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# HIGHER DERIVATIONS AND FIELD EXTENSIONS

BY

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ABSTRACT. Let $K$ be a field having prime characteristic $p$. The following conditions on a subfield $k$ of $K$ are equivalent: (i) $\bigcap_n K^{p^n}(k) = k$ and $K/k$ is separable. (ii) $k$ is the field of constants of an infinite higher derivation defined in $K$. (iii) $k$ is the field of constants of a set of infinite higher derivations defined in $K$. If $K/k$ is separably generated and $k$ is algebraically closed in $K$, then $k$ is the field of constants of an infinite higher derivation in $K$. If $K/k$ is finitely generated then $k$ is the field of constants of an infinite higher derivation in $K$ if and only if $K/k$ is regular.

**Introduction.** The relationship between field extensions and derivations was investigated by Baer [1] in 1927. Baer obtained a characterization of those subfields $k$ of the field $K$ that are the fields of constants of derivations defined in $K$. In the prime characteristic case it was found that $k$ is the field of constants of a nonzero derivation defined in $K$ if and only if $K/k$ is a purely inseparable extension having exponent one. Later Weisfield [7] generalized this result to finite higher derivations and purely inseparable extensions having higher exponent. The works of Weisfield [7] and Sweedler [6] yield the following: Let $K$ be a field having prime characteristic. The following conditions on a subfield $k$ of $K$ are equivalent:

(i) $K/k$ is a purely inseparable modular extension with finite exponent.

(ii) $k$ is the field of constants of a finite higher derivation in $K$.

(iii) $k$ is the field of constants of a set of finite higher derivations in $K$.

The purpose of this paper is to extend the above results to infinite higher derivations. The following is obtained: Let $K$ be a field having prime characteristic $p$. The following conditions on a subfield $k$ of $K$ are equivalent:

(i) $K/k$ is separable and $\bigcap_n K^{p^n}(k) = k$.

(ii) $k$ is the field of constants of an infinite higher derivation in $K$.

(iii) $k$ is the field of constants of a set of infinite higher derivations in $K$.

A few comments should be made concerning the theory for characteristic zero fields. Baer [1] showed that in this case the subfields of $K$ which are fields of constants of derivations in $K$ are precisely those subfields algebraically closed in $K$. These subfields are also the fields of constants of the finite and infinite higher derivations in $K$.

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**Preliminaries.** All fields considered have prime characteristic $p$. Separable will mean separable in the linear disjoint sense. Let $K$ be a field.

**Definition.** An infinite higher derivation in $K$ is a sequence of additive mappings $(d_i)$ of $K$ into itself such that, for all $x$ and $y$ in $K$ and $n = 0, 1, 2, \cdots$

$$d_n(xy) = \sum \{d_i(x)d_j(y) \mid i + j = n\}$$

and $d_0$ is the identity mapping of $K$.

The field of constants of a derivation is its kernel and the field of constants of a higher derivation $(d_i)$ is the intersection of the kernels of the $d_i$ for $i \geq 1$. If $k$ is a subfield of the field of constants of the derivation $d$ in $K$, $d$ is said to be a derivation in $K$ over $k$. The notation $\operatorname{Der}(K/k)$ is adopted for the collection of derivations in $K$ over $k$ and $H(K/k)$ is adopted for the analogous collection of infinite higher derivations in $K$.

The following results will be needed repeatedly in this paper.

**Theorem A** [3, p. 181]. *Let $S$ be a $p$-basis for the field extension $K/k$ and $f\colon S \longrightarrow K$ an arbitrary function. Then there exists a unique derivation $d$ in $K$ over $k$ such that $d(s) = f(s)$ for each $s \in A$.*

**Theorem B** [2, Theorem 1]. *Let $S$ be a $p$-basis for the separable field extension $K/k$ and $f\colon \{1, 2, \cdots\} \times S \longrightarrow K$ an arbitrary function. Then there exists a unique higher derivation $(d_i)$ in $K$ over $k$ such that, for each $s \in S$ and $i \in \{1, 2, \cdots\}$, $d_i(s) = f(i, s)$.*

**Theorem C** [7, p. 436]. *Let $(d_i)$ be a higher derivation in $K$. Then for each $a \in K$:* (i) $d_{pj}(a^p) = (d_j(a))^p$ *for each $j \geq 1$ and* (ii) $d_j(a^p) = 0$ *if $p$ does not divide $j$.*

**Main result.**

**Lemma 1.** *Let $K$ be a purely trancendental extension of $k$. Then there exists $d \in \operatorname{Der}(K)$ having $k(K^p)$ as field of constants and $(d_i) \in H(K)$ with field of constants $k$.*

**Proof.** Let $S$ be a transcendency basis for $K/k$ with $K = k(S)$. If $S = \{s_1, \cdots, s_n\}$ is finite, let $s_0 = 1$ and define $d(s_i) = (s_0 \cdots s_{i-1})^{-1}$ for $1 \leq i \leq n$. Since $S$ is a $p$-basis for $K/k$, this defines a unique derivation $d$ in $K$ over $k$ [Theorem A]. Let $0 < m < n$ and $e^{(m)}$ denote the restriction of $d$ to $k(K^p)(s_0, \cdots, s_m)$. We induct on $m$ to show that the field of constants of $e^{(m)}$ is $k(K^p)$ and $(s_0 \cdots s_m)^{-1} \notin \operatorname{Im}(e^{(m)})$. If $m = 0$, then the result is clearly true since $e^{(0)}$ is the zero derivation in $k(K^p)$. Thus assume the result for $m$ with $0 < m < n$.

Suppose there are $A_i \in k(K^p)(s_0, \cdots, s_m)$ such that

$$e^{(m+1)}\left(\sum_0^{p-1} A_i s_{m+1}^i\right)$$

(1)

$$= \sum_0^{p-1} e^{(m)}(A_i)s_{m+1} + \sum_1^{p-1} iA_i s_{m+1}^{i-1}(s_0 \cdots s_m)^{-1} = 0.$$

If $A_i \neq 0$ for some $i \geq 1$, then there exists a $j \geq 0$ such that $A_j \neq 0$ and $e^{(m)}(A_j) = 0$. From (1) we have

(2) $$e^{(m)}(-A_{j-1}/jA_j) = (s_0 \cdots s_m)^{-1}.$$

This contradicts the induction hypothesis. Thus $A_i = 0$ for each $i \geq 1$ and $A_0 \in k(K^p)$ follows from the induction hypothesis.

Now suppose there are $B_i \in k(K^p)(s_0, \cdots, s_m)$ such that

$$e^{(m+1)}\left(\sum_0^{p-1} B_i s_{m+1}^i\right) = \sum_0^{p-2} (e^{(m)}(B_i) + B_{i+1}(i+1)(s_0 \cdots s_m)^{-1})s_{m+1}^i$$

(3)

$$+ e^{(m)}(B_{p-1})s_{m+1}^{p-1}$$

$$= (s_0 \cdots s_{m+1})^{-1}$$

or

$$\sum_0^{p-2} (e^{(m)}(B_i) + B_{i+1}(i+1)(s_0 \cdots s_m)^{-1})s_{m+1}^{i+1} + e^{(m+1)}(B_{p-1}s_{m+1}^p)$$

(4)

$$= (s_0 \cdots s_m)^{-1}.$$

Consequently, $e^{(m)}(B_{p-1}s_{m+1}^p) = (s_0 \cdots s_m)^{-1}$ and this is a contradiction of the induction hypothesis.

If $S = \{s_\alpha\}$ is infinite, well-order it so that there is no last element. Define $d(s_\alpha) = s_{\alpha+1}$ for each $s_\alpha \in S$. This defines a unique derivation in $K$ over $k$ [Theorem A].

Let $A_i \in k(K^p)$ $(\{s \in S \mid s < s_\alpha\})$ and suppose

(5) $$d\left(\sum_0^{p-1} A_i s_\alpha^i\right) = \sum_0^{p-1} d(A_i)s_\alpha^i + \sum_1^{p-1} iA_i s_\alpha^{i-1} s_{\alpha+1} = 0.$$

Necessarily, $\Sigma_1^{p-1} iA_i s_\alpha^{i-1} = 0$ and from this it follows that $A_1 = \cdots = A_{p-1} = 0$. Iteration of the process yields that $A_0 \in k(K^p)$. Thus the field of constants of $d$ is $k(K^p)$.

Since the action of a higher derivation is completely determined by its action on a $p$-basis [Theorem B], there exists a higher derivation $(d_i) \in H(K/k)$ with $d_1 = d$. Theorem C is used to show that for each $i \geq 1$, the restriction of $d_{p^i}$ to $K^{p^i}(k)$ is a derivation. Since $d_{p^i}(x^{p^i}) = d(x)^{p^i}$ for each $x \in K$, we see that $d_{p^i}$ operates on $S^{p^i}$ in exactly the same manner $d$ operates on $S$. Since $S^{p^i}$ is a transcendency basis for $K^{p^i}(k)/k$ with $K^{p^i}(k) = k(S^{p^i})$, the field of constants of the restriction of $d_{p^i}$ to $K^{p^i}(k)$ is $K^{p^{i+1}}(k)$. Thus the field of constants of $(d_i)$ is $\bigcap_n K^{p^n}(k) = \bigcap_n k(S^{p^n}) = k$.

**Theorem 1.** *The following conditions on a subfield $k$ of $K$ are equivalent.*

(i) *$K/k$ is separable and $\bigcap_n K^{p^n}(k) = k$.*

(ii) *$k$ is the field of constants of a higher derivation in $K$.*

(iii) *$k$ is the field of constants of a set of higher derivations in $K$.*

**Proof.** (i) implies (ii). Let $S$ be a $p$-basis for $K/k$ and $T$ be a $p$-basis for $k$. Since $S \cup T$ is a $p$-basis for $K$, it is algebraically independent over $K_0$, the maximal perfect subfield of $K$. The existence of an $(e_i) \in H(K_0(S \cup T))$ with field of constants $K_0(T)$ is guaranteed by Lemma 1. In the proof of Lemma 1 it was shown that $(e_i)$ can be chosen such that the field of constants of the restriction of $e_{p^j}$ to $K_0(S^{p^j} \cup T)$ is $K_0(S^{p^{j+1}} \cup T)$ for each $j \geq 0$. Take $(d_i)$ to be the unique higher derivation in $K$ agreeing with $(e_i)$ on $S \cup T$. Since $(d_i)$ acts trivially on $T$, $k$ is a subfield of the field of constants of $(d_i)$. Let $U$ be a linear basis for $K/K_0(S \cup T)$. We note that $U^p$ and hence $U^{p^n}$ for any $n$ is a linear basis for $K/K_0(S \cup T)$. It is easily verified that $K^p$ and $K_0(S \cup T)$ are linearly disjoint over $K_0(S^p \cup T^p)$. Thus $U^p$ is linearly independent over $K_0(S)$ and is a linear basis for $K^p(S \cup T) = K$ over $K_0(S \cup T)$.

Let $x \in K$ and assume $d_i(x) = 0$ for all $i \geq 1$. We show that $x \in K^{p^n}(k)$ for each $n$ and consequently $x \in k$. Fix $n \geq 1$ and let $\{A_i\} \subseteq U$ and $\{a_i\} \subseteq K_0(S \cup T)$ be such that $x = \sum a_i A_i^{p^n}$. For each $1 \leq j < n$ we have

$$d_{p^j}(x) = \sum d_{p^j}(a_i) A_i^{p^n} = \sum e_{p^j}(a_i) A_i^{p^n} = 0. \tag{6}$$

As a consequence of the fact that each $e_{p^j}(a_i) = 0$ we have that each $a_i \in K_0(S^{p^n} \cup T)$ and hence $x \in K^{p^n}(k)$. It now follows that the field of constants of $(d_i)$ is $k$.

(ii) implies (iii) is clear.

(iii) implies (i). Let $H$ denote a set of higher derivations and suppose $k$ is the field of constants of $H$. We first show that $\bigcap_n K^{p^n}(k) = k$. Let $x \in K^{p^n}(k)$ and $(d_i) \in H$. Fix $j \geq 1$ and let $p^g > j$. Since $x \in K^{p^g}(k)$, there are $A$ and $B$ in $K^{p^g}[k]$ such that $xB = A$ and $B \neq 0$. We apply $d_j$ to both sides and use Theorem C to obtain that $d_j(x) = 0$.

Now we need to show that $K^p$ and $k$ are linearly disjoint over $k^p$. If this were not the case, then there would exist a minimal subset of $k$ which is linearly independent over $k^p$ and linearly dependent over $K^p$. Thus suppose $\{A_i\}$ is such a minimal subset. Then

$$\sum_1^n a_i^p A_i = 0, \quad \{a_i\} \subseteq K, \quad \text{each } a_i \neq 0, \ n > 1. \tag{7}$$

Without loss of generality we may assume $a_1 = 1$ and $a_2 \notin k$. There exists a $(d_i) \in H$ and $j > 0$ such that $d_j(a_2) \neq 0$.

$$d_{pj}\left(A_1 + \sum_2^n a_i^p A_i\right) = \sum_2^n (d_j(a_i))^p A_i = 0. \tag{8}$$

The minimal nature of $n$ is contradicted by (8). Thus $K/k$ is a separable field extension.

**Theorem 2.** *If $K/k$ is separably generated and $k$ is algebraically closed in $K$, then there exists an infinite higher derivation in $K$ having field of constants $k$.*

**Proof.** Let $S$ be a separating transcendency basis for $K/k$. An application of Theorem 1 to the field extension $k(S)/k$ yields the existence of a higher derivation $(d_i)$ defined on $k(S)$ having $k$ as field of constants. Since $S$ is a $p$-basis for $K$, there is a unique extension of $(d_i)$ to $K/k$ [Theorem B]. We now show that the field of constants of this extension is $k$. Suppose $s$ is an element of the field of constants of $(d_i)$. Let $f(x) = x^m + f_{m-1}x^{m-1} + \cdots + f_0$ denote the minimal polynomial of $s$ over $k(S)$. If each $f_i \in k$, then $s$ is algebraic over $k$ and we are finished. So suppose some $f_i \notin k$. Thus there exists $q > 0$ such that $d_q(f_i) \neq 0$. We may assume $q$ to be minimal and that $d_j(f_g) = 0$ for each $g$ and each $j < q$.

$$d_q(f(s)) = d_q\left(s^m + \sum_0^{m-1} f_i s^i\right) = \sum_0^{m-1} d_q(f_i)s^i = 0. \tag{9}$$

The polynomial $\Sigma_0^{m-1} d_q(f_i)x^i$ is nonzero and has degree less than that of the minimal polynomial of $s$. This is a contradiction. Thus each $f_i \in k$ and hence $s \in k$.

**Corollary 1.** *If $K/k$ is separably generated and $k$ is algebraically closed in $K$, then $\bigcap_n K^{p^n}(k) = k$.*

Recall that a field extension $K/k$ is said to be regular if $K$ and $\bar{k}$ are linearly disjoint over $k$. Regularity is equivalent to $K/k$ being a separable extension and $k$ being algebraically closed in $K$ [4, p. 56]. Thus we have

**Corollary 2.** *Let $K/k$ be finitely generated. Then $k$ is the field of constants of a higher derivation in $K$ if and only if the extension $K/k$ is regular.*

The following example due to Mac Lane [5] illustrates that the converse of

Theorem 2 is false. Let $k$ be a perfect field and let $T = \{t_0, t_1, \cdots\}$ be algebraically independent over $k$, define $Y = \{y_2, y_3, \cdots\}$ by $y_n^p = t_{n-2} + t_{n-1}t_n^p$, $n \geq 2$, and let $K = k(T)(Y)$. Then $K/k$ is a separable extension having $\bigcap_n K^{p^n}(k) = k$; however, $K/k$ is not separably generated.

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