space of $U(\psi)$ is the space of functions f on G such that $f(gan) = f(g)\psi(a)\delta^{-1/2}(a)$. In particular, σ_ψ is in the space. By abuse of notation, let ψ_0^{-1} be the function whose restriction to H is ψ_0^{-1}, and which is zero off of H. Then we see that integration against ψ_0^{-1} is an appropriate linear functional, and so we define $\Sigma_\psi(g) = \int_H \sigma_\psi(gh)\psi_0^{-1}(h)\,dh$.

In view of Lemma 3, in order to know Σ_ψ, it suffices to know it on D. (In fact, it suffices to know it on one member from each $\mathrm{Ad}\, W_P$ orbit in D.)

Let $\Delta(d, H) = \Delta(d)$ be the number of H-cosets in HdH, that is, $\Delta(d)$ is the index of $H \cap d^{-1}Hd$ in H. Let $\{h_i\}$ be a set of representatives for $H \cap d^{-1}Hd$ in H so that $HdH = \bigcup_i Hdh_i$. Then we see $\Sigma_\psi(d) = \Delta(d)^{-1}(\sum_i \sigma_\psi(dh_i)\psi_0^{-1}(h_i))$. (I hope this notation will not cause confusion.) Moreover, we need only sum over those h_i such that $dh_i \in Hd'N$ for $d' \in D$. Let $I(d, d', H) = I(d, d')$ be the number of h_i such that $dh_i \in Hd'N$.

Now we may write $M_P \simeq \Pi_i\, Gl_{n_i}(F)$, where n_i is the rank of the ith diagonal block of M_P. If $K^{(i)} = Gl_{n_i}(R) = Gl_{n_i}(F) \cap K$, then $H \cap M_P = \Pi_i\, K^{(i)} = (M_P)_0$.

Lemma 4. *We have* $\Delta(d, H) = \Delta(d, \widetilde{P})$, *and* $I(d, d', H) = I(d, d', \widetilde{P}) =$

$\Lambda(d)(\Pi_i\, I(d_i, d_i', K^{(i)})$, where $d=\Pi_i\, d_i$, $d_i \in Gl_{n_i}(F)$, is the decomposition of d according to the factors of $\underset{\sim}{M}_P$, and $\Lambda(d) = {}^{\#}((N_P)_0/(N_P)_0 \cap d^{-1}(N_P)_0 d)$ (${}^{\#}$ denotes cardinality). Also $\Delta(d, \tilde{P}) = \Lambda(d)\Lambda(d^{-1})\Delta(d, (M_P)_0) = \Lambda(d)\Lambda(d^{-1})(\Pi_i \Delta(d_i, K^{(i)}))$. Moreover, one may always choose the representatives $\{h_i\}$ of $H \cap d^{-1}Hd$ in H so that $\psi_0(h_i)=1$, and $\sigma_\psi(dh_i) = \sigma_\psi(d')$ if $dh_i \in Hd'N$.

Proof. Write $\tilde{B}^{(i)} = \tilde{B} \cap K^{(i)}$. Write $C = \tilde{B} \cap H$. Then $H/C \simeq \tilde{P}/\tilde{B} \simeq \Pi_i\, K^{(i)}/\tilde{B}^{(i)}$. Therefore $H = \bigcup CwC = \bigcup Cw(\Pi_i\, \tilde{B}^{(i)})$, where w runs through W_P. Therefore $HdH = \bigcup Cw_1 b_1 d b_2 w_2 C$ with $w_1, w_2 \in W_P$, $b_1, b_2 \in \Pi_i\, \tilde{B}^{(i)}$. In particular the (C, C) double cosets in HdH all have representatives in M_P. Now, as we have said, $G = \bigcup \tilde{P}\bar{w}\tilde{P}$ (disjoint union) where the $\bar{w}$ run through a set of (W_P, W_P) double coset representatives in $\overline{W}$. Also $M_P = \bigcup (M_P)_0 d (M_P)_0$, where $(M_P)_0 = \Pi_i\, K^{(i)}$, and d runs through a set of representatives for the Ad W_P-orbits in D. Since $\overline{W} \cap M_P = W_P \cdot D$, we see d is also running through a set of (W_P, W_P) double coset representatives in $\overline{W} \cap M_P$. Therefore, since $\tilde{P} \cap M_P = (M_P)_0$, $(\tilde{P}d\tilde{P}) \cap \underset{\sim}{M}_P = \underset{\sim}{(}M_P)_0 d (M_P)_0$, and since $\tilde{P} \supseteq H \supseteq (M_P)_0$, $HdH \cap M_P = (M_P)_0 d (M_P)_0$. Now if $\underset{\sim}{B}_P = B \cap M_P$, then $(M_P)_0 d (M_P)_0 = \bigcup \tilde{B}_P\, v\, \tilde{B}_P$, where $v \in W_P d W_P$. Similarly $\tilde{P}d\tilde{P} = \bigcup \tilde{B}\, v\, \tilde{B}$. Hence $(\tilde{B}\, v\, \tilde{B}) \cap M_P = \tilde{B}_P\, v\, \tilde{B}_P$. Therefore, since $\tilde{B} \supseteq C \supseteq \tilde{B}_P$, $(C\, v\, C) \cap M_P = \tilde{B}_P\, v\, \tilde{B}_P$. Hence $(HdH) \cap M_P = \bigcup ((C\, v\, C) \cap M_P)$. Since every (C, C) double coset in HdH intersects M_P, we conclude $HdH = \bigcup C\, v\, C$. Since $C \subseteq \tilde{B}$, this union is disjoint. Now one may compute directly, remembering that $v \in W_P \cdot D$, that ${}^{\#}(C/C \cap vCv^{-1}) = {}^{\#}(\tilde{B}/\tilde{B} \cap v\tilde{B}v^{-1})$. Since also ${}^{\#}(H/C) = {}^{\#}(\tilde{P}/\tilde{B})$, we conclude $\Delta(d, H) = \Delta(d, \tilde{P})$.

Next suppose $dh_i \in Hd'N$. Then $dh_i = h'd'n$. Write $h_i = j_i m_i n_i$, where $j_i \in J$, $m_i \in (M_P)_0$ and $n_i \in (N_P)_0$. Similarly, write $h' = j'm'n'$. Then we see that $(m'^{-1}j'^{-1}dj_i d^{-1}m')m'^{-1}dm_i = d'(d'^{-1}n'd')nn_i^{-1}$. For this equation to hold, we must have $j' = dj_i d^{-1}$. This shows in particular that when such an equation holds, we may always choose $h_i \in P_0$. We now see further, that if $n = ab$, with $a \in M_P$, $b \in N_P$, then $(d'a)^{-1}n'(d'a)b = n_i$, and we are left with the equation $dm_i = m'd'a$. The number of solutions of this equation is clearly $\Pi_i\, I(d_i, d_i', K^{(i)})$. Moreover, at this point we may note that since $A_0 \subseteq (M_P)_0$, and d commutes with A_0, and $dm_i = m'd'a$ can have a solution only if $d'd^{-1} \in Sl_n(F)$, and since σ_ψ is right invariant by N, and since $(N_P)_0 \subseteq \ker \psi_0$, and since the restriction of ψ_0 is a character on $K^{(i)}$, we may assume $m_i, m' \in \Pi_j\, Sl_{n_j}(F)$, and the last statement of the lemma is verified.

Now I claim that, if $\{m_j\}$ are a set of representatives of $(M_P)_0 \cap d^{-1}(M_P)_0 d$ in $(M_P)_0$, and if $\{n_l\}$ are a set of representatives of $(N_P)_0 \cap d^{-1}(N_P)_0 d$ in $(N_P)_0$, then a set of representatives for those H cosets in HdH which have representatives in P_0 is exactly the set $\{n_j m_l\}$. For suppose $Hdx = Hdy$, with $y \in P_0$. Then $dx = hdy$. Write $x = j_1 n_1 m_1$, $y = n_2 m_2$ and $h = jnm$. Then $dj_1 n_1 m_1 = jnmdn_2 m_2$, or $dj_1 d^{-1} dn_1 d^{-1} dm_1 = jn(md)n_2(md)^{-1}mdm_2$, and we conclude

$dj_1d^{-1} = j$, $dm_1 = mdm_2$, and $dn_1d^{-1} = nd(mn_2m^{-1})d^{-1}$. Thus $j_1 \in J \cap dJd^{-1}$, $m_1m_2^{-1} \in (M_P)_0 \cap d^{-1}(M_P)_0d$, and finally $n_1mn_2^{-1}m^{-1} \in (N_P)_0 \cap d^{-1}(N_P)_0d$. This establishes the claim.

From this, we see that the total number of solutions to $dh_i = d'n$ is indeed equal to $(\prod_i I(d_i, d_i', K^{(i)}))\Lambda(d)$. Since precisely the same reasoning applies to $\tilde{P}$ (in fact, $\tilde{P}$ is a special case of H), the second statement of the lemma is proved.

It remains to compare $\Delta(d, \tilde{P})$ and $\Delta(d, (M_P)_0)$. Here we may apply formulas developed in [3]. W being a Coxeter group, it has defined on it a length l, derived from word length of expressions in the canonical generators (see [3]). The Poincaré polynomial of W is defined by $\mathcal{P}(t, W) = \sum a_m t^m$, where a_m is the number of elements of W of length m.

Now take $d \in D$, and write $d = \prod_i d_i$, $d_i \in Gl_{n_i}$. Let d_i have diagonal elements $(\pi^{m_\alpha}, \cdots, \pi^{m_\beta})$, where $\alpha = \alpha_i = 1 + \sum_{j<i} n_j$ and $\beta = \beta_i = \sum_{j \le i} n_j$. Then in any Ad W_P orbit, there is a unique d such that $m_\alpha \le m_{\alpha+1} \le \cdots \le m_\beta$.

Let V be the isotropy group of d under Ad W_P. Then $\Delta(d, \tilde{P}) = q^{\mu(d)}\mathcal{P}(q^{-1}, W_P)\mathcal{P}(q^{-1}, V)^{-1}$ where $\mu(d) = \sum_{i<j} |m_i - m_j|$, and $\Delta(d, (M_P)_0) = q^{\nu(d)}\mathcal{P}(q^{-1}, W_P)\mathcal{P}(q^{-1}, V)^{-1}$, where $\nu(d) = \sum_{\alpha_i \le k \le l \le \beta_i} m_l - m_k$. Here d is assumed chosen as above. It is also easy to convince oneself that $\Lambda(d) = q^{\lambda(d)}$, with $\lambda(d) = \sum \max(m_l - m_k, 0)$, where this summation runs over k, l satisfying $k \le \beta_i < \alpha_j \le l$, for some i, j. Now we see easily that $\mu(d) = \nu(d) + \lambda(d) + \lambda(d^{-1})$, and this finishes the lemma.

Returning to our formula for the ψ_0-spherical function Σ_ψ, we find Lemma 4 allows us to rewrite it as follows. Put $\psi'(d) = \psi(d)\delta^{-1/2}(d)$. Then

$$\begin{aligned}
\Sigma_\psi(d) &= \Delta(d, H)^{-1}\left(\sum_i \sigma_\psi(dh_i)\psi_0^{-1}(h_i)\right) \\
&= \Delta(d, \tilde{P})^{-1}\left(\sum \sigma_\psi(d')I(d, d', \tilde{P})\right) \\
&= \Lambda(d)^{-1}\Lambda(d^{-1})^{-1}\left(\prod_i \Delta(d_i, K^{(i)})\right)^{-1}\left(\sum \psi'(d')\Lambda(d)\left(\prod_i I(d_i, d_i', K^{(i)})\right)\right) \\
&= \Lambda(d^{-1})^{-1}\left(\prod_i \Delta(d_i, K^{(i)})^{-1} \sum \prod_i \psi'(d_i')I(d_i, d_i', K^{(i)})\right).
\end{aligned}$$

Now, if E_i is the set of d_i' such that $I(d_i, d_i', K^{(i)})$ is not zero, and if E is the set of d' such that $I(d, d', \tilde{P})$ is not zero, it is clear from the analysis in Lemma 4 that E is the direct product of the E_i's. Thus, we finally have

$$\Sigma_\psi(d) = \Lambda(d^{-1})^{-1}\prod_i\left[\Delta(d_i, K^{(i)})^{-1}\left(\sum \psi'(d_i')I(d_i, d_i', K^{(i)})\right)\right]. \tag{1}$$

Now, ψ_0 is a character on $(M_P)_0$. Let $\tilde{\psi}_0$ be any extension of it to a character of M_P. Then the map $f \to \tilde{\psi}_0 f$ is an isomorphism of the convolution algebra (of locally constant functions) of M_P onto itself, and it takes the 1-spherical (or $(M_P)_0$-spherical) functions to the ψ_0-spherical functions on M_P. Moreover, we observe that the remaining summation in the expression for Σ is just a $\psi_0^{(i)}$ spherical function on $Gl_{n_i}(F)$, where $\psi_0^{(i)}$ is the restriction of ψ_0 to $K^{(i)}$.

Now let $\tilde{\psi}_0$ be the character of A which extends ψ_0 and is trivial on D. Denote by $\hat{D}^{\mathbf{C}}$ the group of all quasi (e.g., complex-valued, not necessarily unitary) characters of D. Then $\hat{D}^{\mathbf{C}}$ is a complex torus of dimension n. The map $\gamma: \psi \to \psi\tilde{\psi}_0^{-1}$ defines a homeomorphism from the coset of quasicharacters of A extending ψ_0 to $\hat{D}^{\mathbf{C}}$. We use it to identify the two, and write $\Sigma_\psi = \Sigma_{\gamma(\psi)}$, so that the spherical functions are parametrized by $\hat{D}^{\mathbf{C}}$. We denote an element of $\hat{D}^{\mathbf{C}}$ by λ. We fix an isomorphism $\alpha: \hat{D}^{\mathbf{C}} \to \mathbf{C}^{\times n}$ by the formula $\alpha(\lambda) = (\lambda(d_1), \cdots, \lambda(d_n))$ where d_j has diagonal entries all 1's, except for π at the jth spot. W_P, the isotropy group of ψ_0, acts on the ψ's, and this action transfers by γ and α to $\hat{D}^{\mathbf{C}}$ and $\mathbf{C}^{\times n}$. On $\mathbf{C}^{\times n}$, W_P acts by permuting the blocks of factors between α_i and β_i, for each i in the decomposition of M_P.

For $f \in \mathcal{S}(\psi_0)$, $\lambda \in \hat{D}^{\mathbf{C}}$, we define $\hat{f}(\lambda) = \int_G f(g)\Sigma_\lambda(g^{-1})\,dg$. The map $\hat{}: f \to \hat{f}$ is a linear map from $\mathcal{S}(\psi_0)$ to a space of functions on $\hat{D}^{\mathbf{C}}$. Since the Σ_λ are ψ_0-spherical, $f \to \hat{f}$ is an algebra homomorphism, taking convolution in $\mathcal{S}(\psi_0)$ and pointwise multiplication on $\hat{D}^{\mathbf{C}}$ as products. $\hat{}$ is called the ψ_0-spherical Fourier transform.

We recall that for an algebra $\mathfrak{A}$, a family $\{U_i\}$ of representations is called sufficient if they separate points of $\mathfrak{A}$—e.g., if $\bigcap_i \ker U_i = 0$.

Theorem 2. (i) *The characters of $\mathcal{S}(\psi_0)$ associated to the spherical functions Σ_ψ, for $U(\psi) \in P.S.(\psi_0)$ form a sufficient set of representations for $\mathcal{S}(\psi_0)$. In fact, they comprise all complex homomorphisms of $\mathcal{S}(\psi_0)$.*

(ii) *$\mathcal{S}(\psi_0)$ is abelian.*

(iii) *The ψ_0-spherical Fourier transform is an isomorphism of $\mathcal{S}(\psi_0)$ onto the subalgebra of $\alpha^*(\mathbf{C}(z_1, \cdots, z_n, z_1^{-1}, \cdots, z_n^{-1}))$ consisting of W_P-invariant functions. (α^* denotes pullback by α.)*

(iv) *The ψ_0-spherical functions are, as functions on D, equal to a function independent of ψ times the product of spherical functions $\Sigma^{(i)}_{\psi'(i)}$ of the algebras $\mathcal{S}(\psi_0^{(i)})$.*

(v) *$\mathcal{S}(\psi_0)$ is naturally isomorphic to the tensor product $\bigotimes_i \mathcal{S}(\psi_0^{(i)})$.*

Proof. (iv) is clear from the explicit formula (1) for Σ_ψ which we developed. This shows then, that if one identifies $\hat{D}^{\mathbf{C}} \simeq \Pi_i \hat{D}^{(i)\mathbf{C}}$, where $D^{(i)} = D \cap Gl_{n_i}(F) \supseteq M_P$, the ψ_0-spherical Fourier transform of the element of $\mathcal{S}(\psi_0)$ living on HdH is, up to a multiple, the same as the product of the $\psi_0^{(i)}$-spherical Fourier transforms of the elements of $\mathcal{S}(\psi_0^{(i)})$ living on Kd_iK. Since $D \simeq \Pi_i D^i$, this establishes (v) on the Fourier transformed side, and we see (i) is reduced to the case when ψ_0 is a character, which case, as we remarked above, is equivalent to the case of trivial ψ_0. But in this case, the first part of (i) has been proved by Bruhat [2], and is therefore true in general. From (i), (ii) follows, since $\mu(f_1 * f_2) = \mu(f_2 * f_1)$ for any character μ associated to some Σ_ψ, and the completeness of the μ's then gives $f_1 * f_2 = f_2 * f_1$, for all $f_1, f_2 \in \mathcal{S}(\psi_0)$. Similarly from (i), the fact that $\hat{}$ is an isomorphism onto its image follows, so (v) is true, and the whole of (iii) is reduced again to the case of trivial ψ_0, where it has been established by Satake. Thus (iii) is true, and from (iii) the second half of (i) follows, and Theorem 2 is proved.

Remark. The formula (1) also allows reduction of the computation of the Plancherel measure to the case of trivial ψ_0, and this case has been explicitly worked out by MacDonald [4].

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