

KERNELS IN DIMENSION THEORY

BY

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ABSTRACT. All spaces are metrizable.

A conjecture of de Groot states that a weak inductive dimension theory beginning with the class of compact spaces will characterize those spaces which can be extended to a compact space by the adjunction of a set of dimension not exceeding n . Nagata has proposed a variant of this conjecture as a means of finding insights into the original conjecture. (See *Internat. Sympos. on Extension Theory*, Berlin, 1967, pp. 157–161.) The proposed variant replaces compact with σ -compact.

The present paper concerns a study of strong inductive dimension theory beginning with an arbitrary class of spaces. The study is motivated by the above two conjectures. It indicates that a theory of kernels is a more natural by-product of inductive theory than a theory of extensions. An example has resulted which, with the aid of the developed theory and the Baire category theorem, resolves the second conjecture in the negative. The original conjecture is still unresolved. It is also shown that the notion of kernels results in a further generalization of Lelek's form of the dimension lowering map theorem (*Colloq. Math.* 12 (1964), 221–227. MR 31 #716).

1. Introduction. At the outset, we agree that all spaces are metrizable.

The present paper is motivated by problems which relate extensions Y of a space X and the dimension of the remainder $Y \setminus X$. Let us state the general problem precisely. Let \mathcal{P} be a topologically closed family of spaces and n be a natural number. Then find a characterization of those spaces X for which there is an extension $Y \in \mathcal{P}$ with $\dim Y \setminus X \leq n$. Classes of spaces which have been considered are the class of compact spaces [14] and the class of topologically complete spaces [1]. For these classes, characterizations have been found. As one might suspect, the characterizations involve dimension theoretic techniques.

There are three fundamental dimension theoretic techniques, weak inductive, strong inductive, and cover. The characterization in the compact case is given by covering techniques. The topologically complete case is handled by the strong inductive technique. It is a long standing conjecture of de Groot that the compact case has a weak inductive technique solution ([4], [5]). Nagata has proposed as a candidate for a class the σ -compact spaces as one in which a weak inductive

Received by the editors August 16, 1971 and, in revised form, March 20, 1972.

AMS (MOS) subject classifications (1970). Primary 54F45, 54D40, 54D45; Secondary 54E35.

Key words and phrases. Dimension theory, extensions, deficiency, kernel.

⁽¹⁾ Both authors were partially supported by the National Science Foundation Grants, NSF GP-12915 and NSF GP-28572.

characterization could be found [12]. Such a characterization would perhaps shed some light on the conjecture of de Groot in the compact case.

With the two weak inductive conjectures and the strong inductive characterization mentioned above in mind, we develop in the present paper an abstract theory of strong inductive invariants. Our theory shows that, in classes such as σ -compact spaces, the notion of extension is not as natural as the notion of a *kernel*, i.e., subspaces Y of X such that $Y \in \mathcal{P}$. This discovery has led to a negative solution to Nagata's problem. A second consequence of the notion of kernel is a generalization of the dimension lowering map theorem of Hurewicz (see §5.2). This generalization is sharper than a similar theorem of Lelek [8]. After an initial section (§2) of definitions, we devote §§3, 4, 5, 7 to an abstract development of the strong inductive theory and its relationship to kernels. §6 is devoted to resolving Nagata's conjecture and finally §8 is devoted to some theorems concerning extensions.

2. Preliminaries. This section is devoted to the notational conventions, basic definitions and statements of simple propositions which will be useful later.

2.1. Conventions. \mathfrak{M} will always be used to denote the class of metrizable spaces.

If A is a subset of a space X , its closure and boundary will be denoted by \bar{A} and $B(A)$, respectively. In the event that the space X needs to be emphasized, we will use the notation $\text{cl}_X(A)$ and $B_X(A)$, respectively.

A subclass \mathcal{P} of \mathfrak{M} will be called *topologically closed* if Y is a member of \mathcal{P} whenever Y is homeomorphic to a member of \mathcal{P} . Unless stated to the contrary, all classes considered in the paper are to be topologically closed. From §3 on, all classes considered are to be nonempty. We shall use the words class and family interchangeably.

Ind and ind will denote the usual strong and weak inductive dimension [11].

2.2. Definition of \mathcal{P} -surplus and \mathcal{P} -deficiency. For each class \mathcal{P} of spaces and each space X , we say a space Y is a \mathcal{P} -kernel (\mathcal{P} -hull) of X if $Y \in \mathcal{P}$ and $Y \subset X$ ($Y \supset X$). By the *strong \mathcal{P} -surplus* of X we mean the infimum of the set of numbers $\{\text{Ind } X \setminus Y : Y \text{ is a } \mathcal{P}\text{-kernel of } X\}$. This infimum will be denoted by $\mathcal{P}\text{-Sur } X$. By the *strong \mathcal{P} -deficiency* of X we mean the infimum of the set of numbers $\{\text{Ind } Y \setminus X : Y \text{ is a } \mathcal{P}\text{-hull of } X\}$. This infimum will be denoted by $\mathcal{P}\text{-Def } X$.

We define in an analogous manner the *weak \mathcal{P} -surplus* and *weak \mathcal{P} -deficiency* using the weak inductive dimension ind . The corresponding symbols will be $\mathcal{P}\text{-sur}$ and $\mathcal{P}\text{-def}$.

Proposition. $\mathcal{P}\text{-Sur } X = \mathcal{P}\text{-sur } X$ whenever X is separable.

Example. Let $\mathcal{S} = \{X : X \text{ is } \sigma\text{-compact}\}$. Let $Z = B \times I^n$ where I is the unit interval and B is the space of irrational numbers. Then $\mathcal{S}\text{-Sur } Z = \mathcal{S}\text{-sur } Z = n$.

This is proved as follows. If K is any \mathcal{S} -kernel of Z , then the natural projection of K into B is σ -compact. Since B is not σ -compact, there is a point $q \in B$ such that $\{q\} \times I^n \cap K = \emptyset$. So $Z \setminus K \supset \{q\} \times I^n$ and $\text{Ind } Z \setminus K = n$.

2.3. **Definition of dimensions which neglect a family \mathcal{P} .** Let \mathcal{P} be a family of spaces. Following Lelek [8] we define *strong (weak) inductive invariants*, $\mathcal{P}\text{-Ind}$ ($\mathcal{P}\text{-ind}$), induced by \mathcal{P} as follows:

If X is a space, then

(1) $\mathcal{P}\text{-Ind } X = -1$ ($\mathcal{P}\text{-ind } X = -1$) if and only if $X \in \mathcal{P}$,

(2) for each integer $n \geq 0$, $\mathcal{P}\text{-Ind } X \leq n$ ($\mathcal{P}\text{-ind } X \leq n$) provided that each nonempty closed subset (each point) of X has arbitrarily small open neighborhoods U such that its boundary $B(U)$ has $\mathcal{P}\text{-Ind } B(U) \leq n - 1$ ($\mathcal{P}\text{-ind } B(U) \leq n - 1$).

For each integer $n \geq 0$, $\mathcal{P}\text{-Ind } X = n$ if $\mathcal{P}\text{-Ind } X \leq n$ and $\mathcal{P}\text{-Ind } X \not\leq n - 1$. If $\mathcal{P}\text{-Ind } X \not\leq n$ for each n , then $\mathcal{P}\text{-Ind } X = \infty$. Similarly, $\mathcal{P}\text{-ind } X = n$, $n = 0, 1, 2, \dots, \infty$, are defined.

The requirement that the closed set be nonempty in condition (2) of the definition of $\mathcal{P}\text{-Ind } X$ is very important in order to cover the cases where $\emptyset \notin \mathcal{P}$. See the proof of Proposition 3 in §2.4.

We will call the function $\mathcal{P}\text{-Ind}$ ($\mathcal{P}\text{-ind}$) defined on the class \mathfrak{M} of metrizable spaces the *strong (weak) inductive dimension which neglects \mathcal{P}* .

In the sequel $\mathcal{P}\text{-Ind } X$ and $\mathcal{P}\text{-ind } X$ will be computed for various spaces X and classes \mathcal{P} . For more examples see [13].

In a manner analogous to $\mathcal{P}\text{-Ind}$, we next define a third type of dimension which neglects \mathcal{P} . This definition uses σ -locally finite bases, an important concept in dimension theory of metrizable spaces (see [11, Theorem II.2]). If \mathcal{P} is a family of spaces then we define the *basic dimension which neglects \mathcal{P}* , $\mathcal{P}\text{-Bind}$, as follows:

If X is a space, then

(1) $\mathcal{P}\text{-Bind } X = -1$ if and only if $X \in \mathcal{P}$,

(2) for each integer $n \geq 0$, $\mathcal{P}\text{-Bind } X \leq n$ if there exists a σ -locally finite open basis \mathcal{B} of X such that for every $U \in \mathcal{B}$ its boundary $B(U)$ has $\mathcal{P}\text{-Bind } B(U) \leq n - 1$.

$\mathcal{P}\text{-Bind } X = n$, $n = 0, 1, \dots, \infty$, are defined as usual.

When $\mathcal{P} = \{\emptyset\}$, we will drop the prefixes of $\mathcal{P}\text{-Ind}$, $\mathcal{P}\text{-ind}$ and $\mathcal{P}\text{-Bind}$. It is well known that $\text{Bind} = \text{Ind} \geq \text{ind}$.

2.4. **Remarks and propositions.** The study of inductive invariants chiefly aims at (1) generalizing dimension theory, (2) finding out what part of dimension theory is due to the inductive nature of the definition, and (3) discovering an interplay of the inductive invariants. These points will become evident as our development unfolds.

Proposition 1. *For any class \mathcal{P} the functions \mathcal{P} -Sur, \mathcal{P} -sur, \mathcal{P} -Def, \mathcal{P} -def, \mathcal{P} -Ind, \mathcal{P} -ind and \mathcal{P} -Bind are topologically invariant.*

Proposition 2. \mathcal{P} -ind \leq \mathcal{P} -Ind, \mathcal{P} -sur \leq \mathcal{P} -Sur and \mathcal{P} -def \leq \mathcal{P} -Def.

Example 1. Let $\mathcal{K} = \{X: X \text{ is compact}\}$. For the real line R we have \mathcal{K} -Sur $R = \mathcal{K}$ -sur $R = \mathcal{K}$ -Ind $R = 1$ and \mathcal{K} -def $R = \mathcal{K}$ -Def $R = \mathcal{K}$ -ind $R = \mathcal{K}$ -Bind $R = 0$. \mathcal{K} -ind X has been denoted by $\text{cmp } X$ —the compactness degree of X —in earlier works ([4], [5]). The conjecture of de Groot [4] referred to in the introduction is the following:

Conjecture. \mathcal{K} -def $X \leq n$ if and only if X is separable and $\text{cmp } X \leq n$.

The conjecture is true for $n \leq 0$. It remains open for all other n [5].

Example 2. Let $\mathcal{C} = \{X: X \text{ is topologically complete}\}$. Then \mathcal{C} -Ind X has been denoted by $\text{Icd } X$ —the strong inductive completeness degree of X —in [1]. The following theorem is proved in [1].

Theorem. $\text{Icd } X \leq n$ if and only if \mathcal{C} -Def $X \leq n$.

Example 3. Let \mathcal{S} be as in the example in 2.2. Then \mathcal{S} -ind X has been denoted by $\sigma\text{-cmp } X$ in [12]. It has been conjectured there that \mathcal{S} -def $X \leq n$ if and only if X is separable and $\sigma\text{-cmp } X \leq n$. We will prove this conjecture false in §6 below.

Proposition 3. (1) *If \mathcal{P} and \mathcal{Q} are two families with $\mathcal{P} \subset \mathcal{Q}$ then \mathcal{P} -Ind \geq \mathcal{Q} -Ind, \mathcal{P} -ind \geq \mathcal{Q} -ind, \mathcal{P} -Bind \geq \mathcal{Q} -Bind, \mathcal{P} -Def \geq \mathcal{Q} -Def, \mathcal{P} -def \geq \mathcal{Q} -def, \mathcal{P} -Sur \geq \mathcal{Q} -Sur and \mathcal{P} -sur \geq \mathcal{Q} -sur.*

(2) $\emptyset\text{-ind } X = \text{ind } X + 1$, $\emptyset\text{-Ind } X = \text{Ind } X + 1 = \emptyset\text{-Bind } X$.

(3) $\emptyset \in \mathcal{P}$ if and only if \mathcal{P} -Ind \leq Ind if and only if \mathcal{P} -ind \leq ind if and only if \mathcal{P} -Bind \leq Ind.

(4) *If $\emptyset \notin \mathcal{P}$, then \mathcal{P} -Ind $X = 0$ if and only if $X = \emptyset$.*

Proof. We shall only indicate a proof of (4). For more details see [2] and [13]. If $X = \emptyset$, then \mathcal{P} -Ind $X \leq 0$ because there are no nonempty closed sets in \emptyset . Since $\emptyset \notin \mathcal{P}$, \mathcal{P} -Ind $X = 0$. If $X \neq \emptyset$, then, since X is both open and closed, we must have $\emptyset \in \mathcal{P}$ in case \mathcal{P} -Ind $X \leq 0$.

Since the case $\mathcal{P} = \emptyset$ is completely determined, we will in the remaining sections always assume $\mathcal{P} \neq \emptyset$. This, of course, is not the same as $\emptyset \in \mathcal{P}$.

Finally, we remark that ideas related to inductive invariants have occurred in the literature prior to the works cited above. We refer the reader to [7, p. 187], and [16, p. 96].

3. Properties of \mathcal{P} -Bind. In §2 we have already observed the relationship between strong and weak inductive dimensions which neglect \mathcal{P} . In this section we discuss the relationship of these two dimensions to \mathcal{P} -Bind.

3.1. **Proposition.** $(n \geq 0)$ $\mathcal{P}\text{-ind } X \leq n$ if and only if either $X \in \mathcal{P}$ or there exists an open basis \mathcal{B} of X such that $\mathcal{P}\text{-ind } B(U) \leq n - 1$ for every $U \in \mathcal{B}$.

Theorem. $\mathcal{P}\text{-ind} \leq \mathcal{P}\text{-Bind}$.

The proof is a straightforward induction on $\mathcal{P}\text{-Bind } X$.

3.2. **Definition of a framework.** In many proofs it will be convenient to introduce the following notion of framework.

The pair $(\{U_\gamma: \gamma \in \Gamma\}, \{F_\gamma: \gamma \in \Gamma\})$ is called a *framework of a σ -locally finite basis of X* if

- (1) $\{U_\gamma: \gamma \in \Gamma\}$ ($\{F_\gamma: \gamma \in \Gamma\}$) is an open (closed) collection;
- (2) $\emptyset \neq F_\gamma \subset U_\gamma$ for every $\gamma \in \Gamma$;
- (3) if $\{V_\gamma: \gamma \in \Gamma\}$ is an open collection such that $F_\gamma \subset V_\gamma \subset U_\gamma, \gamma \in \Gamma$, then it is a σ -locally finite basis of X .

Proposition. In every space there exists a framework of a σ -locally finite basis.

3.3. **Lemma.** If $0 \leq \mathcal{P}\text{-Ind } X \leq n$, then there exists a σ -locally finite open basis \mathcal{B} of X such that $\mathcal{P}\text{-Ind } B(V) \leq n - 1$ for every $V \in \mathcal{B}$.

The lemma easily follows from the definitions of a framework (observe that $\emptyset \neq F_\gamma$ in (2)) and of $0 \leq \mathcal{P}\text{-Ind } X \leq n$.

From this lemma we obtain by induction on $\mathcal{P}\text{-Ind } X$ the following theorem.

Theorem. $\mathcal{P}\text{-Bind} \leq \mathcal{P}\text{-Ind}$.

3.4. **Theorem.** If X is separable, then $\mathcal{P}\text{-ind } X = \mathcal{P}\text{-Bind } X$.

Proof. In view of the theorem of §3.1 we need only prove $\mathcal{P}\text{-Bind } X \leq \mathcal{P}\text{-ind } X$. This holds for $\mathcal{P}\text{-ind } X = -1$. The induction on $\mathcal{P}\text{-ind } X$ is the same as in [11, pp. 90–91], where we must delete the empty set from the constructed basis \mathcal{B} when $\mathcal{P}\text{-ind } X = 0$, since \emptyset need not be a member of \mathcal{P} .

3.5. **Proposition.** Let $\{B_\gamma: \gamma \in \Gamma\}$ be a σ -locally finite closed collection of subsets of a space X . Let, for each $\gamma \in \Gamma$, $\{B_\gamma^\delta: \delta \in \Delta_\gamma\}$ be a σ -locally finite (closed) collection of the subspace B_γ . Then $\{B_\gamma^\delta: \delta \in \Delta_\gamma, \gamma \in \Gamma\}$ is a σ -locally finite (closed) collection of subsets of X .

Theorem. Let $\mathcal{P}\text{-Bind } X \leq n$. Then there exists a σ -locally finite closed collection \mathcal{C} of subsets of X such that

- (1) $\mathcal{C} \subset \mathcal{P}$,
- (2) $\text{Ind}(X \setminus \bigcup \mathcal{C}) \leq n$.

Proof. If $\mathcal{P}\text{-Bind } X = -1$, then let $\mathcal{C} = \{X\}$.

Let $n \geq 0$ and assume the theorem true whenever $\mathcal{P}\text{-Bind } X \leq n - 1$. Let X

be any space with \mathcal{P} -Bind $X = n$. Then there is a σ -locally finite open basis \mathcal{B} of X such that $U \in \mathcal{B}$ implies \mathcal{P} -Bind $B(U) \leq n - 1$. By virtue of the induction hypothesis for each such $B(U)$ there is a σ -locally finite closed collection \mathcal{C}_U of subsets of $B(U)$ satisfying (1) $\mathcal{C}_U \subset \mathcal{P}$ and (2) $\text{Ind}(B(U) \setminus \bigcup \mathcal{C}_U) \leq n - 1$. The above proposition gives $\mathcal{C} = \bigcup \{\mathcal{C}_U : U \in \mathcal{B}\}$ is a σ -locally finite closed collection. Since $B(U) \setminus \bigcup \mathcal{C}$ is closed in $X \setminus \bigcup \mathcal{C}$, the sum theorem [11, Theorem II.1] gives $\text{Ind}(\bigcup \{B(U) : U \in \mathcal{B}\} \setminus \bigcup \mathcal{C}) \leq n - 1$.

Also, $\text{Ind}(X \setminus \bigcup \{B(U) : U \in \mathcal{B}\}) \leq 0$. Hence by [11, II.1.C], we have $\text{Ind}(X \setminus \bigcup \mathcal{C}) \leq n$.

4. Monotone conditions. We devote this section to those theorems which result under some monotone conditions on \mathcal{P} .

4.1. Definition. A family \mathcal{P} is said to be *monotone* if $Y \subset X \in \mathcal{P}$ implies $Y \in \mathcal{P}$. In the event that the implication holds only for closed subspaces Y of X then we say \mathcal{P} is *c-monotone*. Similar concepts result for F_σ , G_δ and open subspaces Y of X . We will prefix monotone with the symbols F_σ , G_δ and *o* in the respective cases.

An extended real valued function f on \mathfrak{M} is said to be *monotone* if $f(X) \leq f(Y)$ whenever $X \subset Y$. *c-monotone*, F_σ -*monotone*, G_δ -*monotone* and *o-monotone* are similarly defined.

4.2. Theorem 1. *A family \mathcal{P} is monotone if and only if \mathcal{P} -Bind is monotone. Corresponding statements hold for closed, F_σ and G_δ -monotone. Analogous statements hold for \mathcal{P} -ind.*

The proof of this theorem is a straightforward induction (cf. [13, Theorem 3.3]).

Remark. If \mathcal{P} is *c-monotone*, then the proposition of §3.1 is greatly simplified (“either $X \in \mathcal{P}$ or” can be dropped).

Theorem 2. *A family \mathcal{P} is c-monotone if and only if \mathcal{P} -Ind is c-monotone.*

4.3. Proposition. $(n \geq 0)$ *Suppose $\emptyset \in \mathcal{P}$. Let A be a subset of X with $0 \leq \mathcal{P}\text{-Ind } A \leq n$. Then for any disjoint closed sets F and G of X there exists an open set U such that $F \subset U \subset X \setminus G$ and $\mathcal{P}\text{-Ind}(B(U) \cap A) \leq n - 1$.*

In case \mathcal{P} is *c-monotone*, the conditions $0 \leq \mathcal{P}\text{-Ind } A$ and $\emptyset \in \mathcal{P}$ can be dropped.

Proof. There are open sets V and W satisfying $F \subset V$, $G \subset W$ and $\bar{V} \cap \bar{W} = \emptyset$. Because $0 \leq \mathcal{P}\text{-Ind } A \leq n$ and $\emptyset \in \mathcal{P}$, there exists an open subset D of A satisfying $\bar{V} \cap A \subset D \subset A \setminus \bar{W}$ and $\mathcal{P}\text{-Ind } B_A(D) \leq n - 1$.

Neither of the disjoint sets $F \cup D$ and $G \cup (A \setminus \bar{D})$ contains a cluster point of the other. Hence there is an open set U such that $F \cup D \subset U \subset X \setminus \overline{B_A(D)}$ and $\bar{U} \cap (G \cup (A \setminus \bar{D})) = \emptyset$. A simple calculation yields $B_A(D) = B(U) \cap A$.

If \mathcal{P} is c -monotone, then $B(U) \cap A \in \mathcal{P}$ if $\mathcal{P}\text{-Ind } A = -1$ and $\emptyset \in \mathcal{P}$ (recall, we have agreed that $\mathcal{P} \neq \emptyset$).

4.4. **Theorem.** *Suppose \mathcal{P} is c -monotone. Let $X = A \cup B$ with $\mathcal{P}\text{-Ind } A \leq n$ and $\text{Ind } B \leq 0$. Then $\mathcal{P}\text{-Ind } X \leq n + 1$.*

Proof. Let F and G be any disjoint closed sets in X . By Proposition 4.3 there exists an open set U satisfying $F \subset U \subset X \setminus G$ and $\text{Ind } B(U) \cap B = -1$. Hence $B(U) \subset A$ which implies $\mathcal{P}\text{-Ind } B(U) \leq n$ in view of Theorem 2 in §4.2.

Corollary. *Suppose \mathcal{P} is c -monotone. Let $X = A \cup B$ with $\mathcal{P}\text{-Bind } A \leq n$ and $\text{Ind } B \leq 0$. Then $\mathcal{P}\text{-Bind } X \leq n + 1$.*

4.5. **Theorem.** *Suppose \mathcal{P} is c -monotone. Then $\mathcal{P}\text{-Ind} \leq \mathcal{P}\text{-Sur}$.*

Proof. The theorem obviously holds for $\mathcal{P}\text{-Sur } X = -1$. Assume the theorem holds for X with $\mathcal{P}\text{-Sur } X \leq n - 1$. Suppose $\mathcal{P}\text{-Sur } X \leq n$. Let Y be a \mathcal{P} -kernel with $\text{Ind}(X \setminus Y) \leq n$. By virtue of Proposition 4.3, for any disjoint closed subsets of X , there exists an open set U such that $F \subset U \subset X \setminus G$ and $\text{Ind}(B(U) \cap (X \setminus Y)) \leq n - 1$. Since \mathcal{P} is c -monotone, $Y \cap B(U)$ is a \mathcal{P} -kernel of $B(U)$. Thus $\mathcal{P}\text{-Sur } B(U) \leq n - 1$. So, by the induction hypothesis, $\mathcal{P}\text{-Ind } B(U) \leq n - 1$. That is, $\mathcal{P}\text{-Ind } X \leq n$.

4.6. **Theorem.** *Suppose \mathcal{P} is c -monotone. Then $\mathcal{P}\text{-Bind} \leq \mathcal{P}\text{-Def}$.*

Proof. The theorem is obviously true for $\mathcal{P}\text{-Def } X = -1$. Assume the theorem holds for all X with $\mathcal{P}\text{-Def } X \leq n - 1$. Suppose $\mathcal{P}\text{-Def } X \leq n$. That is, there is a \mathcal{P} -hull Y of X with $\text{Ind}(Y \setminus X) \leq n$. Let $(\{U_\gamma; \gamma \in \Gamma\}, \{F_\gamma; \gamma \in \Gamma\})$ be a framework of a σ -locally finite basis of Y . For every $\gamma \in \Gamma$, an open set V_γ is selected such that $\text{Ind}(B_Y(V_\gamma) \cap (Y \setminus X)) \leq n - 1$. Since $B_Y(V_\gamma)$ is a \mathcal{P} -hull of $B_Y(V_\gamma) \cap X$, the induction hypothesis gives $\mathcal{P}\text{-Bind}(B_Y(V_\gamma) \cap X) \leq n - 1$. Now $B_X(V_\gamma \cap X)$ is closed in $B_Y(V_\gamma) \cap X$. The c -monotonicity of \mathcal{P} implies $\mathcal{P}\text{-Bind } B_X(V_\gamma \cap X) \leq n - 1$. $\{V_\gamma \cap X; \gamma \in \Gamma\}$ is the required base.

4.7. **Theorem.** *Suppose \mathcal{P} is c -monotone and o -monotone. Then $\mathcal{P}\text{-Ind} \leq \mathcal{P}\text{-Def}$.*

Proof. The proof is similar to that of the preceding theorem. Observe that, if F and G are disjoint closed subsets of X and if Z is any \mathcal{P} -hull of X , then $Y = Z \setminus (c1_Z F \cap c1_Z G)$ is another \mathcal{P} -hull of X in which F and G have disjoint closures. Obviously, $\text{Ind } Y \setminus X \leq \text{Ind } Z \setminus X$.

Remark. As follows from Example 1 in §2.4 the condition that \mathcal{P} be o -monotone cannot be dropped.

4.8. **Theorem.** (1) *$\mathcal{P}\text{-Sur}$ and $\mathcal{P}\text{-sur}$ are monotone if and only if \mathcal{P} is monotone. The corresponding statements hold for closed, F_σ , G_δ and o -monotone.*

(2) If \mathcal{P} is c -monotone, then $\mathcal{P}\text{-Def}$ and $\mathcal{P}\text{-def}$ are c -monotone. The corresponding statements hold for F_σ , G_δ and o -monotone.

(3) If \mathcal{P} is c -monotone, then $\mathcal{P}\text{-def } X = \mathcal{P}\text{-Def } X$ whenever X is separable.

The easy proofs are omitted.

5. **Sum theorems.** In this section we shall develop sum theorems which are analogues of the sum theorem of dimension theory. Besides the condition of c -monotone, we necessarily will have to assume a sum closure for the class \mathcal{P} . We formalize these two conditions on \mathcal{P} as S1 and S2:

S1. \mathcal{P} is c -monotone.

S2. \mathcal{P} is closed for taking of unions of locally countable closed collections from \mathcal{P} .

It is a simple computation to show S1 and S2 are equivalent to S1 and

S'2. \mathcal{P} is closed for taking of unions of σ -locally finite closed collections from \mathcal{P} .

A family which satisfies S1 and S2 is related to normal families of Hurewicz [6] and Morita [10]. Normal families require \mathcal{P} to be monotone as well as S2.

Under the conditions of S1 and S2 we will establish the equivalence of the notions of $\mathcal{P}\text{-Ind}$ and $\mathcal{P}\text{-Sur}$. Using $\mathcal{P}\text{-Sur}$ we shall prove the sum theorem and, in §7, the decomposition theorem for $\mathcal{P}\text{-Ind}$. Thus, the importance of \mathcal{P} -kernels for strong inductive invariants becomes evident (see Theorem 5.1).

5.1. **The excision lemma.** Suppose \mathcal{P} satisfies condition S'2. If $\mathcal{P}\text{-Bind } X \leq n$, then there exists an F_σ \mathcal{P} -kernel Y of X with $\text{Ind } (X \setminus Y) \leq n$.

Proof. Suppose $\mathcal{P}\text{-Bind } X \leq n$. By virtue of the Theorem in 3.5 there exists a σ -locally finite closed collection \mathcal{C} of subsets of X such that $\text{Ind } (X \setminus \bigcup \mathcal{C}) \leq n$ and $\mathcal{C} \subset \mathcal{P}$. Clearly, $\bigcup \mathcal{C}$ is F_σ . Condition S'2 gives $\bigcup \mathcal{C} \in \mathcal{P}$.

Theorem. Suppose \mathcal{P} satisfies conditions S1 and S2. Then $\mathcal{P}\text{-Bind} = \mathcal{P}\text{-Ind} = \mathcal{P}\text{-Sur}$. Furthermore each X has an F_σ \mathcal{P} -kernel Y with $\text{Ind } (X \setminus Y) = \mathcal{P}\text{-Sur } X$.

Proof. Theorems 3.3 and 4.5 give $\mathcal{P}\text{-Bind} \leq \mathcal{P}\text{-Ind} \leq \mathcal{P}\text{-Sur}$. The excision lemma gives $\mathcal{P}\text{-Sur} \leq \mathcal{P}\text{-Bind}$. Furthermore, the \mathcal{P} -kernel of the lemma is F_σ in X .

Corollary 1. Suppose \mathcal{P} satisfies conditions S1 and S2. Then $\mathcal{P}\text{-Ind } X \leq n$ if and only if $X = \bigcup \{X_i : i = 0, 1, \dots, n\}$ for some $n+1$ subsets X_i with $\mathcal{P}\text{-Ind } X_0 \leq 0$ and $\text{Ind } X_i \leq 0$, $i = 1, \dots, n$.

From Theorem 4.8 (1) we obtain

Corollary 2. Suppose \mathcal{P} satisfies S1 and S2. Then $\mathcal{P}\text{-Ind}$ is o -monotone.

Example 1. In Corollary 1, $\text{Ind } X_i \leq 0$, $i = 1, \dots, n$, cannot be replaced by

\mathcal{P} -Ind $X_i \leq 0$. The sufficiency fails as the family $\mathcal{D} = \{X: \text{Ind } X \leq 0\}$ shows. Consider the three-dimensional Euclidean space X . Then \mathcal{D} -Ind $X = 2$. Since X is the union of two subsets of dimension one, X is the union of two subsets with \mathcal{D} -Ind ≤ 0 .

Example 2. Let $\mathcal{A}(1)$ be the family of absolute Borel sets of additive class 1; that is, the family of σ -locally compact spaces [15]. Then $\mathcal{A}(1)$ satisfies conditions S1 and S2. This follows easily since locally compact subsets of a space X are F_σ sets in X .

Example 3. Let \mathcal{C} be the class of topologically complete spaces (Example 2 in §2.4). \mathcal{C} does not satisfy condition S2. We shall exhibit a space X with \mathcal{C} -Ind $X < \mathcal{C}$ -Sur X . Let $X = I \times Q \cup Q \times I$, where $I = \{t: 0 \leq t \leq 1\}$ and $Q = \{t: t \in I \text{ and } t \text{ is rational}\}$. (X is regarded as a subspace of $I \times I$.) Since X is of the first Baire category, $X \notin \mathcal{C}$. That is, \mathcal{C} -Ind $X \geq 0$.

In view of the theorem mentioned in Example 2, §2.4, we have \mathcal{C} -Ind $X = \mathcal{C}$ -Def $X \leq \text{Ind}(I \times I \setminus X) = 0$. Hence \mathcal{C} -Ind $X = 0$.

In view of §2.4, Proposition 3, we have \mathcal{C} -Sur $X \leq \{\emptyset\}$ -Sur $X = \text{Ind } X = 1$. Since each dense subspace of a space of the first Baire category is again of the first Baire category, every \mathcal{C} -kernel of X is not dense in X . Hence \mathcal{C} -Sur $X \geq 1$ because $\text{Ind } U = 1$ for every nonempty open subset U of X .

5.2. Definition. An extended real valued function on the class \mathfrak{M} of metrizable spaces is said to satisfy the *sum condition* if, for each X , $f(X) \leq \sup\{f(E_\gamma): \gamma \in \Gamma\}$ whenever $\{E_\gamma: \gamma \in \Gamma\}$ is a nonempty locally countable closed covering of X .

Proposition. If \mathcal{P} -Sur satisfies the sum condition and $\emptyset \in \mathcal{P}$, then \mathcal{P} satisfies condition S2. The converse holds if \mathcal{P} also satisfies the condition S1.

Proof. Let $\mathcal{C} = \{E_\gamma: \gamma \in \Gamma\}$ be a locally countable closed cover of X and $\mathcal{C} \subset \mathcal{P}$. Either $\mathcal{C} = \emptyset$ or $\mathcal{C} \neq \emptyset$. If $\mathcal{C} = \emptyset$, then $\bigcup \mathcal{C} = \emptyset \in \mathcal{P}$. Suppose $\mathcal{C} \neq \emptyset$. Then $-1 \leq \mathcal{P}$ -Sur $\bigcup \mathcal{C} \leq \sup\{\mathcal{P}$ -Sur $E_\gamma: \gamma \in \Gamma\} \leq -1$. Hence $X = \bigcup \mathcal{C} \in \mathcal{P}$. Thereby the first statement is proved.

Suppose \mathcal{P} satisfies S1 and S2. $\emptyset \in \mathcal{P}$ in view of S1. Let $\{E_\gamma: \gamma \in \Gamma\}$ be a nonempty locally countable closed covering of X . By virtue of the theorem in §5.1, for each $\gamma \in \Gamma$ there is an F_σ \mathcal{P} -kernel G_γ of E_γ such that $\text{Ind } E_\gamma \setminus G_\gamma = \mathcal{P}$ -Sur E_γ . For each $\gamma \in \Gamma$ the set $G_\gamma = \bigcup\{G_\gamma^i: i = 1, 2, \dots\}$ where each G_γ^i is closed in X . By condition S1 we have $G_\gamma^i \in \mathcal{P}$, $\gamma \in \Gamma$, $i = 1, 2, \dots$. Since $\{G_\gamma^i: \gamma \in \Gamma, i = 1, 2, \dots\}$ is a locally countable closed collection we have $G = \bigcup\{G_\gamma^i: \gamma \in \Gamma, i = 1, 2, \dots\} \in \mathcal{P}$ by virtue of condition S2. Now the sum theorem of dimension theory gives $\text{Ind } \bigcup\{E_\gamma \setminus G: \gamma \in \Gamma\} \leq \sup\{\text{Ind}(E_\gamma \setminus G): \gamma \in \Gamma\} \leq \sup\{\text{Ind } E_\gamma \setminus G_\gamma: \gamma \in \Gamma\} = \sup\{\mathcal{P}$ -Sur $E_\gamma: \gamma \in \Gamma\}$. Consequently \mathcal{P} -Sur $X \leq \sup\{\mathcal{P}$ -Sur $E_\gamma: \gamma \in \Gamma\}$.

The proposition above is the sum theorem for \mathcal{P} -Sur. Combined with Theorem 5.1 this sum theorem gives an easy proof of the following theorem, which has already been proved in a more complicated way [2].

Theorem 1. *The sum theorem for \mathcal{P} -Ind. Suppose \mathcal{P} satisfies conditions S1 and S2. Let $\{F_\gamma: \gamma \in \Gamma\}$ be a locally countable closed covering of a space X such that \mathcal{P} -Ind $F_\gamma \leq n$ for $\gamma \in \Gamma$. Then \mathcal{P} -Ind $X \leq n$, i.e., \mathcal{P} -Ind satisfies the sum condition.*

Example. Let \mathcal{L} be the class of all spaces X with the property:

$$\text{Ind } X = \sup\{\text{Ind } C: C \text{ is a compact subset of } X\}.$$

This property was first discussed by Lelek [8]. The class \mathcal{L} does not satisfy condition S1. Nonetheless, it can be proved that \mathcal{L} -Sur satisfies the sum condition and in view of the proposition above, \mathcal{L} satisfies the condition S2. Theorems 3.5 and 3.3 now result in \mathcal{L} -Sur \leq \mathcal{L} -Bind \leq \mathcal{L} -Ind. Since $\mathcal{L} \supset \mathcal{U}(1)$ (see §5.1, Example 2) we have $\mathcal{U}(1)$ -Ind \geq \mathcal{L} -Ind.

The following theorem is an extension of Hurewicz's dimension lowering map theorem.

Theorem 2. *Let $f: X \rightarrow f(X)$ be an arbitrary continuous map. Then $\text{Ind } X \leq \text{Ind } f(X) + \text{Ind } f + \mathcal{L}$ -Sur $X + 1$.*

Proof. Let Y be an \mathcal{L} -kernel of X such that $\text{Ind } X \setminus Y = \mathcal{L}$ -Sur X . Then $\text{Ind } X \leq \text{Ind } Y + \mathcal{L}$ -Sur $X + 1$. For each compact subset C of Y we have $\text{Ind } C \leq \text{Ind } f(X) + \text{Ind } f$ by Hurewicz's theorem.

Because of Theorem 3.4, we have for separable X , \mathcal{L} -Sur $X \leq$ \mathcal{L} -Bind $X =$ \mathcal{L} -ind X . Consequently, the above theorem is an improvement of the corresponding theorem of Lelek [8], proved for separable X , where \mathcal{L} -Sur X is replaced by \mathcal{L} -ind X . To see that \mathcal{L} -Sur \neq \mathcal{L} -Bind, consider a separable space $X = X_1 \cup X_2$, where X_1 and X_2 are disjoint closed sets, X_1 is compact, $\text{Ind } X_1 = 1$, X_2 is totally imperfect and $\text{Ind } X_2 = 2$. (A space is *totally imperfect* if each compact subset is countable.) Then \mathcal{L} -Sur $X = 0$ since X_2 is the union of a one-dimensional subset and a zero-dimensional subset. Also, \mathcal{L} -ind $X = 1$. This follows easily from the fact that $\text{ind } Y - 1 = \mathcal{L}$ -ind Y for separable totally imperfect spaces Y with $\text{ind } Y \geq 0$.

5.3. S_σ -spaces. In this section we show that the inductive invariant analogue of a theorem of Morita [10] remains true under our conditions S1 and S2.

A space X is called an S_σ -space if X is the countable union of closed subspaces with the star-finite property.

Theorem. *Let \mathcal{P} satisfy conditions S1 and S2. If X is an S_σ -space, then \mathcal{P} -ind $X = \mathcal{P}$ -Ind X .*

Proof. X is the countable union of closed subspaces X_i with the star-finite property. Hence, to prove $\mathcal{P}\text{-ind } X \geq \mathcal{P}\text{-Ind } X$ it is enough to prove $\mathcal{P}\text{-ind } Y \geq \mathcal{P}\text{-Ind } Y$ for each space Y with the star-finite property, because $\mathcal{P}\text{-ind}$ is c -monotone and $\mathcal{P}\text{-Ind}$ satisfies the sum condition. The proof given in [9, Theorem 5.2], for spaces Y with the star-finite property, that $\text{ind } Y \geq \text{Ind } Y$, uses only the c -monotone property of ind and Ind and the sum property of Ind . Consequently, the very same proof can be used in the present situation.

6. σ -compactness deficiency. In this section we give a negative solution to the problem posed by Nagata (§2.4, Example 3). Let \mathcal{S} denote the class of σ -compact spaces (see the example in §2.2 and Example 3 in §2.4). Let $\mathcal{A}(1)$ denote the class of σ -locally compact spaces (see Example 2 in §5.1). Since $\mathcal{S} \subset \mathcal{A}(1)$, by Proposition 3 in §2.4 we have $\mathcal{S}\text{-Ind} \geq \mathcal{A}(1)\text{-Ind}, \dots, \mathcal{S}\text{-sur} \geq \mathcal{A}(1)\text{-sur}$ for the seven functions defined in §§2.2 and 2.3.

Since every σ -locally compact *separable* space is also σ -compact we have for each separable space X : $\mathcal{S}\text{-Ind } X = \mathcal{A}(1)\text{-Ind } X, \dots, \mathcal{S}\text{-sur } X = \mathcal{A}(1)\text{-sur}$ for the seven functions.

From the proposition in §2.2, Theorem 3.3 and Theorem 5.1, it follows that for every separable space X : $\mathcal{S}\text{-Ind } X = \mathcal{S}\text{-Bind } X = \mathcal{S}\text{-ind } X = \mathcal{S}\text{-Sur } X = \mathcal{S}\text{-sur } X$. In view of Theorem 4.8 for every separable space X we have $\mathcal{S}\text{-Def } X = \mathcal{S}\text{-def } X$. Then, by virtue of Theorem 4.7 we have $\mathcal{S}\text{-Ind } X \leq \mathcal{S}\text{-def } X$ for every separable space X . We shall show that this inequality can be a strict inequality for some separable space X . Thus we give a negative solution to Nagata's problem.

6.1. Example. Here we present an example of a separable space X with $\mathcal{S}\text{-Ind } X = 0$ and $\mathcal{S}\text{-def } X = 1$.

Let I denote the unit interval, $Q = \{t : t \in I \text{ and } t \text{ is rational}\}$ and $B = I \setminus Q$. X is the subspace of $I \times I \times I$ given by $X = (I \times I \times Q) \cup (B \times B \times B)$.

Since $I \times I \times Q$ is σ -compact we have $\mathcal{S}\text{-Ind } X \leq 0$ by Theorem 4.4 (there is also a direct proof for $\mathcal{S}\text{-ind } X \leq 0$).

$\mathcal{S}\text{-def } X \leq \text{ind}(I \times I \times I) \setminus X = 1$. We shall show $\mathcal{S}\text{-def } X \geq 1$. That is, if Y is an \mathcal{S} -hull of X , then $\text{ind}(Y \setminus X) \geq 1$. We may assume X is dense in Y . Observe that $B \times B \times B$ is an absolute G_δ (i.e., a topologically complete space) which is dense in Y . Recall that a space which contains a dense subset of the second Baire category is a space of the second Baire category. Since Y is a countable union of compact subsets, there is a compact subset Z of Y with nonvoid interior. So, $Z \cap X$ contains a topological copy of X , say W , which is closed in X . Since $\text{ind } Y \setminus X \geq \text{ind } Z \setminus X \geq \text{ind}(c1_Z W) \setminus W$, we need only prove $\text{ind}(c1_Z W) \setminus W \geq 1$. This will follow if we prove the compactness deficiency of W exceeds 0, that is $K\text{-def } W \geq 1$ (see Example 1 in §2.4). In view of Theorem 4.8 we need only find a closed subset T of X with $K\text{-def } T \geq 1$, since K is c -monotone. Let $T =$

$\{0\} \times I \times Q$. T is closed in X and $\mathcal{K}\text{-def } T \geq 1$. The equality $\mathcal{K}\text{-def } T = 1$ is proved in [5]. A more direct proof of $\mathcal{K}\text{-def } T \geq 1$ can be obtained from the equality $\mathcal{K}\text{-ind } T = \mathcal{K}\text{-Bind } T$ (Theorem 3.4), the inequality $\mathcal{K}\text{-Bind } T \leq \mathcal{K}\text{-Def } T$ (Theorem 4.6), and the equality $\mathcal{K}\text{-def } T = \mathcal{K}\text{-Def } T$, provided $\mathcal{K}\text{-ind } T = 1$. To prove $\mathcal{K}\text{-ind } T = 1$ we need only prove $\mathcal{K}\text{-ind } I \times Q = 1$. Let $f: I \times Q \rightarrow Q$ be the natural projection. For every sufficiently small open neighborhood U of $(0, 0) \in I \times Q$, $f(B)$ is a neighborhood of 0 in Q , where B is the boundary of U . Hence B is not compact. That is, $1 \leq \mathcal{K}\text{-ind } I \times Q$. Because $\mathcal{K}\text{-ind } I \times Q \leq \text{Ind } I \times Q = 1$, we have $\mathcal{K}\text{-ind } I \times Q = 1$.

Observe that $\mathcal{S}\text{-def } X \geq 1$ implies $\mathcal{S}\text{-ind } X > -1$. Consequently, $0 \leq \mathcal{S}\text{-ind } X \leq \mathcal{S}\text{-Ind } X \leq 0 < 1 \leq \mathcal{S}\text{-def } X \leq 1$, and the example has the required properties.

7. The decomposition theorem. We have already commented in Example 1 of §5.1 that a decomposition theorem for $\mathcal{P}\text{-Ind}$ cannot hold in general. We will show $\mathcal{P}\text{-Sur}$ satisfies a decomposition theorem under mild conditions.

7.1. Proposition. *If $\mathcal{P}\text{-Sur } X \leq n$, then $X = \bigcup\{X_i: i = 0, 1, \dots, n\}$ such that $\mathcal{P}\text{-Sur } X_i \leq 0$, $i = 0, \dots, n$.*

This proposition follows immediately from the definition of $\mathcal{P}\text{-kernel}$ and $\mathcal{P}\text{-Sur}$.

7.2. Definition. A family \mathcal{P} is said to be *additive* if $X = A \cup B$, $A \in \mathcal{P}$ and $B \in \mathcal{P}$ imply $X \in \mathcal{P}$.

Proposition. *\mathcal{P} is additive if and only if $\mathcal{P}\text{-Sur}(A \cup B) \leq \mathcal{P}\text{-Sur } A + \mathcal{P}\text{-Sur } B + 1$ for any subsets A and B of any space X .*

Corollary 1. *Let \mathcal{P} be an additive family. Then $\mathcal{P}\text{-Sur } X \leq n$ if and only if $X = \bigcup\{X_i: i = 0, \dots, n\}$ with $\mathcal{P}\text{-Sur } X_i \leq 0$, $i = 0, 1, \dots, n$.*

Theorem. *Let \mathcal{P} be a family which satisfies conditions S1 and S2 and is additive. Then $\mathcal{P}\text{-Ind } X \leq n$ if and only if $X = \bigcup\{X_i: i = 0, \dots, n\}$ with $\mathcal{P}\text{-Ind } X_i \leq 0$, $i = 0, \dots, n$.*

Corollary 2. *Let \mathcal{P} be as in the theorem above. Then $\mathcal{P}\text{-Ind}(A \cup B) \leq \mathcal{P}\text{-Ind } A + \mathcal{P}\text{-Ind } B + 1$.*

8. The extension theorem. We give here sufficient conditions under which $\mathcal{P}\text{-Def} = \mathcal{P}\text{-Ind}$. §6 shows that equality does not hold in general.

8.1. Proposition. *Let \mathcal{P} be an additive family. Suppose \mathcal{P} contains the class of topologically complete spaces. If $\mathcal{P}\text{-Sur } X \leq n$ then in every topologically complete space Y which contains X there is a $\mathcal{P}\text{-bull } Z$ of X satisfying*

- (1) $X \subset Z \subset Y$,
- (2) $\text{Ind}(Z \setminus X) \leq n$.

Proof. Let Y be topologically complete and $Y \supset X$. There exists a \mathcal{P} -kernel A of X such that $\text{Ind } X \setminus A \leq n$. By virtue of a theorem of Tumarkin ([11, Theorem II.10]) there exists a G_δ subset B of Y such that $X \setminus A \subset B$ and $\text{Ind } B \leq n$. Since B is topologically complete, $B \in \mathcal{P}$. The additivity of \mathcal{P} gives $A \cup B \in \mathcal{P}$. Then $Z = A \cup B$ is a \mathcal{P} -hull of X contained in Y and $\text{Ind } (Z \setminus X) \leq \text{Ind } B \leq n$.

The extension theorem. *Suppose \mathcal{P} satisfies conditions S1 and S2 and is additive. Suppose \mathcal{P} contains the class of topologically complete spaces. Then $\mathcal{P}\text{-Ind} = \mathcal{P}\text{-Def}$.*

Proof. Corollary 2 of §5.1 and Theorem 4.7 give $\mathcal{P}\text{-Ind} \leq \mathcal{P}\text{-Def}$. The preceding proposition and Theorem 5.1 complete the proof.

The difference theorem. *Let \mathcal{P} be as in the extension theorem. Then $\mathcal{P}\text{-Ind } X \leq n$ if and only if there is a \mathcal{P} -hull Y and a \mathcal{P} -kernel Z of X with $\text{Ind } Y \setminus Z \leq n$.*

Proof. Sufficiency follows from Theorem 5.1. To prove the necessity we observe that the extension theorem gives a \mathcal{P} -hull Y such that $\text{Ind } Y \setminus X \leq n$. We infer from Tumarkin's theorem [11, Theorem II.10] that there is an F_σ subset Z of Y with $Y \setminus X \subset Y \setminus Z$ and $\text{Ind } Y \setminus Z \leq n$. Since $Y \in \mathcal{P}$ and \mathcal{P} is F_σ monotone the necessity follows.

Example. For every ordinal number α , $2 \leq \alpha < \Omega$, let $\mathcal{A}(\alpha)$ denote the family of absolute Borel sets of additive class α . X is said to be an *absolute Borel set of additive class α* if X is a Borel set of additive class α in every (metric) space Y which contains X . It is a classical theorem that $X \in \mathcal{A}(\alpha)$ if (and only if) X is a Borel set of additive class α in some complete space Y which contains X [7]. It is not hard to show that $\mathcal{A}(\alpha)$ is additive, satisfies S1 and contains the class of complete spaces. For the nontrivial condition S2 see [3].

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