

DUAL SPACES OF GROUPS WITH PRECOMPACT CONJUGACY CLASSES

BY

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ABSTRACT. We show that a second countable locally compact type I group with a compact invariant neighborhood of the identity is CCR, and has a Hausdorff dual if and only if its conjugacy classes are precompact. We obtain sharper results if the group is almost connected or has a fundamental system of invariant neighborhoods of the identity. Along the way we show that for a locally compact abelian group A and a group B of topological group automorphisms of A , A has small B invariant neighborhoods at 1 if and only if \hat{A} has precompact orbits under the dual actions of B .

1. **Introduction and preliminaries.** One of the striking features of the Fell topology on the dual space \hat{G} of the locally compact group G is its characterization of important properties of G by simple separation properties of \hat{G} . Thus for second countable G , G is type I if and only if \hat{G} is T_0 and CCR if and only if \hat{G} is T_1 . Moreover, Baggett [2] has recently shown that, for second countable G , \hat{G} discrete is equivalent to G compact. One is naturally led to ask what significance, if any, a Hausdorff dual space has, and while many of our results do not pertain to this question, it is nonetheless the primary motivation of this paper. Our principal results are that a type I $[IN]$ group is CCR and has Hausdorff dual if and only if it is $[FC]^-$; we obtain more explicit results in the case of an $[SIN]$ group or an almost connected group.

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We now establish some conventions and notation. Groups G will always be second countable locally compact, except in §2. Representations of groups will be strongly continuous unitary representations, and representations of C^* algebras will be nondegenerate $*$ representations (hence continuous). The letters \mathbb{R} , \mathbb{Z} and T will denote the real numbers, the integers, and the circle; the circle will

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be regarded as the complex numbers of modulus one under multiplication. We record here some important classes of groups together with their defining properties: $[FC]^-$ (having precompact conjugacy classes), $[FD]^-$ (having a precompact commutator subgroup), $[IN]$ (having a compact neighborhood of 1 invariant under inner automorphisms), $[SIN]$ (having a fundamental system of invariant neighborhoods at 1), and $[Z]$ (being compact modulo the center). The following inclusions hold: $[Z] \subset [SIN] \subset [IN]$ and $[Z] \subset [FD]^- \subset [FC]^- \subset [IN]$. (See [12] and [25].)

An exact sequence of locally compact groups will be an algebraic exact sequence $(1) \rightarrow G \rightarrow H \rightarrow J \rightarrow (1)$ in which the maps are continuous and open. Since we are taking groups to be second countable, it will be enough to have the maps continuous.

The continuous complex valued function f on G will be called central if $f(\alpha x) = f(x)$ for all $x \in G$ and all inner automorphisms α . Notice that if the automorphism β is the point open limit of inner automorphisms then $f(\beta x) = f(x)$ for all $x \in G$. The continuous functional F on the Banach algebra A will be called central if $F(ab) = F(ba)$ for all $a, b, \in A$.

We will take 2 cocycles of G to be those with coefficients in T , where G acts trivially on T . We note that if α is a 2 cocycle of G , α defines a central exact sequence $(1) \rightarrow T \rightarrow H \rightarrow G \rightarrow (1)$ of locally compact groups, and α representations of G are in one to one correspondence with ordinary representations of H whose restrictions to T are multiples of the character $t \mapsto t$. We will denote the irreducible α representations of G by ${}_{\alpha}\hat{G}$ and the image of $Z(H)$ (the center of H) in G by $Z_{\alpha}(G)$. For details and further references on 2 cocycles and their relation to representation theory, we refer the reader to Chapter I of [1]. The same source contains a sketch of the Mackey theory of induced representations, which we will use extensively.

1.1 Proposition. *Let Φ be the continuous positive definite functions ϕ of G with $\phi(e) \leq 1$. Let \mathcal{F} be the positive functionals on $L^1(G)$ of norm ≤ 1 . Then there is a bijection between Φ and \mathcal{F} given by $F(\phi)(f) = \int f(x)\phi(x)dx$ for all $f \in L^1(G)$. Moreover central elements Φ_c of Φ correspond to central elements \mathcal{F}_c of \mathcal{F} under this bijection and Φ and Φ_c are compact metric in the weak* topology inherited from \mathcal{F} and \mathcal{F}_c .*

Proof. The first assertion is part of Théorème 13.4.5 of [5]. Verifying that Φ_c and \mathcal{F}_c correspond under the bijection is completely straightforward. It is clear that \mathcal{F} and \mathcal{F}_c are weak* closed in the unit ball of $L^{\infty}(G)$, and finally, it is well known that the unit ball of the dual space of a separable Banach space is compact metric in the weak* topology (see for example B7 of [5]).

1.2 Definition. We denote by $\mathcal{Q}(G)$ the group of topological group automorphisms of G . The topology on $\mathcal{Q}(G)$ (see [4]) is given as follows: $\alpha_\nu \rightarrow \alpha$ in $\mathcal{Q}(G)$ if and only if $\alpha_\nu \rightarrow \alpha$ and $\alpha_\nu^{-1} \rightarrow \alpha^{-1}$ both uniformly on compacta. One may verify that the evaluation map $\mathcal{Q}(G) \times G \rightarrow G$ is jointly continuous (see p. 40 of Hochschild, *The structure of Lie groups*). We let $I(G)$ be the closure in $\mathcal{Q}(G)$ of the group of inner automorphisms.

1.3 Proposition. $I(G)$ is compact if and only if $G \in [FC]^- \cap [SIN]$.

This is Theorem 4.1 of [10] in the special case that $B^- = I(G)$. In general one has that the subgroup B of $\mathcal{Q}(G)$ is precompact iff the B orbits of G are precompact and the identity of G has a fundamental system of B invariant neighborhoods; moreover, one need not assume G is separable.

1.4 Definition. If f is a continuous complex valued function on G and K is a compact subset of G , let $\|f\|_K = \sup\{|f(x)| \mid x \in K\}$. Let $G \in [FC]^- \cap [SIN]$. Then $f(\alpha x)$ is continuous and bounded in $\alpha \in I(G)$ for each fixed x and we may define the function $f^\#$ by $f^\#(x) = \int_{I(G)} f(\alpha x) d\alpha$, where $d\alpha$ is normalized Haar measure on the compact group $I(G)$.

1.5 Proposition. Suppose f is a continuous complex valued function on the $[FC]^- \cap [SIN]$ group G . Then $f^\#$ is continuous and central. If f is central, $f^\# = f$. If f is positive definite, so is $f^\#$. Moreover, $\|f^\#\|_K \leq \|f\|_{I(G)(K)}$, where K is any compactum in G and $I(G)(K)$ is the compact image of $I(G) \times K$ under the evaluation map. Finally, if f has compact support, then so does $f^\#$.

Proof. Straightforward verification. See [21], for example.

1.6 Proposition. Suppose $G \in [FC]^- \cap [SIN]$. Let π be a continuous finite dimensional irreducible representation of G , v a unit vector in H_π , and let $\phi(x) = (\pi_x v, v)$. Then $\phi^\#(x) = (\deg \pi)^{-1} \text{Tr} \pi_x$ for all $x \in G$.

Proof. First we note that by 1.1, $F(\phi^\#)$ is a central functional on $L^1(G)$. Suppose now that $f \in \ker \pi$; that is, $\int f(x) \pi_x dx = 0$ where $f \in L^1(G)$. Then for each $y \in G$, $\int f(x) \pi_{yxy^{-1}} dx = \pi_y \int f(x) \pi_x dx \pi_y^{-1} = 0$. Moreover if $\alpha \in I(G)$, we have inner automorphisms $\alpha_n \rightarrow \alpha$ and so for each $u, v \in H_\pi$ we have

$$|(\pi_{\alpha_n x} u, v) - (\pi_{\alpha x} u, v)| \leq \|\pi_{\alpha_n x} u - \pi_{\alpha x} u\| \|v\| = \|\pi_{(\alpha_n)^{-1}(\alpha_n x)} u - u\| \|v\| \rightarrow 0$$

uniformly on G compacta because $\alpha_n \rightarrow \alpha$ uniformly on G compacta. It follows that $\int f(x) (\pi_{\alpha x} u, v) dx = 0$ for all u, v and furthermore that

$$\int f(x) \phi^\#(x) dx = \int_{I(G)} \int_G f(x) (\pi_{\alpha x} v, v) dx d\alpha = 0.$$

Thus $F(\phi^\#)$ defines a positive central functional on the image of $L^1(G)$ under π . Since π is irreducible this image is a full matrix algebra $B(H_\pi)$ and so $F(\phi^\#)(f) = \lambda \operatorname{Tr} \pi(f)$ for all $f \in L^1(G)$, for some constant λ . It follows easily that $\phi^\#(x) = \lambda \operatorname{Tr} \pi_x$ for all $x \in G$ and, since $\phi^\#(1) = 1$, $\lambda = (\deg \pi)^{-1}$ and the proposition is proved.

1.7 Proposition. *Let G be locally compact abelian, and let $a \in H^2(G, T)$. Then for any cocycle α representing a , $\alpha^{(2)}(x, y) = \alpha(x, y)\alpha(y, x)^{-1}$ is jointly continuous and bilinear (a character in each variable separately). Moreover, $\alpha^{(2)}$ is independent of the choice of α . Finally, we may choose α so that $\alpha(x, y) = 1$ whenever x, y are in $Z_\alpha(G) = \{w \in G \mid \alpha^{(2)}(w, v) = 1 \text{ for all } v \text{ in } G\}$.*

Proof. The first assertion is Lemma 7.1 of [16] and the second is Lemma 7.2 of [16]. Let β be a cocycle representing a . Then there is a unique locally compact group H which as a Borel group is $T \times G$ with multiplication defined by $\langle s, x \rangle \langle t, y \rangle = \langle st\beta(x, y), x + y \rangle$ for $s, t \in T$ and $x, y \in G$. Of course $(1) \rightarrow T \rightarrow H \xrightarrow{\pi} G \rightarrow (1)$ (up to equivalence of extensions) is the extension defined by a . Since $Z(H) \supset T$, we have $Z(H) = T \oplus A$ for some closed subgroup A of $Z(H)$. Moreover, $\pi: A \rightarrow \pi(Z(H))$ is a topological group isomorphism. Define a Borel cross-section $\sigma: G \rightarrow H$ as follows: $\sigma(x) = (\pi|A)^{-1}(x)$ if $x \in \pi(Z(H))$ and $\sigma(x) = \langle 0, x \rangle$ if $x \notin \pi(Z(H))$. (To see that this is Borel use Lemma 1.2 of [17].) Let $\alpha(x, y) = \sigma(x)\sigma(y)\sigma(x+y)^{-1}$ for all $x, y \in G$. We then have α cohomologous to β (see discussion on pp. 24–25 of [1]) and since $\sigma|_{\pi(Z(H))}$ is a group homomorphism, α is trivial on $\pi(Z(H)) = Z_\alpha(G)$. This proves the proposition.

2. Automorphisms of $[FC]^-$ groups. In this section we prove a result on automorphisms of $[FC]^-$ groups which leads to a result of Robertson [25] and an additional application. In this section only, G will not be assumed second countable. Recall that a periodic group (see [12] for example) is one each element of which is contained in a compact subgroup.

2.1 Proposition. *Let P be a locally compact periodic group whose B orbits are precompact, where $B \subset \mathcal{Q}(G)$ is a subgroup containing the inner automorphisms. Then P contains a compact open B invariant subgroup K .*

Proof. Let N be a compact neighborhood of 1 in P . According to Theorem 3.11 of [12], N generates a compact open subgroup K_1 of P . We assume first that K_1 has countable index in P , and let $x_1 = 1, x_2, x_3, \dots$ be a set of coset representatives for K_1 in P . A set S is precompact in P if and only if $S \subset \bigcup_{i=1}^n x_i K_1$ for some integer n . Now for each $n \geq 1$, let S_n be the closed set

$$K_1 \cap \bigcap_{\alpha \in B} \alpha^{-1} \left(\bigcup_{i=1}^n x_i K_1 \right) = \left\{ x \in K_1 \mid B(x) \subset \bigcup_{i=1}^n x_i K_1 \right\}.$$

Since each element of K_1 has a precompact orbit under B , $K_1 = \bigcup_{n=1}^{\infty} S_n$. Therefore by the Baire category theorem some S_n contains an open set U . Now $B(S_n) \subset \bigcup_{i=1}^n x_i K_1$. Therefore if K is the subgroup generated by $B(S_n)$ and K_2 is the subgroup generated by $\bigcup_{i=1}^n x_i K_1$, we have K open, K_2 compact (by 3.11 of [12]) and $K \subset K_2$. Moreover, $B(B(S_n)) = B(S_n)$, and so K is B invariant.

Now let us drop the assumption that K_1 has countable index in P , and suppose that $B(K_1)$ is not precompact. In particular suppose there is a countably infinite set x_1, x_2, \dots of coset representatives of K_1 in P such that, for each i , there is a $\beta_i \in B$ with $\beta_i(K_1) \cap x_i K_1 \neq \emptyset$. If B' is the countable subgroup of B generated by β_1, β_2, \dots , then $B'(K_1)$ is not precompact. Let H be the open subgroup of P generated by $B'(K_1)$. We claim that $[H: K_1]$ is countable. In fact, for $\beta'_1, \dots, \beta'_n \in B'$ let $K(\beta'_1, \dots, \beta'_n)$ be the group generated by $K_1, \beta'_1(K_1), \dots, \beta'_n(K_1)$. Then by 3.11 of [12] we know $K(\beta'_1, \dots, \beta'_n)$ is compact, and since K_1 is open we have the homogeneous space $K(\beta'_1, \dots, \beta'_n)/K_1$ finite.

Since each element of H is contained in some $K(\beta'_1, \dots, \beta'_n)$ and there are only countably many choices of $\beta'_1, \dots, \beta'_n$ from B' possible, we must have that H is contained in countably many K_1 cosets.

Now let B'' be the subgroup of B generated by B' and the inner automorphisms corresponding to elements of H . Certainly H is B'' invariant and has precompact B'' orbits; moreover H is periodic and contains a compact open subgroup K_1 of countable index. By the reasoning in the first paragraph, H contains a B'' invariant compact open subgroup K . The index $[K_1: K \cap K_1] < \infty$ and so we have coset representatives y_1, \dots, y_n for $K \cap K_1$ in K_1 . Thus

$$B'(K_1) = B' \left(\bigcup y_i (K \cap K_1) \right) = \bigcup B'(y_i) B'(K \cap K_1) \subset \left[\bigcup B'(y_i) \right] K.$$

Each $B'(y_i)$ is precompact, however, and so $B'(K_1)$ is precompact, which is impossible by the construction of B' . This contradiction proves that $B(K_1)$ is precompact. By 3.11 of [12], $B(K_1)$ generates a compact open subgroup of P which is clearly B invariant.

The following is a slight extension of a result due to Robertson [25]. We include a proof for completeness.

2.2 Theorem. *Suppose that G has precompact B orbits, where $B \subset \mathcal{Q}(G)$ is a subgroup containing the inner automorphisms. Then G satisfies an exact sequence $(1) \rightarrow K \rightarrow G \rightarrow \mathbb{R}^n \times D \rightarrow (1)$, where K is compact B invariant, D is discrete FC, and $\mathbb{R}^n \times D$ has precompact orbits under the group B' of induced actions of B . Furthermore \mathbb{R}^n is B' invariant and has small B' invariant neighborhoods of the identity.*

Before proving 2.2 we state two additional results we require.

2.3 Theorem. *Suppose that G has precompact B orbits where $B \in \mathcal{Q}(G)$ is a subgroup including the inner automorphisms. Then G satisfies an exact sequence $(1) \rightarrow P \rightarrow G \rightarrow \mathbf{R}^n \times D \rightarrow (1)$, where P is the closed characteristic subgroup of periodic elements and D is discrete torsion free abelian.*

Proof. This is most of Theorem 3.16 of [12].

2.4 Proposition. *Suppose $(1) \rightarrow P \rightarrow G \rightarrow \mathbf{R}^n \times D \rightarrow (1)$ is an exact sequence of locally compact groups, where P is a discrete torsion group and D is discrete torsion free abelian. If $G \in [FC]^+$, then $G \cong \mathbf{R}^n \times H$, where H is discrete FC.*

Proof of 2.4. Clearly G is locally euclidean and is thus a Lie group. By Theorem 3.20 of [12], the identity component G_0 of G has precompact commutator subgroup; moreover the closed commutator of G_0 is connected, and the abelian quotient of G_0 by its closed commutator is of the form $\mathbf{R}^m \times T^p$. Hence G_0 contains a compact connected, characteristic subgroup K_1 such that G_0/K_1 is a vector group. Thus the periodic elements $P \cap G_0$ of G_0 are connected, and since P is discrete in G , we therefore have $P \cap G_0 = (1)$. We also have that the image of G_0 in G/P is an analytic subgroup of \mathbf{R}^n and is therefore a closed connected subgroup of \mathbf{R}^n ; it is easy to see that the image of G_0 is all of \mathbf{R}^n and in fact the map $G \rightarrow \mathbf{R}^n \times D$ defines a topological group isomorphism of G_0 onto \mathbf{R}^n . Now define H as the preimage of D in G . Clearly $G_0 \cap H \subset P$, so that $G_0 \cap H \subset G_0 \cap P = (1)$. Moreover H is discrete, and therefore the connected subgroup G_0 acts trivially by inner automorphisms on each element of H . Thus G_0 and H commute and since $G = G_0 H$, we have $G = G_0 \times H \cong \mathbf{R}^n \times H$. Finally H as the discrete quotient of an $[FC]^+$ group is itself FC.

Proof of 2.2. Since G has precompact B orbits, Theorem 2.3 implies that G satisfies the exact sequence $(1) \rightarrow P \rightarrow G \rightarrow \mathbf{R}^n \times D' \rightarrow (1)$ where P is the group of periodic elements and D' is discrete torsion free abelian. By 2.1, P contains a compact open B invariant subgroup K , and thus G/K satisfies the exact sequence $(1) \rightarrow P/K \rightarrow G/K \rightarrow \mathbf{R}^n \times D' \rightarrow (1)$ where P/K is a discrete torsion group and G/K has precompact B' orbits, where B' is the group of actions induced in G/K by B . Thus by 2.4 we have $G/K \cong \mathbf{R}^n \times D$ where D is discrete FC. Since \mathbf{R}^n is the identity component of G/K it is invariant under B' . Since \mathbf{R}^n has precompact B' orbits it has small B' invariant neighborhoods at 1 (see 2.6). This proves 2.2.

Corollary to 2.2. *For any locally compact group G let G_F denote the normal subgroup of elements with precompact conjugacy classes. Then $G \in [IN]$ if and only if G_F is open. (See also [29].)*

Proof. If $G \in [IN]$ then G has a compact invariant neighborhood of 1 which

is clearly contained in G_F . Thus G_F is open. If on the other hand G_F is open, then by 2.2 we have an exact sequence $(1) \rightarrow K \rightarrow G_F \rightarrow \mathbf{R}^n \times D \rightarrow (1)$ where K is normal in G and \mathbf{R}^n has a compact neighborhood of the identity invariant under the inner automorphisms of G . The preimage of this neighborhood in G_F is a compact invariant neighborhood of 1 in G .

As another application of 2.1, we show that, for a locally compact abelian G and a group B of automorphisms of G , G has small B invariant neighborhoods if and only if \hat{G} has precompact \hat{B} orbits. Here \hat{B} is the group of dual automorphisms to B acting on \hat{G} . We believe that this result sheds some light on why the [SIN] condition is needed in 1.3.

2.5 Proposition. *Suppose G is locally compact abelian and B is a group of automorphisms of G such that G has small B invariant neighborhoods. Then \hat{G} has precompact \hat{B} orbits.*

Proof. Suppose that G has small B invariant neighborhoods; let $\epsilon > 0$ and let $\chi \in \hat{G}$. Let U be a B invariant neighborhood of 1 in G such that $|\chi(x) - \chi(1)| < \epsilon$ for all $x \in U$. Then $|(\chi \circ \beta)(x + y) - (\chi \circ \beta)(y)| = |\chi(\beta x) - \chi(1)| < \epsilon$ for all $x \in U$ and $\beta \in B$, for each $y \in G$. It follows that the compact open closure \bar{S} of the set $S = \{\chi \circ \beta \mid \beta \in B\}$ is equicontinuous at each point of G . Of course $\text{range } f \subset T$ for each $f \in \bar{S}$ and it follows by the Ascoli Theorem, p. 233 of [15], that \bar{S} is compact. Since χ is arbitrary this shows that \hat{G} has precompact \hat{B} orbits.

2.6. Lemma. *If K is compact abelian and \hat{K} has finite \hat{B} orbits (where $B \subset \mathcal{A}(K)$ is a subgroup) then K has small B invariant neighborhoods. If on the other hand B is a group of automorphisms of \mathbf{R}^n such that \mathbf{R}^n has precompact B orbits, then \mathbf{R}^n has small B invariant neighborhoods and it follows that $\hat{\mathbf{R}}^n$ has precompact \hat{B} orbits and small \hat{B} invariant neighborhoods.*

Proof. First let B be a group of automorphisms of the compact abelian group K such that \hat{K} has finite \hat{B} orbits. Let U be a neighborhood of 1 in K . Then there are χ_1, \dots, χ_n in \hat{K} and $\epsilon > 0$ such that

$$\begin{aligned} W &= W(\chi_1, \dots, \chi_n, \epsilon) \\ &= \{x \in K \mid |\chi_i(x) - 1| < \epsilon \text{ for } i = 1, 2, \dots, n\} \subset U. \end{aligned}$$

This last assertion follows from the fact that \hat{K} is discrete and so $W(\chi_1, \dots, \chi_n, \epsilon)$ defines a compact open neighborhood of 1 in K regarded as the dual of \hat{K} . Now each χ_i has a finite \hat{B} orbit; therefore there is a subgroup B_1 of finite index in B such that $(B_1)^\wedge$ leaves χ_1, \dots, χ_n fixed. Then B_1 leaves W invariant and if b_1, \dots, b_n are coset representatives for B_1 in B we have that $\bigcap_{b \in B} bW = \bigcap_{i=1}^n b_i W \subset U$ is a B invariant neighborhood. This proves the

first assertion. The assertion about \mathbb{R}^n is well known and can be proved by showing B is precompact in $GL(n, \mathbb{R})$ and then using the Weyl unitary trick to show B can be regarded as a group of orthogonal operators on \mathbb{R}^n .

The compact case of the following theorem was used in 4.11 of [12].

2.7 Theorem. *Suppose G is locally compact abelian and B is a group of automorphisms of G . Then G has small B invariant neighborhoods if and only if \hat{G} has precompact B^\wedge orbits.*

Proof. The "only if" part is 2.5. Therefore suppose \hat{G} has precompact B orbits. We may write $G = \mathbb{R}^n \times H$ and $\hat{G} = \hat{\mathbb{R}}^n \times \hat{H}$, where H and \hat{H} contain compact open subgroups. Let P be the periodic subgroup of \hat{H} ; clearly P is characteristic, open, and contains any compact open subgroup of \hat{H} . We may apply 2.1 to P and obtain a compact open subgroup K invariant under B^\wedge . Now the annihilator $K^\perp = (\hat{G}/K)^\wedge = \mathbb{R}^n \times (\hat{H}/K)^\wedge = \mathbb{R}^n \times K_1$, where $K_1 = (\hat{H}/K)^\wedge$ is a compact open subgroup of H (compact because \hat{H}/K is discrete and open because $(H/K_1)^\wedge = K$ is compact). Moreover K^\perp is B invariant because K is B^\wedge invariant. Since K_1 is maximal compact in $K^\perp = \mathbb{R}^n \times K_1$, K_1 is B invariant as well. All together we have $\mathbb{R}^n \times K_1$ open in G and a group B of automorphisms leaving $\mathbb{R}^n \times K_1$ and K_1 invariant. It is easily checked that $(B|_{\mathbb{R}^n \times K_1})^\wedge$ is just the group of actions that B^\wedge induces on $\hat{G}/K = (\mathbb{R}^n \times K_1)^\wedge$ and so $(\mathbb{R}^n \times K_1)^\wedge$ has precompact orbits under $(B|_{\mathbb{R}^n \times K_1})^\wedge$. We want to show $\mathbb{R}^n \times K_1$ has small B invariant neighborhoods and so we are reduced to the following:

2.8 Lemma. *Suppose $G = \mathbb{R}^n \times K$ is locally compact abelian and \hat{G} has precompact B^\wedge orbits, where B is a group of automorphisms of G . Then G has small B invariant neighborhoods.*

Proof. Let $P_1: G \rightarrow \mathbb{R}^n$ and $P_2: G \rightarrow K$ be the projections associated with the direct product. For $\alpha \in B$, let $\alpha_1 = P_1 \alpha P_1$, $\alpha_2 = P_2 \alpha P_1$, and $\alpha_3 = P_2 \alpha P_2$. Since K is necessarily B invariant $P_1 \alpha P_2 = 0$ and so α may be viewed as a matrix

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix}.$$

Moreover if

$$\beta = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & \beta_3 \end{pmatrix}$$

we have

$$\alpha\beta = \begin{pmatrix} \alpha_1\beta_1 & 0 \\ \alpha_2\beta_1 + \alpha_3\beta_2 & \alpha_3\beta_3 \end{pmatrix}.$$

In particular, P_1BP_1 and P_2BP_2 are subgroups of $\mathfrak{U}(\mathbf{R}^n)$ and $\mathfrak{U}(K)$ respectively.

For $(v, k) \in G$ we have $\alpha(v, k) = (\alpha_1(v), \alpha_2(v) + \alpha_3(k))$. A direct calculation shows that, for $(\lambda, d) \in \hat{\mathbf{R}}^n \times \hat{K} = \hat{G}$, we have $\hat{\alpha}(\lambda, d) = (\hat{\alpha}_1(\lambda) + \hat{\alpha}_2(d), \hat{\alpha}_3(d))$. Since $\{\hat{\alpha}(\lambda, d) \mid \alpha \in B\}^-$ is compact for each fixed (λ, d) we have $\{\hat{\alpha}_3(d)\}$ finite for each fixed d , $\{\hat{\alpha}_1(\lambda)\}^-$ compact for each fixed λ , and $\{\hat{\alpha}_2(d)\}^-$ compact for each fixed d . It follows by 2.6 that \mathbf{R}^n has small P_1BP_1 invariant neighborhoods and K has small P_2BP_2 invariant neighborhoods. Since $\{\alpha_2(d)\}^-$ is compact for each fixed d , so is $\bigcup_{\alpha, i} \{\hat{\alpha}_2(d_i)\}^-$ for any finite set $d_1, \dots, d_n \in \hat{K}$. By the Ascoli Theorem (p. 233 of [15]) the family $\bigcup_{\alpha, i} \{\hat{\alpha}_2(d_i)\}$ is equicontinuous at 1; since sets of the form $W(d_1, \dots, d_n, \epsilon)$ form a base of neighborhoods of 1 in K (see proof of 2.6), we may conclude that given any neighborhood U_1 of 1 in K there is a neighborhood U_2 of 0 in \mathbf{R}^n such that $\alpha_2(U_2) \subset U_1$ for every $\alpha \in B$. It follows easily that given any neighborhood U of 1 in G there is a neighborhood W of 1 in G contained in U , with $\alpha(W) \subset U$ for every $\alpha \in B$. Then $\bigcup \alpha(W)$ is a B invariant neighborhood of 1 contained in U . Thus G has small B invariant neighborhoods and 2.8 and 2.7 are proved.

3. Representations of $[FC]^-$ groups. In this section we show that a type I $[FC]^-$ group has a Hausdorff dual space and consequently is CCR. We again assume G is second countable.

3.1 Proposition. *If $G \in [FC]^-$ then $G \in [IN]$. If in addition G is type I then $G \in [FD]^-$ and G satisfies an exact sequence $(1) \rightarrow K \rightarrow G \rightarrow \mathbf{R}^n \times A \rightarrow (1)$, where K is compact and A is discrete abelian.*

Proof. If $G \in [FC]^-$ then $G \in [IN]$ by the Corollary to 2.2, and moreover, by 2.2, G satisfies an exact sequence $(1) \rightarrow K \rightarrow G \rightarrow \mathbf{R}^n \times D \rightarrow (1)$ where K is compact and D is discrete FC . If G is type I, D is type I, and therefore D has a finite commutator by Theorem 5.4 of [11] and Theorem 5.3 of [23]. It follows that $G \in [FD]^-$ and satisfies an exact sequence of the kind described.

3.2 Definition. The character $t \mapsto t$ of T will be denoted χ_0 . If G satisfies the central exact sequence $(1) \rightarrow T \rightarrow G \rightarrow H \rightarrow (1)$ then, for each $\chi \in \hat{T}$, we let

$$\hat{G}_\chi = \{\pi \in \hat{G} \mid \pi|_T = \chi \cdot I\}$$

and

$$G_\chi^{\text{fac}} = \{\pi \mid \pi \text{ a factor representation of } G \text{ and } \pi|_T = \chi \cdot I\}.$$

We denote $\widehat{G}_{\mathbf{x}_0}$ by \widehat{G}_1 and $G_{\mathbf{x}_0}^{\text{fac}}$ by G_1^{fac} .

3.3 Lemma. *Suppose $(1) \rightarrow T \rightarrow G \rightarrow H \rightarrow (1)$ is a central exact sequence of locally compact groups. Then every element of G_1^{fac} is type I if and only if \widehat{G}_1 is a closed and open T_0 subspace of \widehat{G} .*

Proof. For each $\chi \in \widehat{T}$ let m_χ be the central idempotent measure in $M^1(G)$ concentrated on T and given by $\overline{\chi(t)}dt$ there, where dt is normalized Haar measure on T . For each $f \in L^1(G)$ and $\pi \in \widehat{G}$ we have $\|\pi(f * m_\chi)\| \leq \|m_\chi\| \|\pi(f)\|$; it follows that multiplication (convolution) by m_χ is a continuous endomorphism of $C^*(G)$.

If $\pi \in G_{\mathbf{x}}^{\text{fac}}$, then $\pi(m_\chi) = I$ and $\pi(m_{\chi'}) = 0$ for $\chi \neq \chi'$. Moreover it is easy to see that each \widehat{G}_χ is open and closed in \widehat{G} . It is then routine to verify the following: $C^*(G) = \bigoplus_{\widehat{T}} C^*(G)m_\chi$, the C^* direct sum of closed orthogonal two sided $*$ ideals. In particular, we are asserting that for every $a \in C^*(G)$, $am_\chi \rightarrow 0$ as $\chi \rightarrow \pm\infty$. Furthermore if $\pi \in G_{\alpha'}^{\text{fac}}$, then $\ker \pi \supset \bigoplus_{\chi \neq \alpha'} C^*(G)m_\chi$. Thus to say that every element of G_1^{fac} is type I is to assert that $C^*(G)m_{\mathbf{x}_0}$ is type I; it follows that G_1^{fac} being type I is equivalent to $\widehat{G}_1 = (C^*(G)m_{\mathbf{x}_0})^\wedge$ being T_0 .

The proof of the following proposition is similar to a calculation in §6, Chapter V, of [1].

3.4 Proposition. *Consider the central extension of locally compact groups $(1) \rightarrow T \rightarrow G \rightarrow \mathbb{R}^n \oplus D \rightarrow (1)$, where D is countable discrete abelian. If every element of G_1^{fac} is type I, then the map $\pi \mapsto \pi|_{Z(G)}$ defines a homeomorphism of \widehat{G}_1 onto $\widehat{Z(G)}_1$.*

Proof. By Proposition 1.7 we may assume that the cocycle giving G as the extension of T by $\mathbb{R}^n \oplus D$ is chosen so that $\alpha(x, y) = 1$ for all x, y in the image of $Z(G)$ in $\mathbb{R}^n \oplus D$; that is, for all $x, y \in Z_\alpha(\mathbb{R}^n \oplus D)$. Thus we may write $G = T \times (\mathbb{R}^n \oplus D)$ with multiplication defined by $\langle s, x \rangle \langle t, y \rangle = \langle st\alpha(x, y), x + y \rangle$ for all $s, t \in T$ and all $x, y \in \mathbb{R}^n \oplus D$ and the topology the unique locally compact group topology compatible with the Borel group structure and left invariant measure on $T \times (\mathbb{R}^n \oplus D)$. We then have (using the cocycle identity) that $\langle s, x \rangle \langle t, y \rangle \langle s, x \rangle^{-1} = \langle t\alpha(y, x)\alpha(y, x)^{-1}, y \rangle$ for all $s, t \in T$ and $x, y \in \mathbb{R}^n \oplus D$.

Let M be a maximal abelian subgroup of G . Then M is closed and necessarily contains T . Therefore M is normal. Furthermore since T is injective in the category of locally compact abelian groups we may write $M = T \oplus A$ where A is a closed abelian subgroup of M . The group G acts by inner automorphisms on M , hence also on the character group \widehat{M} . This action on \widehat{M} clearly preserves $\widehat{M}_1 = \{\chi_0\} \times \widehat{A}$. We compute the action of G on \widehat{M}_1 : let $(\chi_0, \lambda) \in \widehat{M}_1$, where $\lambda \in \widehat{A}$, and let $\langle s, x \rangle \in G$, $\langle t, 0 \rangle \in T$, and $\langle t', y \rangle \in A$. Then we have

$$\begin{aligned}
 (\chi_0, \lambda)^{(s, x)}(\langle t, 0 \rangle \langle t', y \rangle) &= (\chi_0, \lambda)(\langle t' a(x, y) a(y, x)^{-1}, y \rangle) \\
 &= (\chi_0, \lambda)(\langle t a(x, y) a(y, x)^{-1}, 0 \rangle \langle t', y \rangle) = t a(x, y) a(y, x)^{-1} \lambda(\langle t', y \rangle).
 \end{aligned}$$

In other words, we have $(\chi_0, \lambda)^{(s, x)} = (\chi_0, \lambda + \phi_x)$ where $\phi_x \in \hat{A}$ is defined by $\phi_x(\langle t, y \rangle) = \alpha(x, y) \alpha(y, x)^{-1}$ for all $\langle t, y \rangle \in A$. (ϕ_x is actually a continuous homomorphism by 1.7.) Moreover the map $\Phi: x \rightarrow \phi_x$ is a continuous homomorphism (see Proposition 1.1 of [16]) from $\mathbb{R}^n \oplus D$ to \hat{A} . Since M is maximal abelian it is easy to see that the kernel of Φ is $\{x \in \mathbb{R}^n \oplus D \mid \langle 0, x \rangle \in M\}$. We may summarize our computation by saying that the orbit of (χ_0, λ) under G is the set $\{\chi_0\} \times (\lambda + \text{Range } \Phi)$, where $\lambda + \text{Range } \Phi$ is the coset of the subgroup $\text{Range } \Phi$ in \hat{A} which contains λ , and that the stable group of (χ_0, λ) is M .

We claim that if every element of G_1^{fac} is type I, then $\text{Range } \Phi$ is closed. In fact, if $\text{Range } \Phi$ is not closed, find two distinct cosets of $\text{Range } \Phi$ contained in $\text{Range } \Phi$. These two cosets define two orbits in \hat{M}_1 contained in each other's closure. It follows by Theorem 4.3 of [7] that if π_1 and π_2 are characters chosen from the two orbits in \hat{M}_1 , then the two induced representations U^{π_1} and U^{π_2} of G weakly contain each other. Moreover by Theorems 7.6 and 8.1 of [19] these representations are irreducible and inequivalent. Thus \hat{G}_1 is not T_0 and by 3.3 there are elements of G_1^{fac} which are not type I. Thus if in fact every element of G_1^{fac} is type I, then $\text{Range } \Phi$ is closed.

Since $\text{Range } \Phi$ is closed, the orbits in \hat{M}_1 under G are all closed, and by Theorem 1 of [8] we get that every quasi-orbit concentrated in \hat{M}_1 is transitive. It follows by Theorems 7.6 and 8.1 of [19] that we have a bijection between \hat{G}_1 and the orbit space \hat{M}_1/G as follows: given an element $\pi \in \hat{G}_1$, $\pi|_M = n \int_{\hat{M}_1} \sigma d\mu(\sigma)$ (where n is an integer or ∞) for a unique measure class μ which is concentrated in an orbit $O(\mu)$ in \hat{M}_1 ; we associate the orbit $O(\mu)$ with π . Moreover given any element in $O(\mu)$ the induced representation (up to equivalence) is π . It is easy to see on the basis of Theorem 4.3 of [7] that this bijection is a homeomorphism, where $\hat{M}_1/G = \{\chi_0\} \times \hat{A}/\text{Range } \Phi$ has the topology of $\hat{A}/\text{Range } \Phi$.

Now let $r': \hat{A} \rightarrow (Z(G) \cap A)^\wedge$ be the restriction function. Then by 24.11 of [13], r' is continuous, open, surjective, and has kernel $(Z(G) \cap A)^\perp$, the annihilator of $Z(G) \cap A$ in \hat{A} . However it is easily seen that $Z(G) \cap A = \ker \hat{\Phi}$, where $\hat{\Phi}: A \rightarrow (\mathbb{R}^n \oplus D)^\wedge$ is the adjoint map to $\Phi: \mathbb{R}^n \oplus D \rightarrow \hat{A}$. It follows by 24.38 of [13] that $\ker r' = \text{Range } \Phi$. Thus r' induces a topological group isomorphism $\hat{A}/\text{Range } \Phi \rightarrow (Z(G) \cap A)^\wedge$.

Since $M = \widehat{T \oplus A}$ and $T \subset Z(G) \subset M$, we have $Z(G) = T \oplus (Z(G) \cap A)$. It follows that $\widehat{Z(G)}_1 = \{\chi_0\} \times (Z(G) \cap A)^\wedge$; consequently the restriction $r: \hat{M}_1 \rightarrow \widehat{Z(G)}_1$ induces a homeomorphism s of \hat{M}_1/G onto $\widehat{Z(G)}_1$. It is clear that, for $\pi \in \hat{G}_1$ $\pi|_{Z(G)} = s(G(\mu)) \cdot I$, and the proposition is proved.

Remark. Let $a \in H^2(\mathbb{R}^n \oplus D, T)$ and let α be a cocycle representing a such that $\alpha(x, y) = 1$ for all $x, y \in Z_\alpha(\mathbb{R}^n \oplus D)$. Let H be the group satisfying the central exact sequence $(1) \rightarrow T \rightarrow H \rightarrow \mathbb{R}^n \oplus D \rightarrow (1)$ and defined by α in the usual way. Let $\sigma: \mathbb{R}^n \oplus D \rightarrow H$ be a Borel section and $\pi \mapsto \pi \circ \sigma$ the bijection σ defines between \hat{H}_1 and $\hat{\alpha}(\mathbb{R}^n \oplus D)$. This bijection gives $\hat{\alpha}(\mathbb{R}^n \oplus D)$ a topology which is independent of the choice of σ since $\pi \circ \sigma$ and $\pi \circ \sigma'$ differ only by tensoring with an element of $(\mathbb{R}^n \oplus D)^\wedge$ for different Borel sections σ and σ' (see pp. 25–26 of [1]).

Corollary to 3.4. Let $a \in H^2(\mathbb{R}^n \oplus D, T)$ and let α be a cocycle representing a such that α is trivial on $Z_\alpha(\mathbb{R}^n \oplus D)$. If every factor α representation of $\mathbb{R}^n \oplus D$ is type I, then $\pi \mapsto \pi|_{Z_\alpha(\mathbb{R}^n \oplus D)}$ defines a homeomorphism of $\hat{\alpha}(\mathbb{R}^n \oplus D)$ onto $Z_\alpha(\mathbb{R}^n \oplus D)^\wedge$.

Proof. Let H be the group defined by α , so that $(1) \rightarrow T \rightarrow H \rightarrow \mathbb{R}^n \oplus D \rightarrow (1)$ is central exact. Then H_1^{fac} consists of type I elements, and so by 3.4 the map $\pi \mapsto \pi|_{Z(H)}$ defines a homeomorphism from \hat{H}_1 onto $\widehat{Z(H)}_1$. To prove the corollary, we need only show that for the given Borel section $\sigma: \mathbb{R}^n \oplus D \rightarrow H$ (i.e., $\sigma(x) = \langle 0, x \rangle$), then $\chi \mapsto \chi \circ \sigma$ is a homeomorphism from $Z(H)_1$ onto $Z_\alpha(\mathbb{R}^n \oplus D)^\wedge$. Since α is trivial on $Z_\alpha(\mathbb{R}^n \oplus D)$, $\sigma|_{Z_\alpha(\mathbb{R}^n \oplus D)}$ is a group isomorphism and hence continuous. (In fact, any Borel homomorphism from a second countable locally compact group into a second countable locally compact group is continuous – see pp. 14–15 of [1].) Reasoning as in 1.7 we may find a section σ' which gives rise to α and which is a topological group isomorphism when restricted to $Z_\alpha(\mathbb{R}^n \oplus D)$. Since σ and σ' differ by an element of $(\mathbb{R}^n \oplus D)^\wedge$ we conclude that $\sigma|_{Z_\alpha(\mathbb{R}^n \oplus D)}$ is a topological group isomorphism, so that $\sigma(Z_\alpha(\mathbb{R}^n \oplus D))$ is a closed complementary subgroup to T in $Z(H)$ and $\chi \mapsto \chi \circ \sigma$ provides a homeomorphism from $\widehat{Z(H)}_1$ onto $Z_\alpha(\mathbb{R}^n \oplus D)^\wedge$.

3.5 Proposition. Let G satisfy the central exact sequence $(1) \rightarrow T \rightarrow G \rightarrow \mathbb{R}^n \rightarrow (1)$. Then every element of G_1^{fac} is type I and therefore $\pi \mapsto \pi|_{Z(G)}$ is a homeomorphism from \hat{G}_1 onto $\widehat{Z(G)}_1$.

Remark. This is of course a very special case of Kirillov's results in *Unitary representations of nilpotent Lie groups* (Uspehi Mat. Nauk 17 (1962), 57–110). However the proof we give is short and in the spirit of 3.4, so we include it for completeness.

Proof. The following diagram is exact and commutative.

$$\begin{array}{ccccccc} (0) & \rightarrow & L(T) & \rightarrow & L(G) & \rightarrow & L(\mathbb{R}^n) \rightarrow (0) \\ & & \downarrow & & \downarrow & & \downarrow \\ (1) & \rightarrow & T & \rightarrow & G & \rightarrow & \mathbb{R}^n \rightarrow (1) \end{array}$$

Here the top line consists of Lie algebras and the vertical maps are exponential maps. Let σ' be a linear section from $L(\mathbb{R}^n)$ to $L(G)$, and let $\sigma = \exp_G \circ \sigma' \circ (\exp_{\mathbb{R}^n})^{-1}$. Then $\sigma: \mathbb{R}^n \rightarrow G$ is a section and the associated cocycle is defined by $\alpha(x, y) = \sigma(x)\sigma(y)\sigma(x+y)^{-1}$ for all $x, y \in \mathbb{R}^n$; a simple application of the Campbell-Hausdorff formula yields $\alpha(x, y) = \exp_G(\frac{1}{2}[\sigma'(X), \sigma'(Y)])$ where $\exp_{\mathbb{R}^n}(X) = x$ and $\exp_{\mathbb{R}^n}(Y) = y$. In other words, $\alpha(x, y) = e^{iB(x, y)}$ where B is an alternating bilinear form on \mathbb{R}^n . Since σ is continuous, G is homeomorphic to $T \times \mathbb{R}^n$ and α gives the multiplication of G in the usual way $\langle (s, x), (t, y) \rangle = \langle st\alpha(x, y), x + y \rangle$.

Now take A to be a maximal subspace of \mathbb{R}^n such that $B(x, y) = 0$ for all $x, y \in A$. It is easily seen that the preimage of A in G is a maximal abelian subgroup M . Moreover, $\sigma|_A$ is a topological group isomorphism and $\sigma(A)$ is a closed vector subgroup of M . We have that $M = T \oplus \sigma(A)$; let us regard $\sigma(A)$ as A and $M = T \oplus A$. If we apply the arguments from the preceding proposition to G , we may take the α , M , and A obtained in this proposition to be the corresponding entities used in the proof of the last proposition. In particular, we now have that $\Phi: \mathbb{R}^n \rightarrow \hat{A}$, where \hat{A} is a vector group and Φ is linear with its range automatically closed. Hence the arguments used in 3.4 go through for G , and we obtain that \hat{G}_1 is T_2 and by 3.3 G_1^{fac} is type I. This proves 3.5.

3.6 Theorem. *Suppose that $G \in [FC]^+$ is type I. Then \hat{G} is Hausdorff and G is CCR.*

Proof. By 3.1, G satisfies an exact sequence $(1) \rightarrow K \rightarrow G \rightarrow \mathbb{R}^n \times A \rightarrow (1)$ where K is compact and A is compact and A is countable discrete abelian. Since \hat{K} is countable discrete we certainly have that K is regularly embedded in G ; that is, every element of \hat{G} is obtained from \hat{K} via the Mackey procedure. For each $\sigma \in \hat{K}$ let \hat{G}_σ be the set of $\pi \in \hat{G}$ whose restrictions to K all include σ as a subrepresentation. We then have $\hat{G}_\sigma = \hat{G}_\tau$ if σ is in the orbit of τ under G and \hat{G}_σ and \hat{G}_τ are disjoint if σ is not in the orbit of τ . Using the $\#$ operation of K and the orthogonality relations for characters of K , one may see easily that each \hat{G}_σ is open and closed in \hat{G} . Therefore we will confine our attention to a single \hat{G}_σ .

Let $H(\sigma)$ be the stability group of σ in G . Let $\widehat{H(\sigma)}_\sigma$ be the set of all elements of $\widehat{H(\sigma)}$ whose restrictions to K include (and hence are multiples of) σ , and let $H(\sigma)_{\sigma}^{\text{fac}}$ be the set of all factor representations of $H(\sigma)$ whose restrictions to K are multiples of σ . We see as above that $\widehat{H(\sigma)}_\sigma$ is open and closed in $\widehat{H(\sigma)}$. Now $H(\sigma)/K$ is a closed subgroup of $\mathbb{R}^n \oplus A$ and is therefore of the form $\mathbb{R}^m \oplus D'$ where D' is countable discrete abelian. Therefore by Theorem 8.2

of [19] (see also the discussion on p. 60 of [1]) and by 1.7 there exists a cocycle α of $\mathbf{R}^m \oplus D'$ such that α is trivial on $Z_\alpha(\mathbf{R}^m \oplus D')$ and an $\bar{\alpha}$ representation σ' of $H(\sigma)$ which extends σ . Here $\bar{\alpha}$ denotes the inflation of α to $H(\sigma)$. Moreover, by Theorem 8.3 of [19] the map $\lambda \mapsto \bar{\lambda} \otimes \sigma'$ is a bijection between equivalence classes of irreducible (resp. factor) α^{-1} representations of $H(\sigma)/K$ and $\widehat{H(\sigma)_\sigma}$ (resp. $H(\sigma)_\sigma^{\text{fac}}$), where $\bar{\lambda}$ denotes the lifting of λ to $H(\sigma)$. Since G is type I, ${}_\alpha(\mathbf{R}^m \oplus D')^{\text{fac}}$ is type I, by Theorem 8.4 of [19]. Therefore by the corollary to 3.4, $\pi \mapsto \pi|Z_\alpha(\mathbf{R}^m \oplus D')$ is a homeomorphism from ${}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ to $Z_\alpha(\mathbf{R}^m \oplus D')^\wedge$. We will use this fact to show that the bijection between ${}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ and $\widehat{H(\sigma)_\sigma}$ is a homeomorphism and that consequently $\widehat{H(\sigma)_\sigma}$ is Hausdorff. It will follow easily that \hat{G}_σ is Hausdorff.

To see that the bijection between ${}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ and $\widehat{H(\sigma)_\sigma}$ is a homeomorphism, let us first note that the topology on ${}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ may be described as follows: If $S \subset {}_\alpha(\mathbf{R}^m \oplus D')^\wedge$, then $\lambda \in \bar{S}$ if and only if there are elements $\lambda_n \in {}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ and vectors v_n and v such that the functions $(\lambda_n v_n, v_n) \rightarrow (\lambda v, v)$ uniformly on $\mathbf{R}^m \oplus D'$ compacta. This follows easily from the analogous description of the topology in \hat{H}_1 (where $(1) \rightarrow T \rightarrow H \rightarrow \mathbf{R}^m \oplus D' \rightarrow (1)$ is the extension defined by α) and the fact that a set in $\mathbf{R}^m \oplus D'$ is compact if and only if its preimage in H is compact. Now let $S \subset {}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ and suppose that the image of S in $\widehat{H(\sigma)_\sigma}$ is closed. If $\lambda \in \bar{S}$ there are elements λ_n in S with vectors v_n such that the functions $(\lambda_n v_n, v_n) \rightarrow (\lambda v, v)$ uniformly on $\mathbf{R}^m \oplus D'$ compacta for some vector v . Choose any vector w in the representation space of σ' . Then $(\bar{\lambda}_n \otimes \sigma'(v_n \otimes w), v_n \otimes w) \rightarrow (\bar{\lambda} \otimes \sigma'(v \otimes w), v \otimes w)$ uniformly on $H(\sigma)$ compacta. Hence $\bar{\lambda} \otimes \sigma'$ is in the image of S and $\lambda \in S$. Thus S is closed and $\lambda \mapsto \bar{\lambda} \otimes \sigma'$ is continuous.

Now suppose that $S \subset {}_\alpha(\mathbf{R}^m \oplus D')^\wedge$ is closed and that $\bar{\lambda} \otimes \sigma'$ is in the closure of the image of S in $\widehat{H(\sigma)_\sigma}$. Then there are elements $\lambda_n \in S$ and vectors v_n and v such that the functions $(\bar{\lambda}_n \otimes \sigma' v_n, v_n) \rightarrow (\bar{\lambda} \otimes \sigma' v, v)$ uniformly on $H(\sigma)$ compacta. We will show that $\lambda_n \rightarrow \lambda$ in ${}_\alpha(\mathbf{R}^m \oplus D')^\wedge$. Let H' denote the preimage of $Z_\alpha(\mathbf{R}^m \oplus D')$ in $H(\sigma)$, so that $H'/K = Z_\alpha(\mathbf{R}^m \oplus D')$. Since λ_n and λ are irreducible α^{-1} -representations and $\alpha|Z_\alpha(\mathbf{R}^m \oplus D')$ is trivial, we have by Schur's lemma that $\lambda_n|Z_\alpha(\mathbf{R}^m \oplus D') = \chi_n \cdot I$ and $\lambda|Z_\alpha(\mathbf{R}^m \oplus D') = \chi \cdot I$ for characters χ_n and χ of the abelian group $Z_\alpha(\mathbf{R}^m \oplus D')$. Denote by $\bar{\chi}_n$ and $\bar{\chi}$ the liftings of these characters to H' . We then have

$$\begin{aligned} ((\lambda_n \otimes \sigma')|H' v_n, v_n) &= \bar{\chi}_n \cdot (m_n \sigma'|H' v_n, v_n) \rightarrow \bar{\chi} \cdot (m \sigma'|H' v, v) \\ &= ((\bar{\lambda} \otimes \sigma')|H' v, v) \end{aligned}$$

uniformly on H' compacta where m_n and m are positive integers or $+\infty$. Moreover, $\sigma'|H'$ is an ordinary finite dimensional representation of H' , since σ' is an $\bar{\alpha}$ -representation of $H(\sigma)$ and $\bar{\alpha}$ is trivial on H' . We will show that $\bar{\chi}_n \rightarrow \bar{\chi}$ uniformly on H' compacta. To do this, divide H' by the intersection of the kernels of $\{\chi_n \sigma'\}$ and $\chi \sigma'$, and obtain thereby an $[FD]^- \cap [MAP]$ group H° . It is easy to see (on the basis of 17.3.7 of [5]) that such a group must be $[SIN]$; consequently it has a $\#$ operation by 1.3. Now $\bar{\chi}_n(m_n \sigma'|H'v_n, v_n) \rightarrow \bar{\chi}(m \sigma'|H'v, v)$ uniformly on H° compacta, and using 1.6 we have $(k\sigma'|H')^\# = (\deg \sigma'|H')^{-1} \cdot \text{Tr}(\sigma'|H')$ for k an integer or ∞ . Since $\bar{\chi}_n$ and $\bar{\chi}$ are central we have $\bar{\chi}_n \text{Tr}(\sigma'|H') \rightarrow \bar{\chi} \text{Tr}(\sigma'|H')$ uniformly on H° compacta, hence also on H' compacta. Since $\sigma'|K = \sigma$ is irreducible and finite dimensional, the linear span of $\{\sigma_k|k \in K\}$ is a full matrix algebra. Therefore any K coset in H' contains an x with $\text{Tr} \sigma'_x \neq 0$. It follows by the continuity of σ' that there is a neighborhood U of x with $|\text{Tr} \sigma'_y| \geq \epsilon > 0$ for all $y \in U$. Certainly $\bar{\chi}_n \rightarrow \bar{\chi}$ uniformly on U , and since $\bar{\chi}_n$ and $\bar{\chi}$ are constant on K cosets, a simple compactness argument yields convergence uniformly on H' compacta. Therefore $\lambda_n|Z_\alpha(\mathbb{R}^m \oplus D') = \chi_n \cdot I \rightarrow \chi \cdot I = \lambda|Z_\alpha(\mathbb{R}^m \oplus D')$ uniformly on $Z_\alpha(\mathbb{R}^m \oplus D')$ compacta; it follows by the corollary to 3.4 that $\lambda_n \rightarrow \lambda$ in ${}_\alpha(\mathbb{R}^m \oplus D')$. Since S is closed, $\lambda \in S$ and $\bar{\lambda} \otimes \sigma'$ is in the image of S . Thus this image is closed, and $\lambda \rightarrow \bar{\lambda} \otimes \sigma'$ is a homeomorphism. It follows that $\widehat{H(\sigma)_\sigma}$ is Hausdorff, since ${}_\alpha(\mathbb{R}^m \oplus D')^\wedge$ is Hausdorff. By Theorem 8.1 of [19], inducing gives a bijection between $\widehat{H(\sigma)_\sigma}$ and \hat{G}_σ . By Theorem 4.3 of [7], this bijection is a homeomorphism. (To see that this map is open, one must also use the orthogonality relations for K .) Since \hat{G} is the disjoint union of the open and closed sets \hat{G}_σ , \hat{G} is Hausdorff. This proves 3.6.

3.7 Theorem. *Suppose G is almost connected $[FC]^-$. Then G satisfies an exact sequence $(1) \rightarrow K \rightarrow G \rightarrow \mathbb{R}^n \rightarrow (1)$ for some compact normal subgroup K . Moreover G is type I and the preceding theorem applies.*

Proof. Since G is almost connected, the group D in 2.2 must be finite and so by enlarging K we may assume D is trivial. Since \hat{K} is discrete, K is regularly embedded in G and the Mackey procedure yields all of \hat{G} . Take $\pi \in \hat{K}$. Then since $\mathbb{R}^n \times \hat{K} \rightarrow \hat{K}$ is jointly continuous and \mathbb{R}^n is connected, the stable group $H(\pi) = G$. Thus every element of G^{fac} whose restriction to K includes π is of the form $\lambda \otimes \pi'$, where π' is an \mathbb{R}^n cocycle representation of G extending π and where λ is a cocycle factor representation of \mathbb{R}^n lifted to G . By 3.5, λ must be type I. Thus $\lambda \otimes \pi'$ is type I, and we have that every element of G^{fac} is type I. Thus G is type I.

4. Representations of type I $[IN]$ groups. In this section we show on the basis of the preceding section that type I $[IN]$ groups are CCR. The proof of

4.1 below follows arguments of Thoma in [27] and uses the succeeding three lemmas.

4.1 Proposition. *If $G \in [IN]$ is type I, then $[G:G_F] < \infty$, where G_F is the subgroup of all elements with precompact conjugacy classes.*

4.2 Lemma. *Let f be the characteristic function of a compact invariant neighborhood of 1 in G , and let $\psi = f * f^\sim$. Then ψ is continuous positive definite and vanishes at ∞ . If $k^2\psi \geq \phi$ is a positive definite function for some real k and some continuous positive definite ϕ (i.e., $k^2\psi = \phi + \gamma$ where γ is also positive definite) then ϕ tends to 0 at ∞ . If ϕ is central and continuous, then $\text{supp } \phi \subset G_F$.*

Proof. According to 13.4.11 of [5], ψ is positive definite and continuous and in fact $\psi(x) = (\lambda(x)f, f)$, where λ is the left regular representation of G and $f \in L^2(G)$. Let H be the closed cyclic subspace of $L^2(G)$ generated by $\{\lambda(x)f\}$. Then since $k^2\psi(x) = (\lambda(x)kf, kf)$, it follows by I, p. 398, of [22] that $\phi(x) = (B\lambda(x)f, f)$ where $B: H \rightarrow H$ is a positive selfadjoint operator which commutes with every $\lambda(x)|_H$. Now $A = B^{1/2}$ is a positive operator in the norm closed algebra generated by B , and it follows that $\phi(x) = (\lambda(x)kAf, kAf)$. Since $kAf \in H \subset L^2(G)$, it follows from 13.4.11 of [5] that ϕ tends to zero at ∞ . Now suppose ϕ is central. Since $G \in [IN]$, G_F is open. Since ϕ is constant on conjugacy classes and vanishes at ∞ , it cannot assume a nonzero value at any element of G whose conjugacy class is not precompact. Thus $\text{supp } \phi \subset G_F$.

The following lemma is a special case of Theorem 1 of [3].

4.3 Lemma. *Let N be an open normal subgroup of G and ϕ a positive definite function vanishing off N . Suppose that π_ϕ is the cyclic representation of G corresponding to ϕ and $\pi_\phi|_N$ is the cyclic representation of N corresponding to $\phi|_N$. Then the induced representation $U^{\pi_\phi|_N} \cong \pi_\phi$.*

4.4 Lemma. *We adopt the notation of 1.1. Suppose $G \in [IN]$. Then Φ_c contains a nonzero extreme point ϕ which is approached weak* (in $L^\infty(G) = L^1(G)'$) by elements of Φ_c which vanish off G_F .*

Proof. First let us define the function ψ as follows: let f be the characteristic function of a compact invariant neighborhood of 1 and f^\sim the usual adjoint ($f^\sim(x) = \overline{f(x^{-1})}$). Now let $\alpha = f * f^\sim$. Then α is continuous of compact support and positive definite. Since f and $f^\sim \in Z(L^1(G))$, $\alpha \in Z(L^1(G))$ and so α is central. (See for example p. 157 of Loomis, *Abstract harmonic analysis*.) We take ψ to be α normalized at 1.

The dual space $L^\infty(G)$ of $L^1(G)$ is locally convex in the weak* topology. Moreover, as noted in 1.1, ϕ is compact metric in this topology. It is convex as well, and so by the metrizable version of Choquet's Theorem (pp. 19 ff, of [24])

we have a regular positive Borel measure ν of total mass 1 such that $F(\psi) = \int_{\Phi_c} F(\beta) d\nu(\beta)$ for every weak* continuous linear functional F . Moreover the extreme points K of Φ_c form a Borel set and $\nu(\Phi_c \setminus K) = 0$.

Let us suppose that ϕ is in the topological support of ν ; i.e., that ϕ has no neighborhood U with $\nu(U) = 0$. Given weak* continuous linear functionals F_1, \dots, F_n and $\epsilon > 0$, define the neighborhood U of ϕ by $U = \{\beta \in \Phi_c \mid |F_i(\beta) - F_i(\phi)| < \epsilon \text{ for } i = 1, \dots, n\}$. Then since ϕ is in the topological support of ν there is a continuous positive function g on Φ_c with $\text{supp } g \subset U$ and $\int g(\beta) d\nu(\beta) = 1$. Let $\beta_g = \int g(\beta) \beta d\nu(\beta)$. We mean by this that β_g is the unique continuous central positive definite function in Φ_c with $F(\beta_g) = \int g(\beta) F(\beta) d\nu(\beta)$ for every continuous linear functional F (see Proposition 1.1 of [24]). Now for $i = 1, 2, \dots, n$ we have

$$\begin{aligned} |F_i(\beta_g) - F_i(\phi)| &= \left| \int g(\beta) (F_i(\beta) - F_i(\phi)) d\nu(\beta) \right| \\ &\leq \max_{\beta \in U} |F_i(\beta) - F_i(\phi)| \int g(\beta) d\nu(\beta) \leq \epsilon. \end{aligned}$$

Thus given F_1, \dots, F_n and $\epsilon > 0$ there is a $\beta_g \in \Phi_c$ with $|F_i(\beta_g) - F_i(\phi)| < \epsilon$ for each i . If $\lambda = \max g$ then $\lambda\psi \geq \beta_g$ as positive definite functions (in fact, if \langle, \rangle denotes the pairing between L^1 and L^∞ , then for $f \in L^1$, $\langle f^\sim * f, \beta_g \rangle = \int g(\beta) \langle f^\sim * f, \beta \rangle d\nu(\beta) \leq \lambda \int \langle f^\sim * f, \beta \rangle d\nu(\beta) = \langle f^\sim * f, \lambda\psi \rangle$). Therefore β_g vanishes off G_F by 4.2.

To complete the proof we note that ϕ may be taken to be a nonzero extreme point of Φ_c . Suppose that $(\text{top supp } \nu) \cap K \subset \{0\}$. Then since $\Phi_c \setminus \{0\} = (\Phi_c \setminus K) \cup (K \setminus \{0\})$, we have $\nu(\Phi_c \setminus \{0\}) = 0$ and $\psi = 0$, a contradiction. Therefore the topological support of ν includes nonzero extreme points, and these are approached weak* by elements β_g of Φ_c which vanish off G_F .

Proof of 4.1. Since $G \in [IN]$, G_F is open in G and it is certainly normal. By Proposition 17.3.5 of [5] the extreme points of Φ_c are 0 and the characters of finite type taking the value 1 at 1. Characters of finite type are associated as positive definite functions with cyclic factor representations of type I_n or type II_1 . Since G is type I these factor representations generate finite dimensional von Neumann algebras and since they are cyclic they must be finite dimensional representations of G by 15.5.3 of [5].

By 4.4 there is a character ϕ of finite type which is approached weak* by elements of Φ_c which vanish off G_F . We see easily that $\int_G f(x) \phi(x) dx = 0$ for all $f \in L^1(G)$ which vanish on G_F , and hence ϕ vanishes off G_F . It follows from 4.3 that the representation π_ϕ corresponding to ϕ is induced to G from G_F and since π_ϕ is finite dimensional we have $[G : G_F] < \infty$.

The following was pointed out to the author by J. M. G. Fell.

4.5 Proposition. *An open subgroup H of a type I group G is type I.*

Proof. One may verify easily that $C^*(H)$ is a C^* subalgebra of $C^*(G)$. Then the proposition follows from 4.3.5 of [5].

4.6 Theorem. *A type I $[IN]$ group is CCR. An almost connected $[IN]$ group is type I and hence CCR.*

Proof. If G is type I $[IN]$, G_F is type I $[FC]^-$ by 4.5 and therefore CCR by 3.6. If on the other hand G is almost connected $[IN]$, then G_F is almost connected $[FC]^-$ and so CCR by 3.7. Using 4.1 and the fact that an almost connected group can have no open subgroup of infinite index, we have that $[G : G_F] < \infty$ in either case. Since G_F is CCR, it follows that G is also CCR. (In fact Schochetman in 4.3 of [26] shows that a compact extension of a CCR group is CCR.)

As a corollary, we have the following widely known result. This was obtained also by E. Kaniuth [14], who showed that an $[SIN]$ group with type I regular representation has only finite dimensional irreducible representations, and in the $[FC]^- \cap [SIN]$ case also by R. Mosak [21].

Corollary to 4.6. *A type I $[SIN]$ group has only finite dimensional irreducible representations.*

Proof. In $G \in [SIN]$ then $L^1(G)$ has a central approximate identity and so every irreducible image of $C^*(G)$ has nonzero central elements. Since these images are algebras of all compact operators on various Hilbert spaces, the spaces in question must be finite dimensional.

5. $[IN]$ groups with Hausdorff duals. We have already shown that a type I $[FC]^-$ group has Hausdorff dual. We show in this section that an $[IN]$ group with Hausdorff dual is $[FC]^-$ and that an $[SIN]$ group with Hausdorff dual is the projective limit of $[Z]$ groups.

5.1 Lemma. *If G is a noncompact locally compact abelian group and H_1, \dots, H_n are noncompact closed subgroups, there is a sequence $\{\chi_m\} \subset \hat{G}$ such that $\chi_m|_{H_j}$ is nontrivial for each m and j , and $\chi_m \rightarrow 1$ uniformly on compacta.*

Proof. We dualize. We have \hat{G} is nondiscrete locally compact abelian group, and $(G/H_i)^\wedge$ a closed subgroup for each i . Also, $\hat{H}_i = \hat{G}/(G/H_i)^\wedge$ is not discrete and so $(G/H_i)^\wedge$ is nowhere dense in \hat{G} for each i . Now if the lemma were false there would be an open neighborhood N of 1 in \hat{G} such that

$N \subset \bigcup_{i=1}^n (G/H_i)^\wedge$. We have N locally compact in the relative topology and $(G/H_i)^\wedge \cap N$ nowhere dense in the relative topology. This contradicts the Baire category theorem, and the lemma is proved.

5.2 Theorem. *Suppose $G \in [IN]$. Then \hat{G} is Hausdorff if and only if $G \in [FC]$ and G is type I.*

Proof. The "if" part is 3.6. Now suppose $G \in [IN]$ and \hat{G} is T_2 . Since G is type I, $[G:G_F] < \infty$ by 4.1. By 3.1 and 4.5, $G_F \in [FD]^-$. Let K be the closure of the commutator subgroup of G_F , and let x_1, \dots, x_n be coset representatives for G_F in G excluding a representative for the coset G_F itself. We are assuming here that $G \neq G_F$ and will derive a contradiction. For $i = 1, 2, \dots, n$, let F_i be the closed subgroup of G_F generated by the set $\{x_i b x_i^{-1} b^{-1} \mid b \in G_F\}$. Since $x_i \notin G_F$, x_i^{-1} does not have a precompact conjugacy class, and since G_F has finite index in G , $\{b x_i^{-1} b^{-1} \mid b \in G_F\}$ cannot be precompact either. It follows that, for each i , F_i is not compact.

Now G_F/K is locally compact abelian and contains the noncompact closed images $\bar{F}_1, \dots, \bar{F}_n$ of F_1, \dots, F_n (the images are closed because $F_i K$ is closed for each i). Hence by Lemma 5.1 there is a sequence of characters of G_F/K converging to 1 uniformly on compacta such that each character is nontrivial on $\bar{F}_1, \dots, \bar{F}_n$. This sequence lifts to a sequence $\{\chi_n\}$ of one dimensional characters of G_F converging to 1 uniformly on compacta and such that $\chi_n|_{F_i}$ is nontrivial for each n and i . It follows that for each n and i there is an $b \in G_F$ with $\chi_n(x_i b x_i^{-1} b^{-1}) \neq 1$. Hence $x_i \notin H(\chi_n)$, the stable group of χ_n , for each n and i , and so for each n , $H(\chi_n) = G_F$. The Mackey theory implies that each induced representation U^{χ_n} is irreducible, and by Theorem 4.4 of [7] we have that every subsequence of U^{χ_n} weakly contains U^1 , the regular representation of G/G_F lifted to G , and hence weakly contains every subrepresentation of U^1 . This is easily seen to imply that U^{χ_n} converges to every irreducible subrepresentation of U^1 , and unless $G = G_F$ there will be more than one such subrepresentation. Thus if \hat{G} is T_2 , $G \in [FC]^-$.

5.3 Theorem. *If $G \in [SIN]$, then \hat{G} is Hausdorff if and only if G is the projective limit of $[Z]$ groups.*

Proof. Suppose G is the projective limit of $[Z]$ groups. Proposition 2.2 of [20] states that every factor representation of G is actually a representation of one of the $[Z]$ groups. Thus G has only finite dimensional irreducible representations. It is clear that $G \in [FC]^-$; therefore, by 3.6, \hat{G} is Hausdorff. (See also [21].)

Suppose \hat{G} is Hausdorff. Then since G is $[SIN]$ and type I the corollary to

4.6 implies that G has only finite dimensional irreducible representations. Theorem 3 of [20] states that G must be the projective limit of Lie groups, each containing an open $[Z]$ subgroup of finite index. Moreover by 5.2, $G \in [FC]^-$. Therefore to establish 5.3 we need only prove the following lemma.

5.4 Lemma. *If G is an $[FC]^-$ Lie group which contains a $[Z]$ subgroup of finite index, then G is a $[Z]$ group.*

Remark. The lemma is also true if we replace "Lie" by "compactly generated", but some such hypothesis is necessary. Here is an example of an $[FC]^-$ group which is not a $[Z]$ group but which contains a normal abelian subgroup of index 2. Let $A = (\Pi_1^\infty \mathbb{Z}_2) \oplus \Sigma_1^\infty \mathbb{Z}_2$, where $\Pi \mathbb{Z}_2$ has the product topology and the weak direct sum $\Sigma \mathbb{Z}_2$ has the discrete topology. Define the map α as follows: for $(z_i) \in \Pi \mathbb{Z}_2$ and $(w_i) \in \Sigma \mathbb{Z}_2$, put $\alpha((z_i), (w_i)) = ((z_i + w_i), (w_i))$. Then α is a continuous homomorphism of A into itself and $\alpha^2 = \text{identity}$, so that α is a topological group automorphism. Now let $G = A \times_\eta \mathbb{Z}_2$, where $\eta(x) = \alpha^x$. Then one may verify that the commutator of G is a subgroup of $\Pi \mathbb{Z}_2 \subset A$ and hence is precompact, and that $Z(G) = \Pi \mathbb{Z}_2$. Thus G is an $[FD]^-$ group with an abelian subgroup of index 2, but G is not a $[Z]$ group.

Proof of 5.4. Since open subgroups of $[Z]$ groups are $[Z]$ groups, we may assume that the $[Z]$ subgroup of finite index is normal. We first reduce 5.4 to a problem in abelian group theory. Suppose that G is a finite extension of a $[Z]$ group H , and that $Z(G) \cap Z(H)$ has compact quotient in $Z(H)$. Then $Z(G) \cap Z(H)$ has compact quotient in G , and so does $Z(G)$. Therefore we may simply show $Z(G) \cap Z(H)$ has compact quotient in $Z(H)$.

Now $Z(G)$ is the set of fixed points of $\mathcal{J}(G)$, the inner automorphisms of G . Each element of $\mathcal{J}(G)$ maps H onto itself and hence $Z(H)$ onto itself. Moreover if $\alpha \in \mathcal{J}(G)$ corresponds to an element of H , $\alpha|Z(H)$ is trivial. Since G/H is finite, G acts as a finite group F of automorphisms of $Z(H)$. We say that a topological group automorphism γ of a locally compact group L is of bounded displacement if the set $\{\gamma(x)x^{-1} | x \in L\}$ is precompact (this notion was introduced by Tits in [28]). In the present case we assume G is $[FC]^-$, and so it is simple to verify that each $\alpha \in F$ is of bounded displacement. Moreover, since G is a Lie group so is the closed subgroup $Z(H)$. Therefore we are reduced to 5.5 below.

5.5 Lemma. *Let F be a finite group of automorphisms of the abelian Lie group A . Suppose each element of F has bounded displacement. Then the group of common fixed points of F has compact quotient in A .*

Proof. We must reduce the problem a bit further. Since A is an abelian Lie

group, $A = T^n \oplus \mathbb{R}^m \oplus D$, where D is discrete. Each α maps $T^n \oplus \mathbb{R}^m$ into itself (since $T^n \oplus \mathbb{R}^m$ is the connected component of the identity) and T^n into itself (since T^n is maximal compact in $T^n \oplus \mathbb{R}^m$). Thus α defines an automorphism $\bar{\alpha}$ of D . We denote by \bar{F} the finite group of such automorphisms of D , each of which has finite displacement. For each $\bar{\alpha} \in \bar{F}$, $\ker(\bar{\alpha} - \text{id})$ has finite index in D . It follows that $D' = \bigcap \ker(\bar{\alpha} - \text{id})$ has finite index in D . Let $A' = T^n \oplus \mathbb{R}^m \oplus D'$. Each α maps A' into itself; moreover A/A' is finite, so that if $\bigcap \ker(\alpha|_{A'} - \text{id})$ has compact quotient in A' , $\bigcap \ker(\alpha - \text{id})$ has compact quotient in A . Finally, since each $\alpha|_{A'}$ is of bounded displacement and $\text{range}(\alpha|_{A'} - \text{id}) \subset T^n \oplus \mathbb{R}^m$, we actually have $\text{range}(\alpha|_{A'} - \text{id}) \subset T^n$ for each α . Thus 5.5 would follow from the following.

5.6 Lemma. *Let F be a finite group of automorphisms of the locally compact abelian group $A = T^n \oplus \mathbb{R}^m \oplus D$ such that the range of $(\alpha - \text{id})$ lies in T^n for each $\alpha \in F$. Then $\bigcap \ker(\alpha - \text{id})$, the group of fixed points of F , has compact quotient in A .*

Proof. We dualize (see [13] for details on duals and adjoint maps) and obtain the following equivalent lemma, which we prove outright.

5.7 Lemma. *Let F be a finite group of automorphisms of the locally compact abelian group $A = \mathbb{Z}^n \oplus \mathbb{R}^p \oplus K$, where K is compact. Suppose that $\ker(\alpha - \text{id}) \supset \mathbb{R}^p \oplus K$ for each $\alpha \in F$. Then $\sum_{\alpha \in F} \text{range}(\alpha - \text{id}) = \overline{\sum_{\alpha \in F} \text{range}(\alpha - \text{id})}$ is discrete in the relative topology.*

Proof. It is enough to show $\sum \text{range}(\alpha - \text{id})$ is discrete in the relative topology. Suppose $|F| = m$. The subgroup $B = \mathbb{R}^p \oplus K$ is characteristic in A . Let $\begin{pmatrix} z \\ b \end{pmatrix}$ denote the generic element of A , where z is a column of n integers and $b \in B$, and let $\alpha_1, \dots, \alpha_m$ be the elements of F . Since each $\alpha_i \in F$ is the identity on B , we have

$$\alpha_i \begin{pmatrix} z \\ b \end{pmatrix} = \begin{pmatrix} M_i z \\ Y_i z + b \end{pmatrix}$$

where each M_i is a $n \times n$ integer valued matrix and each Y_i is a n -tuple of elements of B . We may therefore write each

$$\alpha_i = \begin{pmatrix} M_i & 0 \\ Y_i & 1 \end{pmatrix};$$

we then have

$$\alpha_i \alpha_j = \begin{pmatrix} M_i M_j & 0 \\ Y_i M_j + Y_j & 1 \end{pmatrix},$$

and since $\alpha_i \alpha_i^{-1} = \text{id}$, M_i is invertible and in fact $M_i \in \text{SL}^{\pm}(n, \mathbb{Z})$ for each i ; that is, M_i is an automorphism of \mathbb{Z}^n for each i . If we define the integer k_{ij} by $\alpha_{k_{ij}} = \alpha_i \alpha_j$ we have

$$(1) \quad M_i M_j = M_{k_{ij}} \quad \text{and}$$

$$(2) \quad Y_{k_{ij}} = Y_i M_j + Y_j.$$

Now

$$\text{range}(\alpha_i - \text{id}) = \text{range} \begin{pmatrix} M_i - I & 0 \\ Y_i & 0 \end{pmatrix} = \left\{ \begin{pmatrix} M_i z - z \\ Y_i z \end{pmatrix} \mid z \in \mathbb{Z}^n \right\},$$

and

$$\sum \text{range}(\alpha_i - \text{id}) = \left\{ \sum_{i=1}^m \begin{pmatrix} M_i z_i - z_i \\ Y_i z_i \end{pmatrix} \mid z_i \in \mathbb{Z}^n \text{ each } i \right\}.$$

Since B is open in $\mathbb{Z}^n \oplus B$, it would suffice to show $B \cap \sum \text{range}(\alpha_i - \text{id})$ is finite. Let us therefore observe that

$$B \cap \sum \text{range}(\alpha_i - \text{id}) = \left\{ \begin{pmatrix} 0 \\ \sum Y_i z_i \end{pmatrix} \mid z_i \in \mathbb{Z}^n \text{ each } i \text{ and } \sum M_i z_i = \sum z_i \right\}.$$

We point out that $B \cap \sum \text{range}(\alpha_i - \text{id})$ is finitely generated, as a subgroup of the abelian subgroup of B generated by the coordinate entries of Y_1, \dots, Y_m . Next we show that $B \cap \sum \text{range}(\alpha_i - \text{id})$ is torsion. In fact by (2) we have $Y_j z_j = Y_{k_{ij}} z_j - Y_i M_j z_j$ for each i and j and any $z_j \in \mathbb{Z}^n$. It follows that if $\sum Y_j z_j \in B \cap \sum \text{range}(\alpha_i - \text{id})$ (i.e., if $\sum M_j z_j = \sum z_j$) we have

$$\begin{aligned} \sum_{j=1}^m Y_j z_j &= \sum_{j=1}^m Y_{k_{ij}} z_j - Y_i \left(\sum_{j=1}^m M_j z_j \right) = \sum_{j=1}^m Y_{k_{ij}} z_j - Y_i \sum_{j=1}^m z_j \\ &= \sum_{j=1}^m (Y_{k_{ij}} - Y_i) z_j \quad \text{for each } i. \end{aligned}$$

But then

$$m \sum_{j=1}^m Y_j z_j = \sum_{i=1}^m \sum_{j=1}^m (Y_{k_{ij}} - Y_i) z_j = \sum_{j=1}^m \left(\sum_{i=1}^m (Y_{k_{ij}} - Y_i) \right) z_j = 0$$

since k_{ij} runs over $1, \dots, m$ as i does whenever j is fixed. Thus $B \cap \sum \text{range}(\alpha_i - \text{id})$ is finite; it follows that $\sum \text{range}(\alpha_i - \text{id})$ is discrete, and Lemmas 5.7 and 5.4 are proved.

6. $[IN]$ groups with discrete duals. We conclude this paper by describing the second countable $[IN]$ groups with discrete dual spaces. As noted in the introduction, Baggett [2] has shown that any second countable group with

discrete dual is compact; here we sketch a simple proof of this fact in the $[IN]$ case.

The first step is to observe that a separable C^* algebra A with discrete dual is a dual C^* algebra; that is, has the following form: For each $\alpha \in \hat{A}$, let I_α be the unique closed two sided ideal of A whose dual space is $\{\alpha\}$ (see 3.2 of [5].) Then since I_α is a separable C^* algebra with only one irreducible $*$ representation, I_α is isomorphic to the algebra of all compact operators on a separable Hilbert space (see p. 296 of [22]). Let B consist of all sequences $\{x_\alpha\}$ such that $x_\alpha \in I_\alpha$ for each α and $\|x_\alpha\| \rightarrow 0$ as $\alpha \rightarrow \infty$. Define algebraic operations and involution componentwise, and define $\|\{x_\alpha\}\| = \sup \|x_\alpha\|$. Then B is a C^* algebra, and using the elementary theory of C^* algebras, especially the material in 3.2 of [5], one can show readily that A is isomorphic to B as a C^* algebra. It also follows easily that every irreducible $*$ representation of A is trivial on every I_α except one, and there it is faithful. Further any $*$ representation of A decomposes into the discrete direct sum of these irreducible $*$ representations.

Now consider a second countable $[IN]$ group G with discrete dual. Then G is type I $[IN]$ and therefore a finite extension of G_F , an $[FC]^-$ group. Now G_F is the directed union of open compactly generated subgroups, each of which is $[FD]^-$ by 3.20 of [12]. The class of locally compact amenable groups is closed under extension and directed unions (see e.g., Greenleaf, *Invariant means on topological groups and their applications*) and so G must be amenable. Therefore the regular representation of G is faithful on $C^*(G)$ and hence contains the trivial one dimensional representation as a direct summand. This means that $L^2(G)$ has a one dimensional translation invariant subspace, and it follows that G is compact.

REFERENCES

1. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc. No. 62 (1966), 199 pp. MR 34 #7723.
2. L. Baggett, *A separable group having a discrete dual space is compact* (preprint).
3. R. J. Blattner, *Positive definite measures*, Proc. Amer. Math. Soc. 14 (1963), 423–428. MR 26 #5095.
4. J. Braconnier, *Sur les groupes topologiques localement compacts*, J. Math. Pures Appl. (9) 27 (1948), 1–85. MR 10, 11.
5. J. Dixmier, *Les C^* -algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR 30 #1404.
6. J. M. G. Fell, *The dual spaces of C^* algebras*, Trans. Amer. Math. Soc. 94 (1960), 365–403. MR 26 #4201.
7. ———, *Weak containment and induced representations of groups*, Canad. J. Math. 14 (1962), 237–268. MR 27 #242.
8. J. G. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. 101 (1961), 124–138. MR 25 #146.
9. V. M. Gluškov, *The structure of locally compact groups and Hilbert's fifth problem*, Uspehi Mat. Nauk 12 (1957), no. 2 (74), 3–41; English transl., Amer. Math. Soc. Transl. (2) 15 (1960), 55–93. MR 21 #698; MR 22 #5690.

10. S. Grosser and M. Moskowitz, *On central topological groups*, Trans. Amer. Math. Soc. **127** (1967), 317–340. MR 35 #292.
11. ———, *Representation theory of central topological groups*, Trans. Amer. Math. Soc. **129** (1967), 361–390. MR 37 #5327.
12. ———, *Compactness conditions in topological groups*, J. Reine. Angew Math. **246** (1971), 1–40.
13. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR 28 #158.
14. E. Kaniuth, *Die Struktur der regulären Darstellung lokalkompakter Gruppen mit invarianter Umgebungsbasis der Eins* (preprint).
15. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N.J., 1955. MR 16, 1136.
16. A. Kleppner, *Multipliers on abelian groups*, Math. Ann. **158** (1965), 11–34. MR 30 #4856.
17. G. W. Mackey, *Induced representations of locally compact groups*. I, Ann. of Math. (2) **55** (1952), 101–139. MR 13, 434.
18. ———, *The theory of group representations*, Lecture Notes (Summer, 1955), 3 vols., University of Chicago, Dept. of Math., Chicago, Ill., 1955. MR 19, 117.
19. ———, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311. MR 20 #4789.
20. C. C. Moore, *Groups with finite dimensional irreducible representations* (preprint).
21. R. Mosak, *The L^1 and C^* algebras of $[FIA]_{\mathbb{B}}^-$ groups and their representations*, Trans. Amer. Math. Soc. **163** (1972), 277–310.
22. M. A. Naïmark, *Normed rings*, GITTL, Moscow, 1956; English transl., rev. ed., Noordhoff, Groningen, 1964. MR 19, 870; MR 34 #4928.
23. B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. (3) **1** (1951), 178–187. MR 13, 316.
24. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N.J., 1966. MR 33 #1690.
25. L. C. Robertson, *A note on the structure of Moore groups*, Bull. Amer. Math. Soc. **75** (1969), 594–599. MR 39 #7027.
26. I. Schochetman, *Topology and the duals of certain locally compact groups*, Trans. Amer. Math. Soc. **150** (1970), 477–489. MR 42 #422.
27. E. Thoma, *Über unitäre Darstellungen abzählbares, diskreter Gruppen*, Math. Ann. **153** (1964), 111–138. MR 28 #3332.
28. J. Tits, *Automorphismes à déplacement borné*, Topology **3** (1964), suppl. 1, 97–107.
29. D. M. Lee and T. S. Wu, *On existence of compact open normal subgroups of 0-dimensional groups*, Proc. Amer. Math. Soc. **26** (1970), 265–288.

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