

SOME STABLE RESULTS ON THE COHOMOLOGY OF THE CLASSICAL INFINITE-DIMENSIONAL LIE ALGEBRAS

BY

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ABSTRACT. In this paper we compute the cohomology of various classical infinite-dimensional Lie algebras generalizing results of Gel'fand-Fuks for the Lie algebra of all formal power series vector fields.

1. Let k be a field of characteristic zero and $x_1, x_2, \dots, y_1, y_2, \dots$, etc. indeterminants. For us the classical infinite-dimensional Lie algebras will mean simply the Lie algebras occurring on the following list.

(I) $_n$ *Formal power series vector fields.* This is just all expressions of the form $\sum_{i=1}^n f_i(\partial/\partial x_i)$, where $f_i \in k[[x_1, \dots, x_n]]$. The bracket is the usual Lie bracket.

(II) $_n$ *Divergence zero formal power series vector fields.* The subalgebra of (I) $_n$ consisting of all $\vec{f} = \sum f_i(\partial/\partial x_i)$ for which $\text{div } \vec{f} = \sum (\partial f_i / \partial x_i) = 0$.

(III) $_n$ *Divergence constant formal power series vector fields.* The subalgebra of (I) $_n$ consisting of all $\vec{f}$ for which $\text{div } \vec{f} \in k$.

(IV) $_{2n}$ *The Poisson algebra.* As a vector space this is $k[[x_1, \dots, x_n, y_1, \dots, y_n]]$, and the bracket operation is the Poisson bracket:

$$\{f, g\} = \sum \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial y_i}.$$

(V) $_{2n}$ *The Hamiltonian algebra.* The Poisson algebra over its one-dimensional center.

(VI) $_{2n}$ *The derivation algebra of the Hamiltonian algebra.* Since there is just one outer derivation, this contains the Hamiltonian algebra as a codimension one ideal.

(VII) $_{2n+1}$ *The contact algebra.* As a vector space this is $k[[x_1, \dots, x_n, y_1, \dots, y_n, z]]$. The bracket operation is the Lagrange bracket:

$$[f, g] = \{f, g\}_{x, y} + \frac{\partial f}{\partial z} \left(g - \sum y_i \frac{\partial g}{\partial y_i} \right) - \frac{\partial g}{\partial z} \left(f - \sum y_i \frac{\partial f}{\partial y_i} \right).$$

In this paper we will compute the cohomology of each of these algebras (with coefficients in k) for the range $0 \leq i \leq n$. It turns out that in this range the answer in each case is independent of n and rather simple. In certain cases our

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results can be considerably improved. For example Gel'fand and Fuks computed the entire cohomology ring of $(I)_n$ in [2]; and, with slight modifications, their computation also works for $(III)_n$. (See [7].) A complete picture of the range $0 \leq i \leq 2n$ has been obtained by the second author for all the above examples except $(II)_n$ and $(VII)_{2n+1}$. (See [7] and the remarks at the end of §4.) However, these improved results seem to require much more delicate arguments than ours. ⁽²⁾

2. Let L be a Lie algebra over k , ⁽³⁾ A an abelian subalgebra, and $S(A)$ the symmetric algebra over A . We note that L and L^* are both modules for $S(A)$. In fact A acts on both L and L^* by the adjoint representation; therefore, so does the universal enveloping algebra of A , which happens to be $S(A)$ since A is abelian.

We begin with the basic theorem:

Theorem 1. *Let L be a Lie algebra over k , A a finite-dimensional abelian subalgebra, and α an element of L . Assume that L^* is free as a module over $S(A)$. Assume also that α acts semisimply on L and that A is contained in a nonzero eigenspace. Then $H^i(L, k) = 0$ for $0 < i \leq \dim A$.*

Proof. Let $m = \dim A$, and consider the Koszul complex:

$$(2.1) \quad \begin{aligned} 0 \rightarrow S(A) \rightarrow S(A) \otimes A^* \rightarrow \dots \rightarrow S(A) \otimes \Lambda_k^m(A^*) \\ \rightarrow S(A)/S^+(A) \otimes \Lambda_k^m(A^*) \rightarrow 0 \end{aligned}$$

where $S^+(A)$ is the maximal ideal of $S(A)$ of elements of degree > 0 . Taking the tensor product over $S(A)$ with $\Lambda^i L^*$, which is free over $S(A)$, we get an exact sequence:

$$(2.2)_i \quad \begin{aligned} 0 \rightarrow \Lambda^i(L^*) \otimes A^* \xrightarrow{d} \dots \\ \xrightarrow{d} \Lambda^i L^* \otimes \Lambda^m A^* \rightarrow \Lambda^i L^*/S^+ \Lambda^i L^* \otimes \Lambda^m A^* \rightarrow 0. \end{aligned}$$

Let δ be the Hochschild coboundary operator for ΛL^* . We can amalgamate the complexes $(2.2)_i$ into a double complex, whose j th column is

$$0 \rightarrow \Lambda^0 L^* \otimes \Lambda^j A^* \xrightarrow{\delta} \dots \rightarrow \Lambda^k L^* \otimes \Lambda^j A^* \rightarrow \dots$$

We assume α is normalized so that it is $(-1)id$ on A . Then the operator d in $(2.2)_i$ maps $(\Lambda^i L^*)_a \otimes \Lambda^j A^*$ into $(\Lambda^i L^*)_{a+1} \otimes \Lambda^{j+1} A^*$, where the subscript indicates the eigenspace of $\Lambda^i L^*$ with eigenvalue a . Therefore, the double complex we

⁽²⁾ In [b] Rozenfel'd has announced complete results for $(III)_n$ and $(VI)_{2n}$. However it is not clear as he claims that the methods of proof in [2] extend in $(VI)_{2n}$. In [a] Gel'fand et al. show that $(V)_2$ has nontrivial cohomology in dimensions 7 and 10.

⁽³⁾ If $\dim L = \infty$ we assume L is topologized and $L^{**} = L$. See, for example, [3].

have just described breaks up into a countable number of subcomplexes. The a th such complex is described in Figure 1.

$$\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & & \uparrow \\
 0 \rightarrow & (\Lambda^{i+1} L^*)_a & \rightarrow & (\Lambda^{i+1} L^*)_{a+1} \otimes A^* & \rightarrow \dots \rightarrow & \left(\frac{\Lambda^{i+1} L^*}{S^+ \Lambda^{i+1} L^*} \right)_{a+m} \otimes \Lambda^m A^* & \rightarrow 0 \\
 & \uparrow & & \uparrow & & & \uparrow \\
 0 \rightarrow & (\Lambda^i L^*)_a & \rightarrow & (\Lambda^i L^*)_{a+1} \otimes A^* & \rightarrow \dots \rightarrow & \left(\frac{\Lambda^i L^*}{S^+ \Lambda^i L^*} \right)_{a+m} \otimes \Lambda^m A^* & \rightarrow 0 \\
 & \uparrow & & \uparrow & & & \uparrow \\
 0 \rightarrow & (\Lambda^{i-1} L^*)_a & \rightarrow & (\Lambda^{i-1} L^*)_{a+1} \otimes A^* & \rightarrow \dots \rightarrow & \left(\frac{\Lambda^{i-1} L^*}{S^+ \Lambda^{i-1} L^*} \right)_{a+m} \otimes \Lambda^m A^* & \rightarrow 0
 \end{array}$$

Figure 1

As we have already observed, the rows of Figure 1 are exact. We claim that the columns are exact except for the extreme right-hand column and the column indexed by $a + j = 0$. In fact, the standard identity: $(ad\alpha)\omega = \delta(\alpha \lrcorner \omega) + \alpha \lrcorner d\omega$ implies that the only eigenspace of ΛL^* with nontrivial cohomology is the zero eigenspace.

Let us consider Figure 1 with $a = 0$. All the rows are exact, and all the columns are exact except for the extreme right-hand column. Therefore, we get an isomorphism between the cohomology of the first column in position i (which is just $H^i(L, k)$) and the cohomology of the last column in position $i - m$. If $i \leq m$ this cohomology is zero. (It is zero in dimension zero since the complex in question is a relative complex.) Thus $H^i(L, k) = 0$ for $i \leq m$, proving our theorem.

As corollaries we get:

Corollary 1. *The cohomology of the algebra $(I)_n$ is zero in dimensions $0 < i \leq n$.*

Proof. Just take A to be the subalgebra consisting of the constant vector fields, $\sum c_i(\partial/\partial x_i)$, $c_i \in k$, and take $\alpha = \sum_{i=1}^n x_i(\partial/\partial x_i)$.

Corollary 2. *The cohomology of the algebra $(IV)_{2n}$ is zero in dimensions $0 < i \leq n$.*

Proof. Take A to be the subalgebra consisting of all linear forms $\sum_{i=1}^n c_i x_i$, $c_i \in k$, and take $\alpha = \sum x_i y_i$.

Corollary 3. *The cohomology of the algebra $(VII)_{2n+1}$ is zero in dimensions $0 < i \leq n$.*

Proof. Take A and α to be the same as in the preceding example.

3. To apply Theorem 1 to example $(II)_n$ we will show that the dual algebra is free over $S(A)$ where A is the subalgebra spanned by $\partial/\partial x_i$, $i = 1, \dots, n-1$. It is not hard to prove this directly, but we prefer to deduce it from a result which is applicable to other examples besides those discussed here.

We begin with a standard theorem in commutative algebra.

Theorem. *Let M be a graded module over the polynomial ring $k[x_1, \dots, x_n]$. Then the following are equivalent.*

(a) *The Koszul cohomology of M with respect to $x_1, \dots, x_n$ is zero except in dimension zero.*

(b) *M is free as a module over $k[x_1, \dots, x_n]$.*

(c) *For each j , $0 \leq j \leq n$, x_j is a nonzero divisor of $M/(x_1, \dots, x_{j-1})M$.*

See [1, Chapter VIII, Theorem 6.1] and [6, Chapter IV, Proposition 3].

Using the equivalence of (a) and (b) one can prove some general results relating the vanishing of the cohomology of a Lie algebra, L , to the vanishing of its Koszul cohomology with respect to an abelian subalgebra A . The Koszul cohomology is in turn closely related to the Spencer cohomology of L . (See, for example, [4].)

To show that the cohomology of example $(II)_n$ vanishes in dimensions $0 < i < n$ we will use the equivalence of (b) and (c).

Condition (c) dualizes to the following condition:

For each j , let $L_j = \{\vec{f}, \operatorname{div} \vec{f} = \partial \vec{f} / \partial x_1 = \dots = \partial \vec{f} / \partial x_{j-1} = 0\}$; then the map $L_j \rightarrow L_j$, $\vec{f} \rightarrow \partial \vec{f} / \partial x_j$, is surjective. To check this condition we note first of all that L_j is just all divergence free vector fields in the variables $x_j, \dots, x_n$; so it is enough to check the condition for $j = 1$. Given $\vec{f} \in L_1$ we can find a $\vec{g}$ such that $(\partial/\partial x_1)\vec{g} = \vec{f}$. Since $\operatorname{div} \vec{f} = 0$, $(\partial/\partial x_1)\operatorname{div} \vec{g} = 0$; so $\operatorname{div} \vec{g}$ is a power series in $x_2, \dots, x_n$. Let $\vec{g}_1$ be a power series vector field in $x_2, \dots, x_n$ such that $\operatorname{div} \vec{g}_1 = \operatorname{div} \vec{g}$, then $(\partial/\partial x_1)(\vec{g} - \vec{g}_1) = \vec{f}$ and $\operatorname{div}(\vec{g} - \vec{g}_1) = 0$. We have, therefore, proved

Proposition 1. *If L is the Lie algebra $(II)_n$ then L^* is free over $S(A)$ where A is the subalgebra spanned by $\partial/\partial x_i$, $i = 1, \dots, n-1$.*

If $\alpha = (\sum_{i=1}^{n-1} x_i(\partial/\partial x_i)) - (n-1)x_n(\partial/\partial x_n)$, the hypotheses of Theorem 1 are satisfied so we have:

Corollary. *The cohomology of the algebra $(II)_n$ is zero in dimensions $0 < i < n$.*

Remark. One can show that the n th cohomology group of $(\text{II})_n$ is one-dimensional with the volume form, $dx_1 \wedge \dots \wedge dx_n$, as its generator. See [7].

4. We will finally compute the cohomology of the algebras $(\text{III})_n$, $(\text{V})_{2n}$ and $(\text{VI})_{2n}$. We begin by describing a spectral sequence due to Hochschild and Serre: Let L be a Lie algebra over k and M an ideal in L . The Hochschild-Serre spectral sequence has as its E_∞ term $H(L, k)$ and has $E_2^{j,i}$ equal to $H^j(L/M, H^i(M))$. (See for example [1, Chapter XVI, §6]).

If L is the Poisson algebra and M is its one-dimensional center, then $H^i(M) = k$ when $i = 0, 1$ and zero otherwise; so $E_2^{j,0} = E_2^{j,1} = H^j(L/M, k)$, and the other terms are zero. Since $H^j(L, k) = 0$ in dimensions $0 < j \leq n$ the only way these E_2 terms can cancel out is for $d_2 : E_2^{j,1} \rightarrow E_2^{j+2,0}$ to be bijective for $j + 2 \leq n$. So we have proved

Proposition 2. *The cohomology of the Hamiltonian algebra $(\text{V})_{2n}$ is equal to k in all even dimensions in the range $0 \leq i \leq n$ and equal to zero in all odd dimensions in this range.*

Remark. The Poisson algebra is a nontrivial central extension of the Hamiltonian algebra, so it defines an element in H^2 of the Hamiltonian algebra. (See, for example, [5].) It is not hard to see that the i th power of this 2-dimensional element is a basis for the cohomology in dimension $2i$ when $2i \leq n$.

Next we apply the Hochschild-Serre spectral sequence to the Hamiltonian algebra and to its derivation algebra. Let M be the Hamiltonian algebra and L its derivation algebra. It is easy to see that the two-dimensional generator of $H(M)$ is not invariant with respect to L/M , so $H^i(L/M, H^j(M))$ is zero except when $j = 0$; and $H^i(L/M, H^0)$ is equal to k for $i = 0, 1$ and zero otherwise, since L/M is one-dimensional. We conclude with

Proposition 3. *The cohomology of the algebra $(\text{VI})_{2n}$ is equal to k in dimensions zero and one and is zero in the range $1 < i \leq n$.*

A simpler computation of the same kind shows

Proposition 4. *The cohomology of the algebra $(\text{III})_n$ is equal to k in dimensions zero and one, and is zero in the range $1 < i \leq n$.*

Remark. The generator of the first cohomology group of $(\text{III})_n$ is the one cocycle $\vec{f} \rightarrow \text{div } \vec{f}$. The restriction of this is the generator of the first cohomology group of $(\text{VI})_{2n}$.

To conclude, we note that Gel'fand and Fuks have proved the cohomology of $(\text{I})_n$ vanishes in the range $0 < i \leq 2n$. See [2]. The second author, in his thesis, has proved that the cohomology of $(\text{IV})_{2n}$ also vanishes in this range. (The proofs require rather complicated techniques from classical invariant theory.)

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