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OBSTRUCTIONS TO EMBEDDING n-MANIFOLDS IN $(2n-1)$-MANIFOLDS

BY

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ABSTRACT. Suppose $f: (M^n, \partial M^n) \rightarrow (Q^{2n-1}, \partial Q^{2n-1})$ is a proper PL map between PL manifolds M^n and Q^{2n-1} of dimension n and $2n-1$ respectively, M compact. J. F. P. Hudson has shown that associated with each such map f that is an embedding on ∂M is an element $\overline{\alpha}(f)$ in $H_1(M; Z_2)$ when n is odd and an element $\overline{\beta}(f)$ in $H_1(M; Z)$ when n is even. These elements are invariant under a homotopy relative to ∂M. We show that, under slight additional assumptions on M, Q and f, f is homotopic to an embedding if and only if $\overline{\alpha}(f) = 0$ for n odd and $\overline{\beta}(f) = 0$ for n even. This result is used to give a sufficient condition for extending an embedding $f: \partial M^n \rightarrow \partial B^{2n-1}$ (B^{2n-1} denotes $(2n-1)$-dimensional ball) to an embedding $F: (M^n, \partial M^n) \rightarrow (B^{2n-1}, \partial B^{2n-1})$.

In [3] Hudson describes for each proper map $f: (M^n, \partial M^n) \rightarrow (Q^q, \partial Q^q)$ which is an embedding on ∂M an element $\overline{\alpha}(f)$ in $H_{2n-q}(M; Z_2)$ when n is odd and an element $\overline{\beta}(f)$ in $H_{2n-q}(M; Z)$ when n is even. In [3] it is shown that $\overline{\alpha}(f)$ is invariant under a homotopy relative to ∂M and in [2] the necessary lemmas are proven to show $\overline{\beta}(f)$ is invariant under a homotopy relative to ∂M.

The following is the main theorem:

Theorem 1. *If M^n is compact, connected and orientable, Q^{2n-1} is simply connected, $n \geq 4$, and $f_*: \Pi_2(M) \rightarrow \Pi_2(Q)$ is onto, then f is homotopic relative to ∂M to an embedding if and only if $\overline{\alpha}(f) = 0$ when n is odd and $\overline{\beta}(f) = 0$ when n is even.*

Theorem 1 is used to prove

Theorem 2. *Suppose M^n is a compact, connected, orientable, n-manifold, $\partial M \neq \emptyset$, $n \geq 4$ and $f: (M^n, \partial M^n) \rightarrow (B^{2n-1}, \partial B^{2n-1})$ is a proper map with $f | \partial M^n$ an embedding. If $i_*: H_1(\partial M) \rightarrow H_1(M)$ is the zero map using Z coefficients when n is even and Z_2 coefficients when n is odd, then $f | \partial M$ extends to an embedding.*

In [4], one finds a counterexample to Theorem 2 if the hypothesis on $i_*: H_1(\partial M) \rightarrow H_1(M)$ is omitted.

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All work is done in the piecewise linear category. [1] and [3] form a basic reference and all notation is adopted from [3].

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I.

Lemma 1. *Suppose M^n is a compact n-manifold, $n \geq 4$, Q^{2n-1} is a $(2n-1)$-manifold and $f: (M, \partial M) \to (Q, \partial Q)$ is a proper, PL general position map that is an embedding on ∂M. Then f is homotopic relative to ∂M to a PL map $g: (M, \partial M) \to (Q, \partial Q)$ such that $S_2(g)$ is the union of a collection of pairwise disjoint simple closed curves each of which is contained in the interior of M.*

Proof. Without loss of generality one can assume that f is actually an embedding on a neighborhood of ∂M [3, p. 174]. Now use a small general position shift relative to ∂M to homotope f to a new map f' with the following properties:

(1) $f' | \partial M = f | \partial M$,

(2) $\dim S_2(f') \leq 1$ and $S_2(f') \subset \mathring{M}$,

(3) if J is a triangulation of $S_2(f')$, J is a homogenous 1-complex.

Properties (1) and (2) are standard general position arguments and property (3) follows from the fact that $S_2(f') \subset \mathring{M}$ and the fact that intersection between the interiors of the top dimensional simplexes is transverse [1]. In fact $S_2'(f')$ is a one-dimensional manifold without boundary consisting entirely of nice double points. $S_2(f') - S_2'(f')$ consists of a finite number of points which "close off" the noncompact components of $S_2'(f')$. These are the branch points of $S_2(f')$ and if all of the branch points are nice branch points then $S_2(f')$ is a closed one-dimensional manifold and thus a collection of simple closed curves. Therefore, it is the branch points that are limit points for more than one pair of lines of double points that prevent f' from serving as the required map g. The necessary alterations to f' to eliminate all the horrid points are described for the case of a single horrid point p that is a limit point for two pairs of rays of double points. The more general case can be handled in an analogous way. Triangulations and simplicial notation are omitted as they only serve to complicate a simple idea.

Let D^n and D^{2n-1} denote regular neighborhoods of p and $f'(p)$ which are n- and $(2n-1)$-dimensional balls respectively. They are chosen through triangulations of M and Q which guarantee $f | D^n: (D^n, \partial D^n) \to (D^{2n-1}, \partial D^{2n-1})$ is a proper map. Let λ_1 and λ_1' denote one pair of arcs in D^n such that $f(\lambda_1) = f(\lambda_1')$ with $\lambda_1 \cap \partial D^n = b_1$, $\lambda_1' \cap \partial D^n = b_1'$, and one has $\lambda_1 \cap \lambda_1' = p$. Let λ_2 and λ_2' be the other pair of arcs identified by f with $\lambda_2 \cap \partial D^n = b_2$, $\lambda_2' \cap \partial D^n = b_2'$ and $\lambda_2 \cap \lambda_2' = p$. Now join b_1 to b_1' by an arc α in ∂D^n that misses b_2 and b_2'. Then $f(\alpha)$ is a simple closed curve in ∂D^{2n-1}. Since $n \geq 4$, one can span $f(\alpha)$ by an embedded disk Δ in ∂D^{2n-1} using general position and insure that it misses $f(b_2)$. Now take regular neighborhoods S and T of α and Δ in D^n and D^{2n-1} respectively so that $f'|S: (S, \partial S) \to (T, \partial T)$ is a proper map. S and T are n- and $(2n-1)$-dimensional balls respectively. Furthermore, letting S_1 and T_1 be the regular neighborhoods of α and Δ in ∂D^n and ∂D^{2n-1} determined by S and T, $f'|S_1: (S_1, \partial S_1) \to (T_1, \partial T_1)$ is a proper map and S_1 and T_1 are $(n-1)$- and $(2n-2)$-dimensional balls respectively. If $S_2 = \mathrm{cl}(\partial S - S_1)$ and $T_2 = \mathrm{cl}(\partial T - T_1)$, where cl denotes closure, then $f'|S_2: (S_2, \partial S_2) \to (T_2, \partial T_2)$ is also a proper map with $\partial S_1 = \partial S_2$, $\partial T_1 = \partial T_2$ and S_2 and T_2 are $(n-1)$- and $(2n-2)$-dimensional balls respectively. Now one forgets the map f' on $\mathring{D}^n$. Since $f'|\partial S_2$ is an embedding of ∂S_2 into ∂T_2, extend $f'|\partial S_2$ to an embedding $g_2: (S_2, \partial S_2) \to (T_2, \partial T_2)$. Then letting $g_1 = f'|S_1$, g_1 and g_2 together define a map $(g_1 \cup g_2)$ of ∂S into ∂T with $(g_1 \cup g_2)(\partial S)$ containing a single point of intersection. Extend $(g_1 \cup g_2)$ conewise to a map g' of S into T. Now g_2 and $f'|\mathrm{cl}(\partial D^n - S_1)$ extend conewise to a map g'' of $\mathrm{cl}(D^n - S)$ into $\mathrm{cl}(D^{2n-1} - T)$ similar to g' on S. Letting

$$g = \begin{cases} f' & \text{on } M - D^n, \\ g' & \text{on } S, \\ g'' & \text{on } D^n - S \end{cases}$$

gives a map that has replaced p by two nice branch points. Clearly a finite repetition of this step will change any map f' to the required map g. Since each step in the change will be on the interior of a ball in $\mathring{M}$, f is homotopic to g relative to ∂M.

Remark. The components that make up the singular set of the map constructed in Lemma 1 can be divided into three types.

Type 1. *Those circles* Σ *that are folded about two points*; i.e., $\mathrm{Br}(f|\Sigma) = \{a, b\}$.

Type 2. *Those pairs of circles which are identified by* f.

Type 3. *Those circles* Σ *such that* $f|\Sigma$ *is the double covering map of* $f(\Sigma)$.

Suppose $f: (M^n, \partial M) \to (Q^{2n-1}, \partial Q)$ is a PL general position map, Q is oriented and C is a component of $S_2(f)$ of type 1 with b a branch point of $f|C$. Triangulating so that f is simplicial, one can find regular neighborhoods P and R of b and $f(b)$ respectively, such that $f|P: P \to R$ is a proper map. Now $P \simeq B^n$ and $R \simeq B^{2n-1}$. Let R be oriented with the orientation induced from the orientation of Q and orient P (if M is oriented, the orientation for P is the one induced by the orientation of M). Now suppose n is odd.

Definition 1. *The sign of the branch point* b *is the self-intersection number of* $f(\partial P)$ *in* ∂R.

Note the sign of a branch is independent of the particular orientation given P since switching the orientation on ∂P does not change the self-intersection number of $f(\partial P)$ in ∂R.

In the case that n is even, the sign of a branch point is undefined since intersection numbers are skew-commutative in this case which makes the self-intersection number of $f(\partial P)$ meaningless. Suppose, however, that p_1 and p_2 are the points of ∂P such that $f(p_1) = f(p_2)$. Let B_1^{n-1}, B_2^{n-1} and B_3^{2n-2} denote ball neighborhoods of p_1, p_2 and $f(p_1)$ respectively with B_1 and B_2 having orientations induced from the orientation of ∂P and B_3 having orientation induced from ∂R. If $B_1 \# B_2$ denotes the intersection number of B_1 with B_2, then $B_1 \# B_2 = -(B_2 \# B_1)$. The following lemma shows the effect of having a global orientation for M^n on the signs of the branch points when n is odd.

Lemma 2. *Suppose M^n is an oriented n-manifold, n odd, and Q is an oriented $(2n-1)$-manifold. If $f: (M^n, \partial M^n) \rightarrow (Q^{2n-1}, \partial Q^{2n-1})$ is a proper, PL general position map with a component C in $S_2(f)$ of type 1, $C \subset \mathring{M}^n$ with the two branch points of C being denoted by b_1 and b_2, then* (sign b_1) = $-$(sign b_2).

A detailed proof of this is omitted. It follows from the fact that a regular neighborhood of C in $\mathring{M}$ is homeomorphic to $S^1 \times B^{n-1}$. This fact together with the definition of sign of a branch point shows that b_1 and b_2 must have opposite signs.

Lemma 3 and Lemma 4 which follow are taken from [1]. Lemma 3 is just Hudson's Lemma 10 with $t = 1$ and K just a point, and Lemma 4 is just Hudson's Lemma 25 with $t = 1$. The proofs have been omitted.

Lemma 3. *Suppose that there are nondegenerate PL maps*

$$f: B^n \rightarrow B^{2n-1}, \qquad h: \{0\} \times B^1 \rightarrow fS_2f$$

such that f is regular at the boundary, h is a homeomorphism, and $(fS_2f \cap \partial B^{2n-1}) = h(\partial B^1)$. Then if $n \geq 4$, there are PL maps

$$f': B^n \rightarrow B^{2n-1}, \qquad h': I \times \partial B^1 \rightarrow f'S_2f'$$

such that f' is regular at the boundary, h' is a homeomorphism,

$$f' \mid \partial B^n = f \mid \partial B^n, \qquad h' \mid \{0\} \times \partial B^1 = h \mid \{0\} \times \partial B^1$$

and $h'(\{1\} \times \partial B^1)$ is two branch points of f' of opposite sign when n is odd.

Lemma 4. *Suppose that $f: B^n \rightarrow B^{2n-1}$ is a PL map, regular at the boundary and having no horrid locus, and that $h: \partial I \times B^1 \rightarrow fS_2f$ is a PL homeomorphism such that $(fS_2f \cap \partial B^{2n-1}) = h(\partial I \times \{a\})$ where $\partial B^1 = \{a, b\}$, and f Br $f = h(\partial I \times \{b\})$. If n is even and ≥ 4; or if n is odd and > 4 and f has two branch* *points of opposite sign, then there is a PL map* $f'\colon B^n \to B^{2n-1}$ *regular at the boundary with* $S_2 f'$ *consisting entirely of nice double points, and a PL homeomorphism* $h'\colon I \times \{a\} \to f' S_2 f'$ *such that*

$$f' \mid \partial B^n = f \mid \partial B^n, \qquad h' \mid (\partial I \times \{a\}) = h \mid (\partial I \times \{a\}).$$

In order to shorten the hypothesis in the lemmas in the remainder of this section the following will be referred to as the *standard hypothesis*:

> M^n is a compact, connected, orientable, n-manifold with $n \geq 4$, and Q^{2n-1} is a simply connected, $(2n-1)$-manifold; $f\colon (M, \partial M) \to (Q, \partial Q)$ is a proper PL map with $f \mid \partial M$ an embedding.

Lemma 5. *Suppose* M, Q *and* f *fit the standard hypothesis. Then* f *is homotopic relative to* ∂M *to a PL map* $g\colon (M, \partial M) \to (Q, \partial Q)$ *such that* $S_2(g)$ *consists of a single component of type* 1.

Proof. By using Lemma 1, one can suppose that f is a nondegenerate, PL map such that $S_2(f) \subset \mathring{M}$ and further that $S_2(f)$ consists of a finite collection of simple closed curves, hereafter referred to as circles. Recall that the circles of $S_2(f)$ divided into three types. The plan is to eliminate the circles of type 2 and type 3 by introducing additional components of type 1. Then one can "join up" all the components of $S_2(f)$ to a single circle folded at two branch points.

Case 1. Suppose Σ_1 and Σ_2 are two circles identified by f to a single circle C. Let a_1 and a_2 be points of Σ_1 and Σ_2 respectively such that $f(a_1) = f(a_2)$. Let λ be an arc in $\mathring{M}^n$ from a_1 to a_2 not intersecting $S_2(f)$ again. Triangulate so $S_2(f)$, and λ are subcomplexes and f is simplicial. Then $f(\lambda)$ is a circle in $f(\mathring{M}) \subset \mathring{Q}$. $f(\lambda)$ is null homotopic in Q, and since $n \geq 4$, general position in $\mathring{Q}$ gives one a disk D in $\mathring{Q}$ not hitting $f(M)$ except in $f(\lambda) = \partial D$.

Now let W be a 2nd derived neighborhood of D in Q^{2n-1} and let $V = f^{-1}(W)$. Then V is a 2nd derived neighborhood of λ. Thus $W \simeq B^{2n-1}$, $V \simeq B^n$. $f \mid V\colon V \to W$ fits the hypothesis of Lemma 3. Apply the lemma to $f \mid V$ and one gets a map $h\colon V \to W$ such that $h \mid \partial V = f \mid \partial V$ and $f \mid M - \mathring{V}$ and h on V define a new map $(f \cup h)$ on M such that the circles Σ_1 and Σ_2 in $S_2(f)$ have been replaced by a single circle of type 1 in $S_2(f \cup h)$. Since f and $f \cup h$ differ only on $\mathring{V} \subset \mathring{M}$, $f(\mathring{V}) \subset \mathring{W}$ and $f \cup h(\mathring{V}) \subset \mathring{W}$, one has f and $f \cup h$ are homotopic relative to ∂M.

Case 2. Now one must change a component of type 3 in $S_2(f)$ to one of type one. But exactly the same procedure used in Case 1 works here.

Now after a finite number of applications of Case 1 and Case 2, one has altered the map f to a map f' such that $S_2(f')$ consists of a finite collection of circles such that each circle contains two branch points and f is homotopic to f' relative to ∂M. One can use these branch points to join up the circles a pair at a time until one gets a map g such that $S_2(g)$ consists of a single circle of type 1, and g is homotopic to f relative to ∂M.

Let Σ_1 and Σ_2 be a pair of circles in $S_2(f')$, a_1 and b_1 the branch points of $f'|\Sigma_1$, a_2 and b_2 the branch points of $f'|\Sigma_2$. Let λ be an arc from a_1 to a_2 and meeting $S_2(f')$ only in a_1 and a_2.

Now as in Case 1, one finds a regular neighborhood, W of $f'(\lambda)$ and a regular neighborhood V of λ such that $V = (f')^{-1}(W)$. Note that $V \simeq B^n$, $W \simeq B^{2n-1}$, and $f'|V$ is a proper map of V into W. If n is even, then the hypotheses of Lemma 4 are satisfied and one can join up Σ_1 and Σ_2 to have a single circle of type 1. If n is odd, then since M and Q are orientable, Lemma 2 guarantees that each pair of branch points on a circle has opposite signs. So by renaming a_2 and b_2, one can guarantee a_1 and a_2 have opposite signs. Again the hypotheses of Lemma 4 are satisfied and one can join up Σ_1 and Σ_2 to make a single circle. As in Case 1, the alterations to f' are on the interior of an n-ball in $\mathring{M}$ whose image is in the interior of a $(2n-1)$-ball in $\mathring{Q}$ so the new map at each stage is homotopic relative to ∂M to the previous map.

Suppose M is a manifold, C is a simple closed curve in M and x_0 is the base point of M. To say that C can represent the element p in $\Pi_1(M; x_0)$ means there is a map $j: [0, 1] \to C$ with $j(0) = j(1)$ and $j|(0, 1)$ a homeomorphism and a path γ from $j(0)$ to x_0 such that $[\gamma \circ j \circ \gamma^{-1}]$, denoted by $\gamma_*(j)$ is p ([ ] denotes homotopy class).

Lemma 6. *Suppose M, Q and f fit the standard hypothesis and C is a component of $S_2(f)$ of type* 1 *which represents the element ab in $\Pi_1(M; x_0)$. Then f is homotopic relative to ∂M to a proper map $g: (M, \partial M) \to (Q, \partial Q)$ such that $S_2(g) = (S_2(f) - C) \cup C_1 \cup C_2$ where C_1 and C_2 are of type* 1 *with C_1 representing a and C_2 representing b.*

Proof. Let j be a parametrization of C and γ a path from $j(0)$ to x_0 such that $\gamma_*(j) = ab$. For convenience we assume x_0 does not lie on $S_2(f)$ and we require a little more of the parametrization j; namely, that $j(0) = j(1) \notin (\mathrm{Br}(f) \cap C)$ and that $f(j(0)) = f(j(\tfrac{1}{2}))$. It is easy to see such a parametrization exists.

The fact that $[\gamma j \gamma^{-1}] = ab$ means that there exists a path δ from $j(\tfrac{1}{2})$ to x_0 such that $[\delta \circ (j|[0, \tfrac{1}{2}]) \circ \gamma^{-1}] = a$ and $[\gamma \circ (j|[\tfrac{1}{2}, 1]) \circ \delta^{-1}] = b$. Since $n \geq 4$, we can assume γ and δ are embedding such that the images of δ and γ do not intersect $S_2(f) - C$, $\gamma([0, 1]) \cap C = \gamma(0) = j(0)$, $\delta([0, 1]) \cap C = \delta(0) = j(\tfrac{1}{2})$, and $\gamma([0, 1]) \cap \delta([0, 1]) = \gamma(1) = \delta(1) = x_0$. Then the image of $\delta^{-1} \circ \gamma$ is an arc in M from $j(0)$ to $j(\tfrac{1}{2})$. Letting δ and γ represent their images in

M, one has $f(\delta^{-1}\circ\gamma)$ is a circle in $f(M)$. Triangulate so that f is simplicial, and $S_2(f)$, δ and γ are full subcomplexes. $f(\delta^{-1}\circ\gamma)$ bounds a disk D in $\mathring{Q}$, and by shifting to general position relative to ∂D, we have $D\cap f(\mathring{M})=\partial D = f(\delta^{-1}\circ\gamma)$. Now take second derived neighborhoods N of D in $\mathring{Q}$; $N\simeq B^{2n-1}$. $f^{-1}(N)=E$ is a second derived neighborhood of $\delta^{-1}\circ\gamma$; $E\simeq B^n$. Now observe that $f|E\colon E\to N$ fits the hypothesis of Lemma 3. So let $g'\colon E\to N$ be the new map given by Lemma 3. Then $f_1\colon(M,\partial M)\to(Q,\partial Q)$ defined by

$$f_1=\begin{cases} f & \text{on } M-E,\\ g' & \text{on } E\end{cases}$$

has new components C_1 and C_2. But since $x_0\in E$ and $E\simeq B^n$, it is easy to construct parametrizations j_1 and j_2 of C_1 and C_2 and paths γ_1 and γ_2 to x_0 so that $\gamma_{1*}(j_1)=a$, $\gamma_{2*}(j_2)=b$. Furthermore, since $f_1|M-E=f$, i.e. f was changed only on $E\subset\mathring{M}$, and $f_1(E)$ and $f(E)$ are contained in N, f_1 is homotopic to f relative to ∂M. See Figure 1.

Lemma 7. *Suppose M, Q and f satisfy the standard hypothesis; C is a component of $S_2(f)$ of type 1 which represents the element ab in $\pi_1(M;x_0)$ and let d be an element of $\pi_1(M;x_0)$. Then f is homotopic relative to ∂M to a proper map $g\colon(M,\partial M)\to(Q,\partial Q)$ such that $S_2(g)=(S_2(f)-C)\cup C'$ where C' represents $ad^{-1}bd$. Alternatively, if n is odd, then g can be constructed so that C' represents ab^{-1}.*

Proof. First apply Lemma 6 to f and the component C to get a new map f_1 such that $S_2(f_1)=(S_2(f)-C)\cup C_1\cup C_2$ where C_1 represents a and C_2 represents b. (See Figure 1.) The argument now divides into two cases—n odd and n arbitrary.

Suppose n is odd. Following the notation of Lemma 6, one can apply Lemma 4 to $f_1|E$ to join C_1 and C_2 back up by joining e_1 to e_2' and e_1' to e_2. This is possible because when n is odd, sign $b_1=-$ (sign b_2) and Lemma 4 can be applied to $f_1|E$ to join up in any one of the two possible ways [1]. Let g'' denote the map on E given by Lemma 4 and define as before

$$g=\begin{cases} f_1 & \text{on } M-\mathring{E},\\ g'' & \text{on } \mathring{E}.\end{cases}$$

One has that $S_2(g)=(S_2(f)-C)\cup C'$ where C' now can represent ab^{-1}.

In the case that n is even the situation with Lemma 4 is different and this leads to a different conclusion. When one considers $f_1|E$ and prepares to apply Lemma 4, it is impossible to join back up except in the way C was divided by applying Lemma 6. However, one can do the following whether n is even or odd:

Let γ_2' be a path from b_2 to x_0 such that $\gamma_{2*}'(j_2)=d^{-1}bd$. (See [7].)

Now arrange the setting to look like Figure 1 with γ_2' replacing γ_2. Let E' and N' denote second derived neighborhoods of $\gamma_1 \cup \gamma_2'$ and $f(\gamma_1 \cup \gamma_2')$, respectively, such that $f_1 | E' : E' \to N'$ is a proper map. One now applies Lemma 4 to $f_1 | E'$ to get a new map g''', knowing that e_1 is joined up with e_2 and e_1' with e_2' in $S_2(g''')$ since the sign relationships between e_1, e_2, e_1' and e_2' have not been altered by the new path γ_2'. Define

$$g = \begin{cases} f_1 & \text{on } M - \mathring{E}', \\ g''' & \text{on } \mathring{E}'. \end{cases}$$

Let C' denote the component of $S_2(g)$ that replaces $C_1 \cup C_2$. The only question left is can C' represent $ad^{-1}bd$. The choice of γ_2' and the fact that e_1 is joined up with e_2 and e_1' with e_2' guarantee that C' can represent $ad^{-1}bd$ in $\Pi_1(M; x_0)$.

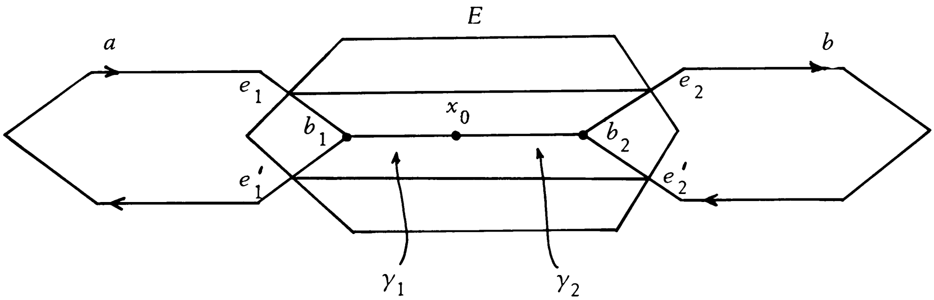

Figure 1

II. The standard hypothesis for §II is the same as the standard hypothesis for §I with one additional assumption. It is necessary to assume that a map $f: (M, \partial M) \to (Q, \partial Q)$ satisfying the standard hypothesis for §I also has the property that $f_*: \Pi_2(M) \to \Pi_2(Q)$ is onto. This assumption about f_* is the minimal substitute for the assumption that $\Pi_2(Q) = 0$ (See [1]).

Lemma 8. *Suppose M, Q and f satisfy the standard hypothesis and further that $S_2(f)$ consists of a single component of type 1. Then if $S_2(f)$ can represent 0 in $\Pi_1(M; x_0)$, f is homotopic to an embedding.*

Proof. If $S_2(f)$ can represent 0 in $\Pi_1(M; x_0)$ then since $n \geq 4$, there is an embedding h of the two-dimensional ball B^2 into M such that, if D denotes the image of B^2, $\partial D = S_2(f)$, and $x_0 \in \mathring{D}$. Now $[f \circ h]$ is an element of $\Pi_2(Q, f(x_0))$. Suppose $[f \circ h] = 0$. Then one may find an embedding k of the three-dimensional ball B^3 into Q such that, if E denotes the image of B^3, $\partial E = f(D) = (f \circ h)(B^2)$.

Now shift E into general position with respect to $f(M)$ keeping $\partial E = f(D)$ fixed and one has (int E) $\cap f(M)$ a finite set of points when $n = 4$ and empty if $n > 4$. When $n > 4$ one finishes the construction in the usual way by letting N be a second derived neighborhood of E in Q. $N \simeq B^{2n-1}$ and $f^{-1}(N) = P$ is a second derived neighborhood of D in M; hence, $P \simeq B^n$ and $f \mid P$: $P \to N$ is a proper map such that $f \mid \partial P$ is an embedding. Extend $f \mid \partial P$ conewise to embed P in N and one gets the required embedding of M in Q.

The case $n = 4$ requires further work. The technique is analogous to that used in Lemma 58 of [8], but is included for completeness. Let K denote the complex $f(M)$. Let $\Sigma \subset K$ denote $f(D)$; Σ is homeomorphic to ∂B^3. Let K' denote the complex $K \cup B^3$ constructed by attaching B^3 to K by sewing its boundary to $\Sigma \subset K$ using the map $k \mid \partial B^3$ defined previously. The essential question is can one extend the inclusion $i: K \to Q$ to an embedding $\phi: K' \to Q$?

Define $\phi': K' \to Q$ as follows: $\phi' \mid K = i$ and $\phi \mid B^3 = k$. Since B^3 is attached to K via $k \mid \partial B^3$, ϕ' is a continuous map of K' into Q, $\phi' \mid K$ and $\phi' \mid B^3$ are each embeddings and $S_2(\phi')$ consists of a finite set of pairs of points, with one point of each pair in $\phi'(K)$ and one point in $\phi'(\text{int } B^3)$. Now join each point of $S_2(\phi') \cap K$ to a point p of $\Sigma - /S_2/$ by arcs whose interiors are mutually disjoint and which intersect Σ only in p. Likewise join each point of $S_2(\phi') \cap B^3$ to the point p of $\Sigma - /S_2/$ by arcs whose interiors are mutually disjoint and which intersect Σ only in p. One has then embedded the cone over $S_2(\phi')$, denoted C_1, into K'. $\phi'(C_1)$ is a collection of simple closed curves in Q meeting in the point $\phi'(p)$. Since Q is simply connected, one can map in the cone over $\phi'(C_1)$, denoted C_2, and since $\dim Q = 7$, C_2 can be embedded and can be made to intersect $\phi'(K')$ only in the base of C_2, namely $\phi'(C_1)$. A derived neighborhood N of C_2 is a seven ball since C_2 is collapsible and its inverse image under a simplicial map is a derived neighborhood P of C_1 which is not a four ball but is a cone. P is homeomorphic to the union of a 4-ball B_1 and a 3-ball B_2 where a face of ∂B_2 is attached to an unknotted, properly embedded 2-ball in B_1. When one observes that ϕ' restricted to base of P is an embedding into ∂N, the usual alteration to $\phi' \mid P$ gives one a new map ϕ of K' which is an embedding. Proceed now as in the case $n > 4$.

One would be through with the proof of Lemma 8 but for the fact that $\Pi_2(Q)$ is not assumed to be zero; hence, it is possible that for an arbitrary choice of the embedding h, $[f \circ h]$ may not be 0 in $\Pi_2(Q; f(x_0))$. However, one can use the assumption that $f_*: \Pi_2(M; x_0) \to \Pi_2(Q; f(x_0))$ is onto to insure that the map h is chosen so that $[f \circ h] = 0$. For suppose h' is chosen arbitrarily and $[f \circ h'] \neq 0$. Since f_* is onto, there is a map k of B^2 into M taking ∂B^2 to X_0 such that $f_*[k] = -[f \circ h']$. Now

$$[f \circ k] + [f \circ h'] = f_*[k] + [f \circ h'] = -[f \circ h'] + [f \circ h'] = 0.$$

But through standard homotopy arguments one uses h' and k to define a map h'' of B^2 into M that equals h' on ∂B^2 and such that $[f \circ h''] = [f \circ h'] + [f \circ k] = 0$. h'' serves the purpose of h.

Remark. With each proper PL general position map $f: (M^n, \partial M^n) \to (Q^{2n-1}, \partial Q)$ such that $f \mid \partial M$ is an embedding and n is odd, Hudson described in [2] an element $\bar{\alpha}(f)$ in $H_1(M^n; Z_2)$ defined by constructing 1-chains using the one-dimensional simplexes of the singular set of f.

It is shown that $\bar{\alpha}(f)$ is independent of the homotopy class of f. When n is even and M and Q are orientable, reference is made to using the same technique to construct an element $\bar{\beta}(f)$ in $H_1(M; Z)$. In [3] one finds the lemmas needed to establish the existence of $\bar{\beta}(f)$ and its invariance under homotopy. Furthermore, assuming that $f(M)$ meets itself *transversely* along $f(S_2(f))$, it follows that the coefficients of 1-simplexes in the natural representative of either $\bar{\alpha}(f)$ or $\bar{\beta}(f)$ are ± 1.

Theorem 1. *Suppose M, Q and f satisfy the standard hypothesis. Then f is homotopic* rel ∂M *to an embedding if and only if*

(a) $\bar{\alpha}(f) = 0$ *if n is odd, or*

(b) $\bar{\beta}(f) = 0$ *if n is even.*

Proof. The necessity is obvious so we proceed with the sufficiency. In view of Lemma 5 and the fact that $\bar{\beta}(f)$ and $\bar{\alpha}(f)$ are independent of the homotopy class of f, one may assume without loss of generality that $S_2(f)$ consists of a single component C of type 1. The proof is now divided into two cases: (a) n even; (b) n odd.

Case (a). Suppose n is even. Then let $x_0' \in C$ and let $H: \Pi_1(M; x_0') \to H_1(M; x_0')$ be the Hurewicz homomorphism. Then the kernel of H is the commutator subgroup of $\Pi_1(M; x_0')$. There exists a map $j: ([0, 1], \{0, 1\}) \to (C, x_0')$ such that if $[j]$ denotes the homotopy class of the map j, based at x_0', $H([j]) = \bar{\beta}(f)$. Since the coefficients of $\bar{\beta}(f)$ are ± 1, we can choose j to take $(0, 1)$ homeomorphically onto $C - \{x_0'\}$. That is, C represents an element of the commutator subgroup of $\pi_1(M, x_0')$. In order to make the construction easier, it is best to now shift the base point away from C. Let $x_0 \in (M - C)$. Then by a choice of path γ from x_0' to x_0, one gets an induced isomorphism $\gamma_*: \Pi_1(M; x_0') \to \Pi_1(M; x_0)$. Then $\gamma_*[j]$ is in the commutator subgroup of $\Pi_1(M; x_0)$. Thus, $\gamma_*[j] = V_1 \cdot V_2 \cdot \cdots \cdot V_t$ where for each $1 \leq i \leq t$, $V_i = a_i b_i a_i^{-1} b_i^{-1}$, a_i, b_i in $\Pi_1(M; x_0)$. Suppose $t = 1$. Then $\gamma_*(j) = (ab)(a^{-1}b^{-1})$. As one might expect, Lemma 7 is going to be used, and the illustration accompanying this lemma will be helpful in following the argument. Apply Lemma 7 to $(ab)(a^{-1}b^{-1})$ with

$d = b$ and f is homotopic $\operatorname{rel} \partial M$ to a map g with $S_2(g) = (S_2(f) - C) \cup C' = C'$ and C' represents $(ab)(b^{-1})(a^{-1}b^{-1})b = 0$. Now g, M, and Q fit the hypothesis for Lemma 8 and g is homotopic $\operatorname{rel} \partial M$ to an embedding.

If $t > 1$, first apply Lemma 6 $(t-1)$ times to homotope f $\operatorname{rel} \partial M$ to a map f_1 with $S_2(f_1) = C_1 \cup C_2 \cup \cdots \cup C_t$ where C_i represents V_i for $1 \leq i \leq t$. Now one simply observes that one can perform the preceding operation on each C_i, $1 \leq i \leq t$, independently and after performing the operation on all the V_i's, one gets a map g homotopic $\operatorname{rel} \partial M$ to f such that g, M and Q fit the hypothesis of Lemma 8. Thus g is homotopic $\operatorname{rel} \partial M$ to an embedding.

Case (b). Suppose the same setting as in case a and that n is odd. The universal coefficient theorem gives $H_1(M; Z_2) \simeq H_1(M; Z) \otimes Z_2$. Consider the sequence

$$\Pi_1(M; x_0') \xrightarrow{H} H_1(M; x_0'; Z) \xrightarrow{T} H_1(M; x_0'; Z_2)$$

where H denotes the Hurewicz map and T the map induced by tensoring with Z_2. Let $j: ([0, 1], \{0, 1\}) \to (C, x_0')$ be a map such that $(T \circ H)([j]) = \overline{\alpha} = 0$. If $H([j]) = 0$, $[j]$ is in the commutator subgroup of $\Pi_1(M; x_0')$ and one has the same situation as in Case (a) and one gets the theorem. If $H([j]) \neq 0$, then $H([j])$ is in the kernel of T and one has $H([j]) = y^2$ for some y in $H_1(M; x_0; Z)$. Let k be a map of $([0, 1], \{0, 1\})$ into (M, x_0') such that $H([k]) = y$. Then $H([k]^2) = y^2 = H([j])$. Thus there is an element λ in the commutator subgroup of $\Pi_1(M; x_0')$ such that $\lambda \cdot [k]^2 = [j]$. Now one applies Lemma 6 to f to get a new map f_1 homotopic to f $\operatorname{rel} \partial M$ such that $S_2(f_1) = C_1 \cup C_2$ where C_1 can represent λ and C_2 can represent $[k]^2$. Now apply the appropriate part of Lemma 7 to both λ and $[k]^2$ and one gets a new map g, homotopic to f_1 $\operatorname{rel} \partial M$ such that g, M and Q fit the hypothesis of Lemma 8. So g is homotopic $\operatorname{rel} \partial M$ to an embedding, hence f is.

Corollary 1. *Suppose $f: (M^n, \partial M) \to (Q^{2n-1}, \partial Q)$ is a proper PL map with $f | \partial M$ an embedding, M^n is compact, orientable, connected with $n \geq 4$, and Q is orientable and 2-connected. If n is odd and $H_1(M; Z_2) = 0$ then f is homotopic to an embedding and, if n is even and $H_1(M; Z) = 0$, then f is homotopic to an embedding.*

Proof. Since Q^{2n-1} is 2-connected, f, M, Q fit the standard hypothesis of §II. In particular $f_*: \Pi_i(M) \to \Pi_i(Q)$ is onto for $i = 1, 2$. But the groups containing $\overline{\alpha}(f)$ or $\overline{\beta}(f)$ are both zero. Theorem 1 now applies.

Remark. Since every closed orientable, PL n-manifold M^n can be embedded in E^{2n-1} or in S^{2n-1} [6], it follows that if f is a PL map of M into S^{2n-1},

$\bar{\alpha}(f) = 0$ if n is odd and $\bar{\beta}(f) = 0$ if n is even. This fact is used to prove the following theorem.

Theorem 2. *Suppose M^n is a compact, connected, orientable, PL n-manifold, $\partial M \neq \emptyset$ and $n \geq 4$, and $f: (M^n, \partial M^n) \to (B^{2n-1}, \partial B^{2n-1})$ is a proper PL map with $f|\partial M^n$ an embedding. If $i_*: H_1(\partial M) \to H_1(M)$ is the zero map using Z coefficients when n is even and Z_2 coefficients when n is odd, then $f|\partial M$ extends to an embedding.*

Proof. Let M_1 and M_2 denote two copies of the manifold M. Let $f: (M, \partial M) \to (B^{2n-1}, \partial B^{2n-1})$ be a general position map, with $f|\partial M$ an embedding. Let DM denote the double of M, i.e. the closed manifold that results from sewing M_1 and M_2 together along the boundary by the identity map. Let $F: DM \to S^{2n-1}$ denote the mapping of DM into S^{2n-1} that comes from doubling the map f, i.e. use f to embed M_1 into B_1^{2n-1}, one copy of the standard $(2n-1)$ ball, and M_2 into B_2^{2n-1}, a second copy of B^{2n-1}. Then F results from sewing B_1 and B_2 together by the identity map. DM is orientable and $S_2(F)$ will just be two copies of $S_2(f)$. Using the Mayer-Vietoris sequence for the pair (M_1, M_2) and the fact that $i_*: H_1(\partial M; Z) \to H_1(M; Z)$ is the zero map gives that

$$(j_* \oplus k_*): H_1(M_1; Z) \oplus H_1(M_2; Z) \to H_1(DM; Z)$$

is injective; j and k denoting the inclusions into DM of M_1 and M_2 respectively. $\bar{\beta}(F) = j_*(\bar{\beta}(f)) \oplus k_*(\bar{\beta}(f))$. By [6], $\bar{\beta}(F)$ must be zero, and since $j_* \oplus k_*$ is injective, $\bar{\beta}(f)$ must be zero. By Theorem 1, f is homotopic to an embedding.

For the case where n is odd, the universal coefficient theorem together with the fact that $i_*: H_1(\partial M; Z_2) \to H_1(M; Z_2)$ is the zero map implies that the map

$$(j_* \oplus k_*): H_1(M_1; Z_2) \oplus H_1(M_2; Z_2) \to H_1(DM; Z_2)$$

is injective, so as before the fact that $\bar{\alpha}(F)$ must be zero gives $\bar{\alpha}(f)$ must be zero and the map extends to an embedding.

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