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ENVELOPES OF HOLOMORPHY AND HOLOMORPHIC CONVEXITY

BY
ROBERT CARMIGNANI

ABSTRACT. This paper is primarily a study of generalized notions of envelope of holomorphy and holomorphic convexity for special (algebraically restricted) subsets of $\mathbf{C}^n$ and in part for arbitrary subsets of $\mathbf{C}^n$. For any special set S in $\mathbf{C}^n$, we show that every function holomorphic in a neighborhood of S not only can be holomorphically continued but also holomorphically extended to a neighborhood in $\mathbf{C}^n$ of a maximal set $\tilde{S}$, the "envelope of holomorphy" of S, which is also a special set of the same type as S. Formulas are obtained for constructing $\tilde{S}$ for any special set S. "Holomorphic convexity" is characterized for these special sets. With one exception, the only topological restriction on these special sets is connectivity. Examples are given which illustrate applications of the theorems and help to clarify the concepts of "envelope of holomorphy" and "holomorphic convexity."

Introduction. Using Rossi's representation in [10] of the envelope of holomorphy of an arbitrary Riemann domain as the spectrum of a Fréchet algebra, Harvey and Wells introduce in [6] the notion of *envelope of holomorphy* of an arbitrary subset of a Stein manifold which is characterized as a projective limit of Stein manifolds. In this paper we derive, since envelopes of holomorphy are unique up to biholomorphism, an equivalent construction of *envelopes of holomorphy* of arbitrary subsets of $\mathbf{C}^n$ via H. Cartan's construction in [4] of the envelope of holomorphy of an arbitrary Riemann domain, although we could have extended this construction to arbitrary subsets of Riemann domains. It should be pointed out that both Cartan's and Rossi's constructions depend on Oka's solution to the Levi problem in [9] to prove that the envelope of holomorphy is a Stein manifold. Also, the characterization of the envelope of holomorphy of a Riemann domain as a projective limit of Stein manifolds is similar to the notion mentioned by Bremermann ("... for the proper notion of intersection ...") in [3, p. 176] and in a subsequent paper we show that this can be done.

In §II we construct the *envelopes* of *holomorphy* of *Reinhardt sets* and *tube sets* in $\mathbf{C}^n$ and *complete Hartogs sets* in $\mathbf{C}^2$; with the exception of *tube* sets the only topological restriction is connectivity. Also, we prove that each of these special sets has a *schlicht envelope of holomorphy*, that is, the *envelope of holomorphy* can be represented as a subset of $\mathbf{C}^n$. Without connectivity we could show holomor-

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phic continuation to a maximal subset of $\mathbf{C}^n$ but not in a univalent manner. Except for *Reinhardt* domains which intersect the variety $\{z_1 \cdot z_2 \cdots z_n = 0\}$ and do not contain the origin, the construction of the envelopes of holomorphy of *Reinhardt* domains and *tube* domains in $\mathbf{C}^n$ and *complete Hartogs* domains in $\mathbf{C}^2$ can be found in [12]. The lemmas of §I construct the envelope of holomorphy of any *Reinhardt* domain. Using this and by showing that the projective limit reduces to an intersection of domains of holomorphy, we construct in §II the *envelope of holomorphy*, $\tilde{S}$, of any compact connected *Reinhardt set* S and then obtain $\tilde{S}$ for connected *Reinhardt sets*. In a similar fashion we handle *complete Hartogs sets* in $\mathbf{C}^2$. We show the extension property for polygonally connected *tube sets*, but for maximality we require that the base of the *tube* be compact. We prove that if S is connected and is a *Reinhardt set*, *complete Hartogs set* in $\mathbf{C}^2$, or a polygonally connected *tube set* with compact base, then S is *holomorphically convex* ($S = \tilde{S}$) if and only if S is an intersection of domains of holomorphy. This is not true in general.

§III consists of eight examples. In particular, 3.1 is an example of a domain in $\mathbf{C}^2$ whose envelope of holomorphy is not spread over a domain of holomorphy.

Capital letters are used to indicate theorems which can be found in the literature, the exceptions being 1.7 and 1.10. A different proof of 1.7 appears in [12, p.174]. In the same reference [p. 182] essentially the same proof is given for 1.10.

$\complement A$ is a notation for the complement of A, A/B is the set $A \cap \complement B$, and $A \subset\subset B$ means $\overline{A}$ is a compact subset of B.

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I. Basic definitions and lemmas.

Definition 1.1. A subset $T \subset \mathbf{C}^n$ is called a *tube set* if there exists a subset $\tau \subset \mathbf{R}^n$ such that

$$T = \tau + i\mathbf{R}^n \quad (i\mathbf{R}^n = \{i(x_1, x_2, \ldots, x_n) : (x_1, \ldots, x_n) \in \mathbf{R}^n\},$$

τ is called the base of the *tube* T).

Let Exp be the map: $\mathbf{C}^n \to \mathbf{C}^n$ defined by $\operatorname{Exp}(z_1, \ldots, z_n) = (e^{z_1}, \ldots, e^{z_n})$.

Definition 1.2. A subset $S \subset \mathbf{C}^n$ is called a *Reinhardt set* if $(z_1, \ldots, z_n) \in S$ implies $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in S$ for arbitrary real $\theta_1, \theta_2, \ldots, \theta_n$.

Definition 1.3. Let V be the complex analytic variety defined by

$$V = \{(z_1, \ldots, z_n) : z_1 \cdot z_2 \cdot z_3 \cdots z_n = 0\}.$$

If T is a *tube set* then $\operatorname{Exp} T = \{(e^{t_1}, \ldots, e^{t_n}) : (t_1, \ldots, t_n) \in T\}$ is a *Reinhardt set* such that $(\operatorname{Exp} T) \cap V = \varnothing$ and if S is a *Reinhardt set* then $\operatorname{Exp}^{-1}\{S\}$ is a *tube set* with base $\mathbf{R}^n \cap \operatorname{Exp}^{-1}\{S\} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : (e^{x_1}, \ldots, e^{x_n}) \in S\}$.

Definition 1.4. A *Reinhardt set* S is said to be *logarithmically convex* if $\operatorname{Exp}^{-1}\{S\}$ is the intersection of open convex sets. The *logarithmic hull* of S, $\hat{S}^{\log}$, is the intersection of all open *logarithmically convex Reinhardt sets* containing S.

The following lemma for open convex sets in $\mathbf{R}^n$ is the analog of Theorem 2.6.11, p.48, Hörmander [7] for pseudoconvex open sets in $\mathbf{C}^n$.

Lemma 1.5. *If Ω is an open convex set in $\mathbf{R}^n$, then there exists a strictly convex function $f \in C^\infty(\Omega)$ such that*:

(1) $\Omega_c = \{x \in \Omega : f(x) < c\} \subset \subset \Omega$ *for all* $c \in \mathbf{R}$.

(2) $\Omega = \bigcup_{k=0}^{\infty} \Omega_k$, $\Omega_0 \neq \varnothing$ *and* $(\nabla f) \neq 0$ *on* $\partial\Omega_k$ *for each* $k = 0, 1, 2, \ldots$, *where* $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ *and* $\partial\Omega_k$ *is the boundary of* Ω_k.

Remark. Sard's theorem shows that Lemma 1.5 is still valid if we delete the words convex and strictly convex.

For a detailed account of the analogies of geometric and complex convexity see Bremermann [2].

Definition 1.6. An open set $S \subset \mathbf{C}^n$ is called a domain of holomorphy if there are no open sets S_1 and S_2 in $\mathbf{C}^n$ with the following properties:

(a) $\varnothing \neq S_1 \subset S_2 \cap S$.

(b) S_2 is connected and not contained in S.

(c) For every $f \in \mathcal{O}(S)$ there is a function $f_2 \in \mathcal{O}(S_2)$ such that $f = f_2$ in S_1.

Lemma 1.7. *If S is an open logarithmically convex Reinhardt set such that $S \cap V = \varnothing$ (V is the variety of Definition* 1.3), *then S is a domain of holomorphy.*(1)

Proof. One could prove this lemma using the theorem of Cartan and Thullen; however here we prove it via the Levi condition and Lemma 1.5.

Let $\Omega = \mathbf{R}^n \cap \operatorname{Exp}^{-1}\{S\}$; then Ω is an open convex set in $\mathbf{R}^n$. Take f to be the function in Lemma 1.5 and define the functions ρ_k $(k = 0, 1, 2, \ldots)$ as follows:

$$\rho_k(z_1, \ldots, z_n) = \begin{cases} f(\log|z_1|, \ldots, \log|z_n|) - k, & \text{for } z \text{ in } S, \\ \delta(z, S) + 1, & \text{for } z \text{ in } \mathrm{C}S \end{cases}$$

(where δ is the Euclidean distance function in $\mathbf{C}^n$). Because $f \in C^\infty(\Omega)$ we have $\rho_k \in C^\infty(S)$. Let $S_k = \{z : \rho_k(z) < 0\}$; then $S = \bigcup_{k=0}^{\infty} S_k$. Since $\nabla\rho_k = \frac{1}{2}((1/z_1)\partial f/\partial x_1, \ldots, (1/z_n)\partial f/\partial x_n)$ we have that $\nabla\rho_k \neq 0$ on $\partial S_k = \{z : \rho_k(z) = 0\}$ for each $k = 0, 1, 2, \ldots$. It is easy to verify that $\partial^2\rho_k/\partial z_i \partial\bar{z}_j = (1/4)(1/z_i) \cdot (1/\bar{z}_j)\partial^2 f/\partial x_i \partial x_j$ for all $z \in S_k$. Since $f \in C^\infty(\Omega)$ is strictly convex, its Hessian at each point in Ω is a positive definite Hermitian operator on $\mathbf{R}^n$. Hence its extension must be positive definite on $\mathbf{C}^n$. We obtain the following:

$$\sum_{i,j=1}^{n} \frac{\partial^2\rho_k}{\partial z_i \partial\bar{z}_j}(z) w_i \bar{w}_j = \frac{1}{4} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\log|z_1|, \ldots, \log|z_n|) \frac{w_i}{z_i}\left(\frac{\overline{w_j}}{\bar{z}_j}\right)$$
$$> 0 \quad \text{for all } w \text{ in } \mathbf{C}^n \text{ and } z \in \partial S_k.$$

(1) Lemma 1.7 can also be proved using the fact that the map Exp is a regular covering map.

So by the solution to the Levi problem S_k is pseudoconvex. Because $S_k \subset S_{k+1}$ ($k = 0, 1, 2, \dots$) and pseudoconvex domains are domains of holomorphy, the result follows.

Lemma 1.8. *If S is an open Reinhardt set, then the logarithmic hull of S, $\hat{S}^{\log}$, is open.*

Proof. Since S is open, $\mathrm{Exp}^{-1}\{S\}$ is an open *tube* in $\mathbf{C}^n$. Let $T = \mathrm{Exp}^{-1}\{S\}$, then $\mathrm{Exp}\, \hat{T} = \hat{S}^{\log}/V$. It follows that $\hat{S}^{\log}/V$ is open because Exp is a local biholomorphism from $\mathbf{C}^n$ into itself. Thus $\hat{S}^{\log} = \mathrm{Exp}\,(\hat{T}) \cup (S \cap V)$.

Definition 1.9. For S an open set in $\mathbf{C}^n$ denote by $\mathcal{O}(S)$ the space of holomorphic functions in S.

Lemma 1.10. *Let S be an open connected Reinhardt set such that $S \cap V = \varnothing$. Then for each $f \in \mathcal{O}(S)$ there exists a function $\hat{f} \in \mathcal{O}(\hat{S}^{\log})$ such that $\hat{f}/S = f$.*

Proof. We need the following theorem due to Bochner. For a proof see [1] or [7].

Theorem A. *If T is an open connected tube set, then for each $f \in \mathcal{O}(T)$ there exists $\tilde{f} \in \mathcal{O}(\hat{T})$, where $\hat{T}$ is the convex hull of T, such that $\tilde{f} = f$ on T.*

Let

$R = \{z : \exists\ i \text{ with } z_i + \bar{z}_i \le 0 \text{ and } z_i - \bar{z}_i = 0\}$ and
$\overline{\log}$ be the function: $\mathbf{C}^n - R \to \mathbf{C}^n$ defined by $\overline{\log}(z_1, \dots, z_n) = (\log z_1, \dots, \log z_n)$ where $\log z_i$ is the principal branch of logarithm for $i = 1, 2, \dots, n$.

If $f \in \mathcal{O}(S)$, where S is a *Reinhardt* domain such that $S \cap V = \varnothing$, then $f \circ \mathrm{Exp} \in \mathcal{O}(\mathrm{Exp}^{-1}\{S\})$ and if we let $T = \mathrm{Exp}^{-1}\{S\}$, by Bochner's theorem we know there exists $\tilde{f} \in \mathcal{O}(\hat{T})$ such that $\tilde{f}/T = f \circ \mathrm{Exp}$. Thus $\tilde{f} \circ \overline{\log} \in \mathcal{O}(\hat{S}^{\log}/R)$ can be extended to a single-valued function $\hat{f} \in \mathcal{O}(\hat{S}^{\log})$ (recall $S \cap V = \varnothing$) such that $\hat{f}|_{\hat{S}^{\log}/R} = \tilde{f} \circ \overline{\log}$. Because $\mathrm{Exp} \circ \overline{\log}$ extends to the identity map on $\mathbf{C}^n$ we have

$$\hat{f}\,|_{S/R} = \tilde{f} \circ \overline{\log}\,|_{S/R} = f \circ \mathrm{Exp} \circ \overline{\log}\,|_{S/R} = f\,|_{S/R}\,.$$

Consider the funtion $g = \hat{f}\,|_S - f \in \mathcal{O}(S)$; since S is connected and $g \equiv 0$ on the open set S/R, we have $g = \hat{f}\,|_S - f \equiv 0$ on S. This completes the proof.

For any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{R}^n$, let α_{i_k} be ≥ 0 $(0 \le k \le n)$ and $\alpha_{i_{j+1}}$ be < 0 $(k \le j \le n)$ such that for each k, α_{i_k} is one and only one component of α.

Lemma 1.11. *If C is any positive constant and $\alpha \neq 0$ is as above, then the following hyperbolic half-spaces are domains of holomorphy*:

$$H^{\alpha}_{C>} = \{z : |z_{i_1}|^{\alpha_{i_1}} \dots |z_{i_k}|^{\alpha_{i_k}} < C|z_{i_{j+1}}|^{-\alpha_{i_{j+1}}} \dots |z_{i_{n-k+j}}|^{-\alpha_{i_{n-k+j}}}\},$$

$$H^{\alpha}_{C<} = \{z : |z_{i_1}|^{\alpha_{i_1}} \dots |z_{i_k}|^{\alpha_{i_k}} > C|z_{i_{j+1}}|^{-\alpha_{i_{j+1}}} \dots |z_{i_{n-k+j}}|^{-\alpha_{i_{n-k+j}}}\}.$$

Note that if $\alpha_i \geq 0$ $(i = 1, 2, \ldots, n)$, $H^{\alpha}_{C>}$ is a complete *Reinhardt* domain.

Proof. We prove it for $H^{\alpha}_{C>}$; the case $H^{\alpha}_{C<}$ is proved similarly. Let

$$p(z) = \alpha_{i_1} \log|z_{i_1}| + \ldots + \alpha_{i_k} \log|z_{i_k}|$$

$$- \alpha_{i_{j+1}} \log\left|\frac{1}{z_{i_{j+1}}}\right| - \ldots - \alpha_{i_{n-k+j}} \log\left|\frac{1}{z_{i_{n-k+j}}}\right| - \log C;$$

then $H^{\alpha}_{C>} = \{z : p(z) < 0\}$ and we have that $p(z)$ is plurisubharmonic in $\Omega = \{z : |z_{i_{j+1}}| > 0 \text{ for } k \leq j < n\}$ since a linear combination with nonnegative coefficients of plurisubharmonic functions is plurisubharmonic. Thus because Ω is a domain of holomorphy, it follows from the following theorem that $H^{\alpha}_{C>}$ is a domain of holomorphy.

Theorem B. *If Ω is a domain of holomorphy and p is a plurisubharmonic function in Ω, then $\{z \in \Omega : p(z) < 0\}$ is a domain of holomorphy.*

For a proof see [12, p. 103].

Let P_i be the map defined by $P_i(z_1, \ldots, z_i, \ldots, z_n) = (z_1, \ldots, 0, \ldots, z_n)$.

Lemma 1.12. *Let S be an open connected Reinhardt set in $\mathbf{C}^n$; then*

$$\tilde{S} = \hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\}$$

is a domain of holomorphy.

Proof. First to show $\tilde{S}$ is open: $\hat{S}^{\log}$ is open by Lemma 1.8. Suppose there exists a point $z^\circ = (z_1^\circ, \ldots, z_{i-1}^\circ, 0, z_{i+1}^\circ, \ldots, z_n^\circ) \in S$ and let ξ be any element of $\hat{S}^{\log}$; then it follows by the *logarithmic convexity* of $\tilde{S}$ that there exists a polydisc $\{z : |z_j - \xi_j| < \delta, j = 1, \ldots, n\} \subset \hat{S}^{\log}$ such that the polydisc $\{z : |z_j - \xi_j| < \delta, j \neq i, |z_i| < |\xi_i| + \delta\} \subset \tilde{S}$. Thus for each point $P_i(\xi) \in P_i(\hat{S}^{\log})$ there exists a neighborhood of $P_i(\xi)$ contained in $\tilde{S}$. This shows $\tilde{S}$ is open.

We shall complete the proof of the lemma by induction over the dimension n. Since every open set in $\mathbf{C}$ is a domain of a holomorphy, it is true for $n = 1$. Suppose it has been proved for $n - 1$ dimensions. We shall prove it for n dimensions by invoking the following theorem of Cartan and Thullen:

Theorem C. *Let S be an open subset of $\mathbf{C}^n$; then to each $p \in \partial S$, there exists a function $f_p \in \mathcal{O}(S)$ which cannot be holomorphically continued to p if and only if S is a domain of holomorphy.*

Theorem C follows immediately from the definition of domain of holomorphy 1.6.

We need to consider the following three cases:

Case 1. Suppose $z^\circ \in \partial\tilde{S}$ and $z^\circ \cap V = \varnothing$; then $\text{Exp}^{-1}(z^\circ) \in \partial T$, where $T = \text{Exp}^{-1}(\tilde{S})$. So by the geometric form of the Hahn-Banach theorem and the *logarithmic convexity* of $\tilde{S}$ there exists a hyperbolic half-space $H^{\alpha}_{C>}$ (or $H^{\alpha}_{C<}$ as defined in Lemma 1.11) such that $\tilde{S} \subset H^{\alpha}_{C>}$ and $z^\circ \in \partial H^{\alpha}_{C>}$ ($\tilde{S} \subset H^{\alpha}_{C>}$ since

$\operatorname{Exp} T \subset H_{C>}^{\alpha}$). By Lemma 1.11, $H_{C>}^{\alpha}$ is a domain of holomorphy; hence by Theorem C there exists a function $F \in \mathcal{O}(H_{C>}^{\alpha})$ such that F cannot be continued to z°.

Case 2. Suppose $z^{\circ} \in \partial\tilde{S}$, $z_j^{\circ} = 0$ for some integer j $(1 \leq j \leq n)$ and $S \cap P_j(S) = \varnothing$. Let $F(z_1, \ldots, z_n) = 1/z_j$. Then $F \in \mathcal{O}(\tilde{S})$ and cannot be analytically continued to z°.

Case 3. Suppose $z^{\circ} \in \partial\tilde{S}$, $z_j^{\circ} = 0$ for some integer j $(1 \leq j \leq n)$ and $S \cap P_j(S) \neq \varnothing$. Here we use the inductive hypothesis that the lemma is true for $(n-1)$ dimensions. Let g_j be the map: $\mathbf{C}^n \to \mathbf{C}^{n-1}$ defined by

$$g_j(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n) = (z_1, \ldots, z_{j-1}, z_n, z_{j+1}, \ldots, z_{n-1})$$

and let f_j be the map from $\mathbf{C}^n$ into itself defined by

$$f_j(z_1, \ldots, z_j, \ldots, z_n) = (z_1, \ldots, z_n, \ldots, z_{n-1}, z_j).$$

Let π_n be the map from $\mathbf{C}^n$ onto $\mathbf{C}^{n-1}$ defined by

$$\pi_n(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}).$$

Then $g_j = \pi_n \circ f_j$ is an open map since the projection π_n is an open map and f_j is a biholomorphism. It is easy to see g maps *Reinhardt sets* onto *Reinhardt sets.* Since the restriction of g to $\mathbf{R}^n$ maps convex sets in $\mathbf{R}^n$ onto convex sets in $\mathbf{R}^{n-1}$, it follows that $g_j(\tilde{S})$ is *logarithmically convex.*

Let $P_i^{n-1} : \mathbf{C}^{n-1} \to \mathbf{C}^{n-1}$ be the map defined by

$$P_i^{n-1}(z_1, \ldots, z_i, \ldots, z_{n-1}) = (z_1, z_2, \ldots, 0, \ldots, z_{n-1}) \qquad (1 \leq i \leq n-1).$$

Suppose P_i^{n-1} has a fixed point in $g_j(\tilde{S})$; then there exists $z^{\circ} \in \tilde{S}$ such that $z_k = 0$ and the ith component of $g_j(z_1, \ldots, z_n)$ is z_k. Since $P_k(\tilde{S}) \subset \tilde{S}$ we have $P_i^{n-1}(g_j(\tilde{S})) \subset g_j(\tilde{S})$. Thus $g_j(\tilde{S})$ satisfies the conditions of the lemma and is $\subset \mathbf{C}^{n-1}$, so by our inductive hypothesis it is a domain of holomorphy. $g_i(\tilde{S}) \times \mathbf{C}$ is a domain of holomorphy in $\mathbf{C}^n$ since the product of domains of holomorphy is a domain of holomorphy. Since f_j is a biholomorphism of $\mathbf{C}^n$ onto itself $f_j(g_i(\tilde{S}) \times \mathbf{C})$ is a domain of holomorphy and contains $\tilde{S}$ since $(z_1, \ldots, z_n) \in \tilde{S} \Rightarrow f_j(z_1, \ldots, z_n, \ldots, z_{n-1}, \lambda) = (z_1, \ldots, \lambda, \ldots, z_n) \in f_j(g_i(\tilde{S}) \times \mathbf{C})\ \forall\ \lambda \in \mathbf{C}$. If z° were in $f_j(g_i(\tilde{S}) \times \mathbf{C})$, then $z^{\circ} = P_j(z^{\circ})$ would be in $\tilde{S}$. But this is impossible since $z^{\circ} \in \partial\tilde{S}$. It follows that $z^{\circ} \in \partial f_j(g_i(\tilde{S}) \times \mathbf{C})$ because every neighborhood of z° contains points of $\tilde{S}$. By Theorem C there exists a function $F \in \mathcal{O}(f_j(g_i(\tilde{S}) \times \mathbf{C}))$ such that F cannot be continued to z°.

Any point $z^{\circ} \in \partial\tilde{S}$ is as in Case 1, 2 or 3; thus by the sufficiency part of Theorem C, $\tilde{S}$ is a domain of holomorphy in $\mathbf{C}^n$. This completes the proof of Lemma 1.12.

Definition 1.13. Let S be an open set in $\mathbf{C}^n$ and $f \in \mathcal{O}(S)$; then f can be *holomorphically continued* to a point $z^\circ \in \partial S$ if there exist open sets S_1 and S_2 in $\mathbf{C}^n$ with $z^\circ \in S_2$ and S_2 connected such that (1) $\varnothing \neq S_1 \subset S \cap S_2$, (2) there exists $\tilde{f} \in \mathcal{O}(S_2)$ such that $f = \tilde{f}$ in S_1.

Definition 1.14. A subset $S \subset \mathbf{C}^n$ is called a *Hartogs set* in the ith component if $(z_1, \ldots, z_i, \ldots, z_n) \in S$ implies $(z_1, \ldots, e^{i\theta} z_i, \ldots, z_n) \in S$ for arbitrary real θ.

Theorem D. *If S is an open, connected Hartogs set in z_i and if there exists $z^\circ \in S$ such that $z_i^\circ = 0$, then every function in $\mathcal{O}(S)$ can be holomorphically continued to each point of $\tilde{S} = \{(z_1, \ldots, \lambda z_i, \ldots, z_n) : z \in S$ and $|\lambda| \leq 1\}$.*

See [12] for a proof of D.

Lemma 1.15. *Let S be an open connected Reinhardt set; then for each $f \in \mathcal{O}(S)$ there is a function $\tilde{f} \in \mathcal{O}(\tilde{S})$ where $\tilde{S}$ is as in Lemma 1.12 such that $f = \tilde{f}$ on S.*

Proof. Given $f \in \mathcal{O}(S)$, consider $f|_{S/V} \in \mathcal{O}(S/V)$, where V, recall, is the variety $\{z : z_1 z_2 \ldots z_n = 0\}$. By Lemma 1.10 there exists a function $f^1 \in \mathcal{O}(\hat{S}^{\log}/V)$ such that $f|_{S/V} = f^1$ on S/V. Define the function f^2 on $S \cup (\hat{S}^{\log}/V) = \hat{S}^{\log}$ as follows:

$$f^2(z) = \begin{cases} f^1(z), & z \in \mathcal{O}(\hat{S}^{\log}/V), \\ f(z), & z \in S. \end{cases}$$

This is permissible since $S \cap (\hat{S}^{\log}/V) = S/V$ and S/V is connected. The connectivity of S/V follows because V is a complex analytic subset of S and S is connected.

Thus $f^1 = f$ on S/V a connected open set; this implies $f^2 \in \mathcal{O}(\hat{S}^{\log})$ and $f = f^2$ on S.

Suppose P_j has a fixed point in S; then consider $\hat{S}^{\log}$ as a *Hartogs* domain in z_j. If $w \in P_j(\hat{S}^{\log})$ but $w \notin \hat{S}^{\log}$, it follows from the *logarithmic convexity* of $\hat{S}^{\log}$ that $w \in \partial\hat{S}^{\log}$ and so by Theorem D there exists an open connected set S_2 containing w, an open set S_1 such that $\varnothing \neq S_1 \subset \hat{S}^{\log} \cap S_2$, and a function $f_w \in \mathcal{O}(S_2)$ where $f_w = f^2$ on S_1. Since $\tilde{S}$ is a domain of holomorphy $S_2 \subset \tilde{S}$ and

$$S_2 \supset S_2 \cap \hat{S}^{\log} \subset S_2 \cap \left(\tilde{S}/ \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\}\right)$$

$$= S_2 \cap (\tilde{S}/V) = S_2/V$$

because $S_2 \subset \tilde{S}$.

We have that S_2/V is connected since S_2 is connected. $\overline{S_2/V} = \overline{S}_2$ implies $S_2 \cap \hat{S}^{\log}$ is connected because $S_2/V \subset S_2 \cap \hat{S}^{\log} \subset S_2$. Let

$$\tilde{f}_w(z) = \begin{cases} f^2(z), & z \in \hat{S}^{\log}, \\ f_w(z), & z \in S_2. \end{cases}$$

Since $S_2 \cap \hat{S}^{\log}$ is connected, $f_w = f^2$ on $S_2 \cap \hat{S}^{\log}$. Thus for each $w \in P_j(\hat{S}^{\log})$ where $1 \leq j \leq n$ and P_j has a fixed point in S we can extend $f_2 \in \mathcal{O}(\hat{S}^{\log})$ to a function $\tilde{f}_w \in \mathcal{O}(\hat{S}^{\log} \cup N_w)$ where N_w is a neighborhood of w contained in $\tilde{S}$ and $\hat{S}^{\log} \cap N_w$ is connected.

Define $\tilde{f}$ as follows. Let

$$\tilde{f}(z) = \begin{cases} f^2(z), & z \in \hat{S}^{\log}, \\ \tilde{f}_w(z), & z \in \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\}. \end{cases}$$

Suppose $w \in N_{w_1} \cap N_{w_2}$ where $f_{w_1} \in \mathcal{O}(\hat{S}^{\log} \cup N_{w_1})$ and $f_{w_2} \in \mathcal{O}(\hat{S}^{\log} \cup N_{w_2})$. Then we have

$$\tilde{S} \supset (\hat{S}^{\log} \cup N_{w_1}) \cap (\hat{S}^{\log} \cup N_{w_2}) = \hat{S}^{\log} \cup (N_{w_1} \cap N_{w_2}) \supset \tilde{S}/V$$

since $N_{w_1} \cup N_{w_2} \subset \tilde{S}$. $\tilde{S}$ being connected implies $\tilde{S}/V$ is connected, so $\hat{S}^{\log} \cup (N_{w_1} \cap N_{w_2})$ is connected because $\overline{\tilde{S}/V} = \overline{\tilde{S}}$. This means $f_{w_1} = f_{w_2}$ on $(\hat{S}^{\log} \cup \tilde{N}_{w_1}) \cap (\hat{S}^{\log} \cup N_{w_2})$. Thus $\tilde{f} \in \mathcal{O}(\tilde{S})$ and $f = f_{w_2}|_S = \tilde{f}|_S$.

II. Envelopes of holomorphy and holomorphic convexity.

2.1. *Open sets.* We now state for open subsets of $\mathbf{C}^n$ the fundamental existence theorem for envelopes of holomorphy first proved by Cartan [4] and Oka [9].

Theorem E. *Given any open subset $U \subset \mathbf{C}^n$ there exist*:

(1) *a Stein manifold $\tilde{U}$,*

(2) *a biholomorphism $\chi : U \to \chi(U) \subset \tilde{U}$ such that for each $f \in \mathcal{O}(U)$ one can find $\tilde{f} \in \mathcal{O}(\tilde{U})$ so that $\tilde{f} \circ \chi = f$ on U,*

(3) *a local biholomorphism $\pi : \tilde{U} \to \pi(\tilde{U}) \subset \mathbf{C}^n$ such that $\pi \circ \chi$ = identity on U (π is sometimes called the projection or spread of $\tilde{U}$ in $\mathbf{C}^n$).*

Furthermore, $\tilde{U}$ is unique up to biholomorphism.

Remarks. Uniqueness is due to B. Malgrange [8]. Also, $\pi(\tilde{U})$ need not be a domain of holomorphy. For an example see 3.1.

Definition 2.11. For U open in $\mathbf{C}^n$ any Stein manifold satisfying (2) and (3) in Theorem E is called the envelope of holomorphy of U, which we shall denote by $\tilde{U}$. If π is injective, then U is said to have a schlicht envelope of holomorphy and furthermore if U is connected then the domain of holomorphy $\pi(\tilde{U}) \subset \mathbf{C}^n$ is the schlicht envelope of holomorphy.

Theorem 2.12. *If S is an open connected Reinhardt set in $\mathbf{C}^n$, then S has a schlicht envelope of holomorphy $\tilde{S}$ given by*:

$$\tilde{S} = \hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\}.$$

Remark. In [5], it is proved that if K is a compact connected *Reinhardt* (*circled*) *set* and if int K meets $\{z_j = 0\}$ whenever K does, then K is holomorphically convex if and only if K is rationally convex. Also, the rationally convex hull of K is explicitly constructed. From this we can obtain the envelope of holomorphy of any *Reinhardt* domain. The author would like to thank the referee for pointing this out.

Proof. By Lemma 1.12, $\tilde{S}$ is a domain of holomorphy. The extension property is Lemma 1.15.

Definition 2.13. A *Hartogs set* S in z_i is said to be *complete* if $(z_1, \ldots, z_i, \ldots, z_n) \in S$ implies $(z_1, \ldots, \lambda z_i, \ldots, z_n) \in S$ for all $\lambda \in \mathbf{C}$ such that $|\lambda| \leq 1$.

Lemma 2.14. *If $S \subset \mathbf{C}^n$ is a complete Hartogs domain in z_n, then there exists an upper semicontinuous function g on $\pi_n(S)$ (where $\pi_n(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1})$) such that*

$$S = \{z : |z_n| < g(z_1, \ldots, z_{n-1}) : (z_1, \ldots, z_{n-1}) \in \pi_n(S)\}.$$

Proof. $g(z_1, \ldots, z_{n-1}) = \operatorname{Sup}_{(z_1, \ldots, z_n) \in S} |z_n|$ is the function.

Theorem F. *If $S \subset \mathbf{C}^2$ is a complete Hartogs domain in z_2, then S has a schlicht envelope of holomorphy $\tilde{S}$ given by*

$$\tilde{S} = \{z : |z_2| < e^{V(z_1)}, z_1 \in \pi_2(S)\}$$

where V is the least superharmonic majorant for $\ln g$ *in $\pi_{z_2}(S)$ (g the defining function for S given in Lemma* 2.14).

Proof. The extension property can be found in [12, p. 183]. The function P, defined by $P(z_1, z_2) = |z_2| - e^{V(z_1)}$, is plurisubharmonic in $\pi_{z_2}(S) \times \mathbf{C}$, a domain of holomorphy. Thus by Theorem B, $\tilde{S} = \{z : P(z) < 0\} \subset \pi_{z_2}(S) \times \mathbf{C}$ is a domain of holomorphy.

Theorem G. *If T is a tube domain, then T has a schlicht envelope of holomorphy, $\hat{T}$, the convex hull of T.*

Theorem G follows from Bochner's Theorem A and the fact that open convex sets are domains of holomorphy.

2.2. *Reinhardt sets.* The following is due to Cartan:

Theorem H. *If U_α and U_β are open subsets of $\mathbf{C}^n$ and $U_\alpha \subset U_\beta$, then there exists a local biholomorphism $\varphi_{\alpha,\beta}$ of $\tilde{U}_\alpha$ into $\tilde{U}_\beta$ such that the following diagram is commutative*:

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\quad i \quad} & U_\beta \\ \chi_\alpha \big\downarrow & & \big\downarrow \chi_\beta \\ \tilde{U}_\alpha & \xrightarrow[\varphi_{\alpha,\beta}]{} & \tilde{U}_\beta \end{array}$$

(*where i is the inclusion map*).

For a proof see [8, p. 36]. Let S be any subset of $\mathbf{C}^n$ and $\{U_\alpha : \alpha \in \mathcal{A}\}$ be the family of open neighborhoods of S, where $U_\alpha \subset U_\beta$ if and only if $\alpha > \beta$. If $\alpha > \beta > \gamma$, then it follows from Theorem H that $\varphi_{\beta,\gamma} = \varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta}$. Thus it is possible to define:

Definition 2.21. For S any subset of $\mathbf{C}^n$ and $\{U_\alpha : \alpha \in \mathcal{A}\}$ as above, we shall call the projective limit, $\tilde{S} = \varprojlim_\alpha \tilde{U}_\alpha$, the *envelope* of *holomorphy* of S. (Compare Harvey and Wells [6].)

The maps, $\chi_\alpha : U_\alpha \to \tilde{U}_\alpha$ and $\pi_\alpha : \tilde{U}_\alpha \to \pi_\alpha(\tilde{U}_\alpha) \subset \mathbf{C}^n$ defined in Theorem E induce maps

$$\chi : S \to \tilde{S} \quad \text{and} \quad \pi : \tilde{S} \to \bigcap_\alpha \pi_\alpha(\tilde{U}_\alpha) \subset \mathbf{C}^n$$

such that χ is injective and $\pi \circ \chi =$ identity on S because χ_α is injective and $\pi_\alpha \circ \chi_\alpha =$ identity on U_α for each $\alpha \in \mathcal{A}$.

By $f \in \mathcal{O}(S)$ we mean $\exists\ U_\alpha$ such that $f \in \mathcal{O}(U_\alpha)$. Then $\exists\ \tilde{f} \in \mathcal{O}(\tilde{U}_\alpha)$ that lifts f, i.e., $\tilde{f} \circ \chi_\alpha = f$ on U_α. Let p_α be projection from $\tilde{S}$ onto $\tilde{U}_\alpha$, then $\tilde{f} \circ p_\alpha$ lifts $f \in \mathcal{O}(S)$ to $\tilde{S}$.

Definition 2.22. S is said to be *holomorphically convex* if χ is surjective.

Definition 2.23. If π is injective, then S is said to have a *schlicht envelope of holomorphy*.

Remark. Theorems E and H are valid for Riemann domains so we could have extended these definitions to arbitrary subsets of Riemann domains. However, the spreads χ_α and hence the map χ need not be injective.

If $\{V_\beta : \beta \in \mathcal{B}\}$ is a fundamental system of open neighborhoods of S, then $\tilde{S}$ can be equivalently characterized as $\varprojlim_\beta \tilde{V}_\beta$. Furthermore, if each V_β is connected and has a schlicht envelope of holomorphy and Ω_β is the domain of holomorphy associated to V_β, then S has a *schlicht envelope of holomorphy* $\tilde{S} = \bigcap_\beta \Omega_\beta$. More precisely, we have the following:

Lemma 2.24. *If S is a connected Reinhardt set and $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a fundamental system of open connected Reinhardt neighborhoods of S, then S has a schlicht envelope of holomorphy $\tilde{S}$ given by*:

$$\tilde{S} = \bigcap_\alpha \tilde{U}_\alpha = \bigcap_\alpha \left(\hat{U}_\alpha^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{U}_\alpha^{\log}) : P_i \text{ has a fixed point in } U_\alpha\} \right)$$

$$= \hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\}.$$

Proof. By Theorem 2.12 each U_α has a schlicht envelope of holomorphy $\tilde{U}_\alpha = \hat{U}_\alpha^{\log} \cup \bigcup_{i=1}^n \{P_i(\hat{U}_\alpha^{\log}) : P_i$ has a fixed point in $U_\alpha\}$. Thus the map $\pi : \tilde{S} \to \mathbf{C}^n$ must be injective since each π_α is injective. This shows S has a *schlicht envelope of holomorphy*

$$\tilde{S} = \bigcap_{\alpha} \tilde{U}_\alpha = \bigcap_{\alpha} \left(\hat{U}_\alpha^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{U}_\alpha^{\log}) : P_i \text{ has a fixed point in } U_\alpha\} \right).$$

$\hat{S}^{\log}$ is by definition the intersection of all *logarithmically convex Reinhardt* domains containing S, and since $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a fundamental neighborhood system for S it follows that $\hat{S}^{\log} = \bigcap_\alpha \hat{U}_\alpha^{\log}$. For each $i = 1, \ldots, n$ we have $P_i(\bigcap_\alpha \hat{U}_\alpha^{\log}) = \bigcap_\alpha P_i(\hat{U}_\alpha^{\log})$ since P_i is a projection. It follows that

$$\begin{aligned} &\hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\} \\ &\quad = \left(\bigcap_{\alpha} \hat{U}_\alpha^{\log} \right) \cup \left(\bigcap_{\alpha} \left(\bigcup_{i=1}^{n} \{P_i(\hat{U}_\alpha^{\log}) : P_i \text{ has a fixed point in } U_\alpha\} \right) \right) \\ &\quad = \bigcap_{\alpha} \left(\hat{U}_\alpha^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{U}_\alpha^{\log}) : P_i \text{ has a fixed point in } U_\alpha\} \right) \end{aligned}$$

because if $z \in \hat{U}_\alpha^{\log}$ and z is not in $P_i(\hat{U}_\alpha^{\log})$ for some i and $\alpha \in \mathcal{A}$ such that P_i has a fixed point in U_α; and if z is not in $\hat{U}_\beta^{\log}$ and $z \in \bigcup_{i=1}^{n} \{P_i(\hat{U}_\beta^{\log}) : P_i$ has a fixed point in $U_\beta\}$, then we obtain a contradiction since i can be chosen so that P_i has a fixed point in S.

Lemma 2.25. *If K is a compact Reinhardt set in $\mathbf{C}^n$ and if $K_j = \{z : \delta(z,K) < 1/j\}$ $(j = 1,2,\ldots)$, then the family $\{K_j : j = 1,2,\ldots\}$ forms a fundamental neighborhood system of open connected Reinhardt neighborhoods of S.*

Proof. Since K is compact and δ is continuous it follows that given any $U \supset K$ there exists an integer N such that if $j \geq N$, then $K \subset K_j \subset U$. It remains to show K_j is a connected *Reinhardt set*. If $w \in K_j$, then there exists $k \in K$ such that $\delta(w,k) < 1/j$. It follows that $\{z : |z_i| = |w_i|$ for each $i = 1, \ldots, n\} \subset K_j$ since K is *Reinhardt*; that K_j is connected follows from the facts $\{tw + (1-t)k : t \in [0,1]\} \subset K_j$ and the union of connected sets with a point in common is connected. This completes the proof of the lemma.

In [5, p. 513] it is proved that $\tilde{K}$ is the projective limit of compact Hausdorff spaces. Hence $\tilde{K}$ is compact since the projective limit of compact spaces is compact.

Theorem 2.26. *If K is a compact connected Reinhardt set, then K has a schlicht envelope of holomorphy $\tilde{K}$ given by:*

$$\tilde{K} = \hat{K}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{K}^{\log}) : P_i \text{ has a fixed point in } K\}.$$

Moreover, $\tilde{K}$ has a fundamental neighborhood system of domains of holomorphy.

Proof. By Lemma 2.25, K has a fundamental neighborhood system of open connected *Reinhardt sets* K_j. Lemma 2.24 implies

$$\tilde{K} = \bigcap_{j=1} \tilde{K}_j = \hat{K}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{K}^{\log}) : P_i \text{ has a fixed point in } K\}.$$

By the preceding remark $\tilde{K}$ is compact, and since $\Omega_m = \bigcap_{j=1}^m \tilde{K}_j$ is a domain of holomorphy for each m we have that $\{\Omega_m : m = 1, 2, \ldots\}$ is a fundamental neighborhood system of domains of holomorphy for $\tilde{K}$.

Theorem 2.27. *If K is a compact connected Reinhardt set in $\mathbf{C}^n$, then the following are equivalent*:

(1) *K is holomorphically convex.*

(2) *K has a fundamental neighborhood system of domains of holomorphy.*

(3) *$K = \hat{K}^{\log} \cup \bigcup_{i=1}^n \{P_i(\hat{K}^{\log}) : P_i$ has a fixed point in $K\}$.*

Proof. (1)⇒(3). K being *holomorphically convex* means $\chi : K \to \tilde{K}$ is a bijection onto $\tilde{K}$; thus $\pi : \tilde{K} \to \mathbf{C}^n$ is injective and so it follows that K has a *schlicht envelope of holomorphy* $K = \tilde{K}$. By Theorem 2.26 we obtain the formula in (3).

(2)⇒(3). Let $\{\Omega_\beta : \beta \in \mathcal{B}\}$ be a fundamental neighborhood system of domains of holomorphy for K. By Lemma 2.25 for each Ω_β there exists K_j such that $K \subset K_j \subset \Omega_\beta$. The definition of domain of holomorphy (1.6) shows that $\tilde{K}_j \subset \Omega_\beta$. Thus $K = \bigcap_j \tilde{K}_j$ and from this, Lemma 2.24 implies

$$K = \hat{K}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{K}^{\log}) : P_i \text{ has a fixed point in } K\}.$$

(3)⇒(1). Theorem 2.26 and (3) together imply $K = \tilde{K}$. Thus $\chi : K \to \tilde{K}$ must be surjective, but this means K is *holomorphically convex*.

(3)⇒(2). This is clear.

For a point $p = (p_1, \ldots, p_n) \in S$, a *Reinhardt set*, let $K_p = S_{|p_1|} \times \ldots \times S_{|p_n|} = \{z : |z_i| = |p_i| \text{ for each } i = 1, \ldots, n\}$; then K_p is a compact connected *Reinhardt set* and $S = \bigcup_{p \in S} K_p$. From this we obtain:

Lemma 2.28. *If S is a connected Reinhardt set in $\mathbf{C}^n$, then S has a fundamental neighborhood system of Reinhardt domains.*

Proof. If U is an open set containing S, then $U \supset K_p$ for each $p \in S$. By Lemma 2.25 there exists a *Reinhardt* domain V_p such that $U \supset V_p \supset K_p$ for each $p \in S$. It follows that $U \supset \bigcup_{p \in S} V_p \supset \bigcup_{p \in S} K_p = S$. If $V = \bigcup_{p \in S} V_p$, then V is an open *Reinhardt set* because the union of *Reinhardt sets* is *Reinhardt* and each V_p is open. Since $\mathbf{C}^n$ is locally connected the component of V containing S is open. This completes the proof of the lemma.

Theorem 2.29. *If S is a connected Reinhardt set, then S has a schlicht envelope of holomorphy $\tilde{S}$ given by*:

$$\tilde{S} = \hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \textit{ has a fixed point in } S\}.$$

Moreover, $\tilde{S} = \bigcap_\alpha \tilde{U}_\alpha$, where each $\tilde{U}_\alpha$ is a domain of holomorphy.

Proof. By Lemma 2.28, S has a fundamental neighborhood system of open connected *Reinhardt sets* U_α each having a schlicht envelope of holomorphy $\tilde{U}_\alpha$. The result follows from Lemma 2.24.

Theorem 2.29.1. *If S is a connected Reinhardt set in* $\mathbf{C}^n$, *then the following are equivalent*:

(1) *S is holomorphically convex.*

(2) *S is an intersection of domains of holomorphy*

(3) $S = \hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i$ *has a fixed point in* $S\}$.

Proof. (1)⇒(3). This is proved just as in Theorem 2.27.

(2)⇒(3). Suppose $S = \bigcap_\alpha \mathcal{D}_\alpha$ where $\{\mathcal{D}_\alpha : \alpha \in \mathcal{A}\}$ is a family of domains of holomorphy, then by Lemma 2.28, S has a fundamental neighborhood system of open connected *Reinhardt sets* $\{R_\beta : \beta \in \mathcal{B}\}$, just as in the proof for the compact case we obtain $S = \bigcap_\beta \tilde{R}_\beta$. So by Lemma 2.24 we have

$$S = \hat{S}^{\log} \cup \bigcup_{i=1}^{n} \{P_i(\hat{S}^{\log}) : P_i \text{ has a fixed point in } S\}.$$

(3)⇒(1). The same proof as in Theorem 2.27 works here.

(3)⇒(2). This is clear.

2.3. *Hartogs sets.* Let $S \subset \mathbf{C}^2$ be a *complete Hartogs set* in z_2. For each $p = (p_1, p_2) \in S$, let $\{p_1\} \times D_{|p_2|} = \{z : z_1 = p_1 \text{ and } |z_2| \le |p_2|\}$; then $S = \bigcup_{p \in S} \{p_1\} \times D_{|p_2|}$. From the proof of Lemma 2.25 it follows that the sets $S_{p,j} = \{z : |z - p_1| < 1/j\} \times \{|z| < |p_2| + 1/j\}$ form a fundamental neighborhood system for $\{p_1\} \times D_{|p_2|}$. Because for each p in S and each positive integer j, $S_{p,j}$ is a *complete Hartogs* domain, and the union of *complete Hartogs sets* is a *complete Hartogs set* we have:

Lemma 2.31. *Every connected complete Hartogs set has a fundamental neighborhood system of complete Hartogs domains.*

Proof. By the remarks above one can find a fundamental neighborhood system of open *complete Hartogs sets*. The lemma now follows from the local connectivity of $\mathbf{C}^n$.

Theorem 2.32. *If $S \subset \mathbf{C}^2$ is a connected complete Hartogs set, then S has a schlicht envelope of holomorphy $\tilde{S}$ given by*:

$$\tilde{S} = \{(z_1, z_2) : |z_2| < \tilde{g}(z_1), z_1 \in \pi_{z_2}(S)\}$$
$$\cup \{(z_1, z_2) : |z_2| = \tilde{g}(z_1) \text{ and } z_1 \in \pi_{z_2}(S)\}$$

where $\tilde{g}(z) = \inf\{h(z) : \ln h$ *is superharmonic in a neighborhood of* $\pi_{z_2}(S)$ *and* $h \ge g\}$ (*g is the defining function for S given in Lemma* 2.14).

Proof. Theorem F and Lemma 2.31 imply S has a fundamental neighborhood system of open sets whose envelopes of holomorphy are schlicht. The result now follows just as in the *Reinhardt* case.

The proof of the following theorem is also similar to the *Reinhardt* case:

Theorem 2.33. *If S is a connected complete Hartogs set in $\mathbf{C}^2$, then the following are equivalent*:

(1) *S is holomorphically convex.*

(2) *S is an intersection of domains of holomorphy.*

(3) *$S = \tilde{S}$ as given in Theorem* 2.32.

Moreover, if S is compact, then S has a fundamental neighborhood system of domains of holomorphy.

2.4. *Tube sets.* A close examination of the proof of Theorem A as given in [7, p. 41] yields the following:

Theorem 2.41. *If T is a polygonally connected tube set, then for each function f holomorphic in a neighborhood of T one can find a function F holomorphic in a neighborhood of $\hat{T}$, the convex hull of T, such that $F = f$ on T.*

If a *tube set* is connected and has a compact base, then $\hat{T}$ is a closed convex set. Thus $\hat{T}$ is the intersection of domains of holomorphy because open convex sets are domains of holomorphy. This helps to establish the following:

Theorem 2.42. *If T is a polygonally connected tube set with base τ either open or compact in $\mathbf{R}^n$, then T has a schlicht envelope of holomorphy $\hat{T}$. Furthermore, the following are equivalent*:

(1) *T is holomorphically convex.*

(2) *T is an intersection of domains of holomorphy.*

(3) $T = \hat{T}$.

Proof. The preceding remark makes (2)$\Leftrightarrow$(3) obvious. Since $\pi(\hat{T}) = T$ we have that (2)$\Rightarrow$(1). Finally Theorem 2.41 shows that (1)$\Rightarrow$(3).

Because *tube sets* do not have fundamental neighborhood systems of open *tube sets* we do not obtain results for arbitrary *tube sets*. For an example see 3.6.

Remark. Just as in 2.21, one can define the notion of *envelope of meromorphy* of an arbitrary subset S of a Riemann domain Ω. Since the envelope of meromorphy and envelope of holomorphy of any Riemann domain $U \subset \Omega$ coincide, it follows that the *envolope of meromorphy* and *envelope of holomorphy* of S, also, coincide. Thus all the theorems and lemmas given here remain valid if we change the words holomorphic, holomorphically and *envelope of holomorphy* to meromorphic, meromorphically and *envelope of meromorphy* respectively whenever they appear.

III. **Examples.**

3.1. *A domain in $\mathbf{C}^2$ whose envelope of holomorphy is not spread over a domain of holomorphy.* Let $D = \{(z_1, z_2) : \exists\ \rho \in (\frac{1}{2}, 1)$ and $\exists\ t \in (0, 2n\pi)$ with $n > 1$ where $z_1 = \rho e^{it}$, and $-\frac{1}{2} + t < |z_2| < \frac{1}{2} + t\}$ and $\Omega = \{(z_1, z_2, z_3) : \frac{1}{2} < |z_1| < 1, 0 < \operatorname{Im} z_3 < 2n\pi$, and $|z_2| < \frac{1}{2} + \operatorname{Im} z_3\}$. Then the envelope of holomorphy, $\tilde{D}$, of D can be described as follows:

$$\tilde{D} = \{z \in \Omega : z_1 - e^{z_3} = 0\}.$$

Since Ω is a domain of holomorphy, $\tilde{D}$ is obviously Stein. To see the extension property note that $\tilde{D} = \{(z_1, z_2, z_3) : \exists\ \rho \in (\frac{1}{2}, 1)$ and $\exists\ t \in (0, 2n\pi)$ with $z_1 = \rho e^{it}$, $|z_2| < \frac{1}{2} + t$, and $z_3 = \log \rho + it\}$. An application of Theorem D for *Hartogs* domains shows each $f \in \mathcal{O}(D)$ extends to $\tilde{D}$.

Let $\pi(z_1, z_2, z_3) = (z_1, z_2)$; then $\pi(\tilde{D}) = \{(z_1, z_2) : \exists\ \rho \in (\frac{1}{2}, 1)$ and $\exists\ t \in (0, 2n\pi)$ with $z_1 = \rho e^{it}$, $|z_2| < f(t)$ where

$$f(t) = \begin{cases} \frac{1}{2} + 2(n-1)\pi, & \text{for } t \in [0, 2(n-1)\pi), \\ \frac{1}{2} + t, & \text{for } t \in [2(n-1)\pi, 2n\pi)\}. \end{cases}$$

By Theorem F it follows that the envelope of holomorphy of $\pi(\tilde{D})$ is

$$\{(z_1, z_2) : \tfrac{1}{2} < |z_1| < 1, |z_2| < \tfrac{1}{2} + 2n\pi\}$$

because $\pi(\tilde{D})$ is a *complete Hartogs* domain and the least superharmonic majorant of $\ln f$ is the constant function $\ln(\frac{1}{2} + 2n\pi)$. D was given by Thullen in [11, pp. 64–76] as an example of a domain having a nonschlicht envelope of holomorphy.

3.2. *A convex set $S \subset \mathbf{C}^2$ which is not holomorphically convex.* Let $S = \{(z_1, z_2) : |z_1| < 1$ and $|z_2| < \infty\} \cup \{(z_1, z_2) : |z_1| = 1$ and $|z_2| < 1\}$. S is a connected *Reinhardt set* and $\hat{S}^{\log} = \{(z_1, z_2) : |z_1| \leq 1$ and $|z_2| < \infty\} = D \times \mathbf{C}$ (where $D = \{z : |z| \leq 1\}$) because each open convex set containing $\{(\log|z_1|, \log|z_2|) : \log|z_1| < 0$ and $\log|z_2| < \infty\} \cup \{(\log|z_1|, \log|z_2|) : \log|z_1| = 0$ and $\log|z_2| < 0\}$ contains $\{(\log|z_1|, \log|z_2|) : \log|z_1| \leq 0$ and $\log|z_2| < \infty\}$.

Thus by Theorem 2.29 each f holomorphic in a neighborhood of S extends to a function holomorphic in a neighborhood of $\hat{S}^{\log}$.

3.3. *A closed lower dimensional set in $\mathbf{C}^2$ whose envelope of holomorphy is a domain of holomorphy which is not simply connected.* If

$$\begin{aligned} S = \{(z_1, z_2) &: e \leq |z_2| < e^2, 1 \leq |z_1| \text{ and } \log|z_1| = (1/(2 - \log|z_2|)) - 1\} \\ &\cup \{(z_1, z_2) : 1 < |z_2| \leq e, 1 \leq |z_1| \text{ and } \log|z_1| = (1/\log|z_2|) - 1\} \\ &\cup \{(z_1, z_2) : |z_2| = e, |z_1| \leq 1\}, \end{aligned}$$

then S is a connected *Reinhardt set*. So if $A = \{z : 1 < |z| < e^2\}$, then $\hat{S}^{\log} = \{(z_1, z_2) : 1 < |z_2| < e^2, z_1 \in \mathbf{C}\} = \mathbf{C} \times A$, a domain of holomorphy.

3.4. *A set in $\mathbf{C}^2$ whose envelope of holomorphy is neither open nor closed.* If $S = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}/\{z : |z_1| = 1, |z_2| = 1\}$, then S is a connected *Reinhardt set* such that $S = \hat{S}^{\log} \cup P_1(S) \cup P_2(S)$. Thus by Theorem 2.29 we have $\tilde{S} = S$.

3.5. *An open convex set whose closure does not have a fundamental neighborhood system of domains of holomorphy.* For $\rho > 0$ let $S_\rho = \{(z_1, z_2) : |z_1| < \rho\}$; then $\overline{S}_\rho = \{(z_1, z_2) : |z_1| \leq \rho \}$ is a *holomorphically convex Reinhardt set*, but $\overline{S}_\rho$ does not have a fundamental neighborhood system of domains of holomorphy since every open convex set containing $\{(\log|z_1|, \log|z_2|) : \log|z_1| \leq \log \rho\} = \{(x,y) : x \leq \log \rho\} \subset \mathbf{R}^2$ contains a uniform neighborhood of $\{(x,y) : x \leq \log \rho\}$. However, the variety $\lim_{\rho \to 0} \overline{S}_\rho = \{(z_1, z_2) : z_1 = 0\}$ has a fundamental neighborhood system of domains of holomorphy.

3.6. *A closed convex tube set which does not have a fundamental neighborhood system of domains of holomorphy.* Let $S = \mathbf{C} \times i\mathbf{R} = \{(z_1, z_2) : z_1 \in \mathbf{C}$ and $z_2 + \bar{z}_2 = 0\} = R + i\mathbf{R}^2$; then S is a connected *complete Hartogs set* and since any subharmonic function that is $-\infty$ on $\mathbf{R} \subset \mathbf{C}$ is $\equiv -\infty$, we have by Theorem 2.32 that every function holomorphic on a neighborhood of S extends to a function holomorphic on $\mathbf{C} \times U$, where U is an open set in $\mathbf{C}$ containing the real numbers. However, it is easy to find neighborhoods of $\mathbf{C} \times \mathbf{R}$ not containing $\mathbf{C} \times U$ for any U open in $\mathbf{C}$.

3.7. *A holomorphically convex set which is not the intersection of domains of holomorphy.* Let

$$S = \{(z_1, z_2) : |z_1| \leq 1, |z_2| = 0\} \cup \{(z_1, z_2) : z_1 = 0, |z_2| \leq 1\} \cup \bigcup_{i=1}^{\infty} \left\{(z_1, z_2) : \frac{1}{2^{i+1}} \leq |z_1| \leq \frac{1}{2^i}, |z_2| = 1 - \frac{1}{i}\right\}.$$

S is a compact *Reinhardt set*. Every domain of holomorphy containing S must contain $\{(z_1, z_2) : 1/2^{i+1} \leq |z_1| \leq 1/2^i, |z_2| \leq 1 - 1/i\}$. But S is *holomorphically convex* since each component of S is *holomorphically convex*. This example is due to J. E. Bjork.

3.8. *A logarithmically convex Reinhardt domain which is not a domain of holomorphy.* Let $S = \{(z_1, z_2) : |z_1| < 1, 0 < |z_2| < 1\} / \{(z_1, z_2) : z_1 = 0, |z_2| = 1/2\}$; then $\hat{S}^{\log} = S$, but $\tilde{S} = \{(z_1, z_2) : |z_1| < 1, 0 < |z_2| < 1\}$ by 2.12. Note that the extension property also follows from Hartogs theorem [7, p. 30] since S is connected and $\{(z_1, z_2) : z_1 = 0, |z_2| = 1/2\}$ is compact.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65201