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P-COMMUTATIVE BANACH *-ALGEBRAS(1), (2)

BY

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ABSTRACT. Let A be a complex *-algebra. If f is a positive functional on A, let $I_f = \{x \in A: f(x^*x) = 0\}$ be the corresponding left ideal of A. Set $P = \bigcap I_f$, where the intersection is over all positive functionals on A. Then A is called P-commutative if $xy - yx \in P$ for all $x, y \in A$. Every commutative *-algebra is P-commutative and examples are given of noncommutative *-algebras which are P-commutative. Many results are obtained for P-commutative Banach *-algebras which extend results known for commutative Banach *-algebras. Among them are the following: If $A^2 = A$, then every positive functional on A is continuous. If A has an approximate identity, then a nonzero positive functional on A is a pure state if and only if it is multiplicative. If A is symmetric, then the spectral radius in A is a continuous algebra seminorm.

1. **Introduction.** The theory of commutative Banach *-algebras is, generally speaking, much better understood than the corresponding noncommutative theory. The purpose of this paper is to study a new class of noncommutative Banach *-algebras, called P-commutative Banach *-algebras, which properly contains the class of commutative Banach *-algebras; as we shall see, many of the well-known properties of commutative Banach *-algebras remain true for P-commutative algebras.

The structure of the paper is as follows. In §2 we introduce the notation and terminology which we shall use throughout. In §3 we define P-commutative algebras, prove some theorems concerning subalgebras, ideals, and quotients, and then discuss several examples. In §4 we show that if A is a P-commutative Banach *-algebra satisfying $A^2 = A$, then every positive functional on A is automatically continuous. N. Varopoulos [9] proved the same result for commutative Banach *-algebras with continuous involution. I. Murphy [5] gave a different proof which did not require continuity of the involution. We also show that if A is a P-commutative Banach *-algebra with an approximate identity, then the existence of a nonzero positive functional on A implies the existence of a multiplicative linear functional on A; and in fact, a nonzero positive linear functional is a pure state if

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and only if it is multiplicative. In §5 we study P-commutative Banach *-algebras which are symmetric. We show, among other things, that the spectral radius in such an algebra is a continuous algebra seminorm. An example will show that the assumption of symmetry cannot be dropped. Finally we show that an arbitrary Banach *-algebra with an approximate identity is P-commutative if its spectral radius $\nu(\cdot)$ satisfies $\nu(x^*x) \leq (x)^2$ for all x.

2. **Preliminaries.** We shall consider only associative complex linear algebras. By a *-algebra we will mean an algebra with an involution $x \longrightarrow x^*$; that is, a conjugate-linear anti-automorphism of period two. In normed algebras identities will be assumed to have norm one, and approximate identities will be two-sided and bounded by one. If A is a *-algebra, we denote by A_e the algebra obtained from A by adjoining an identity. We let J (or J_A) denote the Jacobson radical of A, and R (or R_A) the reducing ideal of A; that is, the intersection of the kernels of all irreducible *-representations of A on Hilbert spaces (R is called the *-radical in [7]). For typographical convenience we let J_e denote the radical of A_e (similarly, we write R_e). The words "multiplicative linear functional" will be abbreviated "MLF". If f is a positive functional on A, we set $I_f = \{x \in A: f(x^*x) = 0\}$. A positive functional f on A will be called "pure" if for every positive functional g on A satisfying $g(x^*x) \leq f(x^*x)$ for all $x \in A$, it follows that $g = \lambda f$ for some $0 \leq \lambda \leq 1$.

Now let A denote a Banach *-algebra. A continuous positive functional f on A will be called a "state" if $\|f\| = 1$. We denote by $\sigma(x)$ (or $\sigma_A(x)$) the spectrum of an element x in A, and by $\nu(x)$ (or $\nu_A(x)$) the spectral radius of x. We remark that if A has an approximate identity, then every positive functional f on A is representable [8]; a positive functional f on A is said to be representable if there exists a *-representation π of A on a Hilbert space H and a cyclic vector $\eta \in H$ such that $f(x) = (\pi(x)\eta \mid \eta)$ for every $x \in A$. In particular, f is Hermitian, continuous, and satisfies $f(R) \equiv 0$. Finally, we remark that any results we use from C. Rickart's book [7] will not require the assumption of local continuity of the involution by virtue of J. Ford's square root lemma [3].

3. **The definition, immediate consequences, and some examples.** Let A be a *-algebra, and let $P = \bigcap I_f$, where the intersection is taken over all positive functionals on A.

Proposition 3.1. *P is a two-sided ideal in A.*

Proof. Since P is the intersection of a family of left ideals, it suffices to show that P is a right ideal. If f is a positive functional on A and $y \in A$, set $f_y(x) = f(y^*xy)$ for each $x \in A$. Then f_y is a positive functional on A, and if

$x \in P$, we have $f[(xy)^*(xy)] = f(y^*x^*xy) = f_y(x^*x) = 0$. Hence $xy \in P$ and P is a right ideal. Q.E.D.

The following definition was suggested to the author by Professor R. S. Doran.

Definition 3.2. A *-algebra A is said to be P-commutative if $xy - yx \in P$ for every $x, y \in A$.

Proposition 3.3. *Let A be a P-commutative *-algebra. Then A_e is P-commutative.*

Proof. It is clear that $P_A \subset P_e$. Now let $(x, \lambda), (y, \alpha) \in A_e$. Then $(x, \lambda)(y, \alpha) - (y, \alpha)(x, \lambda) = (xy - yx, 0) \in P_A$ since A is P-commutative. Therefore, A_e is P-commutative. Q.E.D.

Proposition 3.4. *Let A be a P-commutative *-algebra, B a *-algebra, and ϕ a *-homomorphism of A onto B. Then B is P-commutative.*

Proof. Let f be an arbitrary positive functional on B. Then $f \circ \phi$ is a positive functional on A; so if $x \in P_A$, then $0 = (f \circ \phi)(x^*x) = f[\phi(x)^*\phi(x)]$, and thus, $\phi(x) \in P_B$. Hence, if $\phi(x), \phi(y)$ are arbitrary elements of B, then $\phi(x)\phi(y) - \phi(y)\phi(x) = \phi(xy - yx) \in \phi(P_A) \subset P_B$. Therefore, B is P-commutative. Q.E.D.

It follows from Proposition 3.4 that if I is a *-ideal in A, then A/I is a P-commutative *-algebra.

For the remainder of this paper, unless stated otherwise, A will denote a Banach *-algebra.

Since for every Banach *-algebra A, it is true that $R = \bigcap I_f$, f representable (see [7, 4.6.8, p. 226]), it is true that $P \subset R$. So we see immediately that if A is P-commutative, then A/R is commutative. In particular, if A is P-commutative and reduced (i.e., $R = \{0\}$), then A is commutative. Now if A has an approximate identity, then every positive functional on A is representable, and hence, $P = R$ and A is P-commutative if and only if A/R is commutative.

It is not surprising that a closed *-subalgebra of a P-commutative Banach *-algebra need not be P-commutative. We will give an example later in this section to illustrate this fact. However, we have the following result.

Proposition 3.5. *Let A be a P-commutative symmetric Banach *-algebra with an approximate identity $\{e_\alpha\}$. If B is a closed *-subalgebra of A containing $\{e_\alpha\}$, then B is P-commutative.*

Proof. We adjoin identities to A and B to obtain A_e and B_1 (we use the notation B_1 in this proposition only so that we may distinguish between the reducing ideals R_e of A_e and R_1 of B_1). Now A_e is P-commutative and symmetric [7, 4.7.9, p. 233], and B_1 is a closed *-subalgebra of A_e. Since both A and

B possess approximate identities, all positive functionals are representable, and it follows that $P_A = R_A = R_e$ and $P_B = R_B = R_1$. So by [7, 4.7.19, p. 237], we have that $P_B = R_1 = B_1 \cap R_e = B_1 \cap R_A = B \cap P_A$; it follows that B is P-commutative. Q.E.D.

A related result is given next.

Proposition 3.6. *Let A be a P-commutative symmetric Banach *-algebra. If I is a closed *-ideal of A having an approximate identity, then I is P-commutative.*

Proof. We have that $P_A \subset R_A = J_A$ and $P_I = R_I = J_I$. Since I is an ideal, it follows that $J_I = I \cap J_A$. Thus, we obtain $P_I \supset I \cap P_A$. The inclusion $P_I \subset I \cap P_A$ is obvious; hence I is P-commutative. Q.E.D.

We now give some examples. We begin by giving a few examples of noncommutative, P-commutative Banach *-algebras.

Example 3.7. Let A be a noncommutative radical Banach *-algebra (i.e., $J_A = R_A = A$). Then, for A_e, we have that $P_e = R_e = A$. Now if $(x, \lambda), (y, \alpha) \in A_e$, then $(x, \lambda)(y, \alpha) - (y, \alpha)(x, \lambda) = (xy - yx, 0) \in A = P_e$ which implies A_e is P-commutative. Obviously, A_e is noncommutative.

We next construct an example which is semisimple.

Example 3.8. Let A' be a semisimple, noncommutative Banach algebra with norm $\|\cdot\|'$ and involution $x \to x'$. Let $A = \{(x, y): x, y \in A'\}$. We denote elements of A by either (x, y) or a, b, etc. Furnish A with pointwise algebraic operations, norm $\|(x, y)\| = \|x\|' + \|y\|'$, and involution $(x, y)^* = (y', x')$. Then A is noncommutative and semisimple. We now show that A is P-commutative. Let f be a positive functional on A, let x be an arbitrary element in A', and set $a = (x, 0)$. Then $a^*a = 0$ implies $f(a^*a) = 0$; hence $a \in I_f$ and since f was arbitrary, $a \in P$. Similarly, every element of A of the form $(0, y)$ is in P. But, since P is an ideal in A, it follows that $(x, y) \in P$ for every $x, y \in A'$; i.e., $A = P$. Therefore, A is P-commutative.

In Example 3.8, if A' has an identity e, then A is an example of a Banach *-algebra having no nonzero positive functionals. Indeed, let f be a positive functional on A, choose an arbitrary $a \in A$, and let $1 = (e, e)$ denote the identity of A. Then $|f(a)|^2 = |f(1 \cdot a)|^2 \le f(1)f(a^*a) = 0$ and so $f \equiv 0$ on A.

We next use Example 3.8 to show that a closed *-subalgebra of a P-commutative algebra need not be P-commutative.

Example 3.9. Let A' be as in 3.8 above but with the added assumption that it is reduced. Now let $B = \{(x, x): x \in A'\}$. Then B is a closed noncommutative *-subalgebra of A. We show that B is not P-commutative. Let π be a *-representation of A' on some Hilbert space H. Then the map $\pi_B: B \to B(H)$ defined by

$\pi_B[(x, x)) = \pi(x)$ is a *-representation of B on H and $x \in \ker(\pi)$ if and only if $(x, x) \in \ker(\pi_B)$. Since A' is reduced, B is reduced. Therefore, since B is noncommutative, it follows that B cannot be P-commutative.

Example 3.10. Let B be any noncommutative, symmetric Banach *-algebra with continuous involution, identity e, and center Z, and assume that R_B is noncommutative. Let A be the closed *-subalgebra of B generated by Z and R_B. Then A is noncommutative and has identity e. We show that A is P-commutative. Since B is symmetric and $e \in A$, we have that $P_A = R_A = A \cap R_B = R_B$. Now let $z_1 + r_1, z_2 + r_2 \in A$. Then $(z_1 + r_1)(z_2 + r_2) - (z_2 + r_2)(z_1 + r_1) = r_1 r_2 - r_2 r_1 \in R_A$. Since R_A is closed in A and since every element of A is the limit of elements of the form $z + r$, where $z \in Z$, $r \in R_B$, it follows that $xy - yx \in R_A$ for every $x, y \in A$, i.e. A is P-commutative.

We next give an example showing that P is not always equal to R. It also shows that, in general, nothing can be said about the relationship between J and P.

Example 3.11. Let A be the set of all formal power series, $\Sigma a_n z^n$, satisfying $\Sigma |a_n|/n! < \infty$ (it will be understood that the sum runs from $n = 1$ to ∞). Give A the usual algebraic operations, norm defined by $\|\Sigma a_n z^n\| = \Sigma |a_n|/n!$, and involution $\Sigma a_n z^n \to \Sigma \bar{a}_n z^n$. Then A is a commutative, radical Banach *-algebra, so $J = R = A$. We show that $P \neq A$. Consider $f: A \to C$ defined by $f(\Sigma a_n z^n) = a_2$. Then f is clearly linear, and f is positive since $f[\Sigma \bar{a}_n z^n)(\Sigma a_n z^n)] = |a_1|^2 \geq 0$. Since there exist power series in A for which $a_1 \neq 0$, it follows that $I_f \neq A$ which implies $P \neq A$.

4. Multiplicative linear functionals and continuity of positive functionals. If A is a Banach *-algebra, we denote by A^2 the set of all finite sums of products of pairs of elements from A. I. Murphy [5] proved the following result:

Theorem. *Assume $A^2 = A$ and let every nonzero positive functional on A dominate a continuous nonzero positive functional. Then every positive functional on A is continuous.*

As a corollary, Murphy proved that if A is commutative and satisfies $A^2 = A$, then every positive functional on A is continuous. We extend this result to P-commutative algebras. The proof follows Murphy's essentially, the main deviation being in showing that $f_u(x^*x) = f_x(u^*u)$.

Theorem 4.1. *If A is a P-commutative Banach *-algebra satisfying $A^2 = A$, then every positive functional on A is continuous.*

Proof. Let f be a nonzero positive functional on A. Then f is Hermitian since every x in A can be written in the form $x = \Sigma_{i=1}^n \lambda_i x_i^* x_i$ (see [5]). For every $u \in A$, let f_u be the positive functional defined in the proof of Proposition 3.1. Then f_u is continuous on A (see [7, 4.5.4, (ii), p. 215]). Now if $f_u \equiv 0$ for every $u \in A$, then $|f(u^*x^*y)|^2 \leq f(u^*x^*xu)f(y^*y) = 0$ for every $u, x, y \in A$; hence, $f(A^3) \equiv 0$ which is false since $A^3 = A$. So we may choose $u \in A$ such that $f_u \not\equiv 0$, and since $f_{\alpha u} = |\alpha|^2 f_u$, we may assume that $\|u^*u\| < 1$. Ford's square root lemma then provides an element $v \in A_e$ such that $v^*v = e - u^*u$. Now we show that $f_u(x^*x) = f_x(u^*u)$ for every $x \in A$. Recall that I_f is a left ideal in A. It is well known that the linear space A/I_f, together with the inner product $(\cdot\,|\,\cdot)$ defined by $(x + I_f | y + I_f) = f(y^*x)$, is a pre-Hilbert space. Furthermore, since A is P-commutative, it follows that $ux + I_f = xu + I_f$ for every $x \in A$. Therefore, we have

$$\begin{aligned} f_u(x^*x) &= f(u^*x^*xu) = (xu + I_f \,|\, xu + I_f) \\ &= (ux + I_f \,|\, ux + I_f) = f(x^*u^*ux) = f_x(u^*u). \end{aligned}$$

Now for every $x \in A$,

$$\begin{aligned} (f - f_u)\,(x^*x) &= f(x^*x) - f_u(x^*x) = f(x^*x) - f_x(u^*u) \\ &= f[x^*(e - u^*u)x] = f(x^*v^*vx) = f[(vx)^*(vx)] \geq 0 \end{aligned}$$

since f is positive on A and A is an ideal in A_e. So f dominates f_u, and by an application of Murphy's theorem, we see that every positive functional on A is continuous. Q.E.D.

The following lemma is well known [2, 2.2.10, p. 28] for the case when A has isometric involution. A slight modification of the proof in [2] shows that it is true for Banach algebras with arbitrary involution. We therefore omit the proof.

Lemma 4.2. *Let A be a Banach *-algebra with approximate identity $\{e_\alpha\}$. Let π be a nondegenerate *-representation of A on a Hilbert space H, and let 1 denote the identity operator on H. Then $\lim \pi(e_\alpha) = 1$, where the limit is in the strong operator topology on $B(H)$.*

It is well known that on a commutative Banach *-algebra a nonzero positive functional is a pure state if and only if it is multiplicative.

Theorem 4.3. *Let A be a P-commutative Banach *-algebra with approximate identity $\{e_\alpha\}$. Then a nonzero positive functional f on A is a pure state if and only if it is multiplicative.*

Proof. Let f be a pure state on A. Since A has an approximate identity, $f(R) \equiv 0$ and so we may define a function $f': A/R \to C$ by $f'(x + R) = f(x)$. Clearly f' is linear and positive, and it is easy to check that it is also pure and that $\|f\| \leq \|f'\|$. Now positive scalar multiples of pure positive functionals are pure positive functionals; hence, there exists a scalar λ, $0 < \lambda \leq 1$, such that $\lambda f'$ is a pure state on A/R. Since A/R is commutative, it follows that $\lambda f'$ is multiplicative which implies that λf is multiplicative on A. Now f is representable, so we may write $f(x) = (\pi(x)\eta \mid \eta)$, where π is a *-representation on the Hilbert space H and $\eta \in H$. Then by Lemma 4.2, we see that $\lim_\alpha f(e_\alpha) = \|\eta\|^2$; and $\|\eta\|^2 \leq \|f\|$ since $\|e_\alpha\| \leq 1$ for every α. Now if we choose $x \in A$ such that $f(x) \neq 0$, then $(\lambda f)(x) = \lim(\lambda f)(xe_\alpha) = (\lambda f)(x) \lim(\lambda f)(e_\alpha)$ which implies $\lim(\lambda f)(e_\alpha) = 1$ or $\lim f(e_\alpha) = 1/\lambda \geq 1$ since $\lambda \leq 1$. But $\lim f(e_\alpha) = \|\eta\|^2 \leq \|f\| = 1$, and hence $1/\lambda = \lim f(e_\alpha) = 1$; thus $\lambda = 1$ and so f is multiplicative on A. This proves the first half of the theorem.

Now assume that f is a nonzero positive MLF on A. Again $f(R) \equiv 0$, so we define f' as before. Then f' is positive and multiplicative on A/R and, hence, is a pure state. Therefore, f is pure. If we choose $x \in A$ such that $f(x) \neq 0$, then $f(x) = \lim f(xe_\alpha) = f(x) \lim f(e_\alpha)$ which implies $\lim f(e_\alpha) = 1$; hence $\|f\| \geq 1$. But $\|f\| \leq \|f'\| = 1$ implies $\|f\| = 1$, and therefore f is a state. Q.E.D.

If A is P-commutative and has an approximate identity, we set

$$M = \{f : f \text{ is a nonzero positive } MLF \text{ on } A\} = \{f : f \text{ is a pure state}\}$$

(M may be empty). Since the image of every irreducible *-representation of A must be commutative, and since the commutant of each image is just the set of scalar operators [7, 4.4.12, p. 211], we see that M is precisely the set of irreducible *-representations of A. Therefore, $R = \bigcap \ker(f)$ $(f \in M)$. If we furnish M with the relative weak *-topology, then, as in the commutative case, M is a locally compact Hausdorff space. If A has an identity, then M is compact. If we define the map $x \to \hat{x}$ of A into $C_0(M)$ (the continuous functions on M which vanish at infinity) by $\hat{x}(f) = f(x)$, then $x \to \hat{x}$ is a norm decreasing *-homomorphism. It is injective if and only if A is reduced.

5. **Symmetric, P-commutative Banach *-algebras.** In this section we study P-commutative Banach *-algebras which are symmetric. Two results for such algebras have already been obtained (see 3.5, 3.6).

Proposition 5.1. *Let A be symmetric and P-commutative. Then a two-sided ideal in A is primitive if and only if it is maximal modular.*

Proof. The "if" part is true for any algebra. Now let I be a primitive ideal in A. Then since A is symmetric $R = J$, and therefore, $R \subset I$. If θ is the quotient map of A onto A/R, then $\theta(I) = I/R$ is a primitive ideal in A/R. But A/R commutative implies I/R is maximal modular which implies $I = \theta^{-1}(I/R)$ is maximal modular in A. Q.E.D.

It is well known that if A is a commutative Banach *-algebra, then A is symmetric if and only if every MLF on A is Hermitian.

Theorem 5.2. *If A is symmetric and P-commutative, then every MLF on A is Hermitian (hence positive).*

Proof. Let f be a MLF on A. Then $R = J \subset \ker(f)$ since $\ker(f)$ is a primitive ideal in A. If we define $f': A/R \to C$ by $f'(x + R) = f(x)$, then f' is a Hermitian MLF on A/R since A/R is commutative. Therefore, f is Hermitian. Q.E.D.

The converse of Theorem 5.2 is not true; that is, there exist nonsymmetric P-commutative algebras on which every MLF is Hermitian.

Example 5.3. Let A be a noncommutative Banach *-algebra having identity e, involution $x \to x'$, and no nonzero MLF's. Let B be a commutative, symmetric, semisimple Banach *-algebra with identity (denoted by e also) and involution $z \to z^*$. Now set $D = \{(x, y, z): x, y \in A, z \in B\}$. Give D pointwise algebra operations, involution $(x, y, z)^* = (y', x', z^*)$, and l_1-norm. Then D is just the product of B with Example 3.8; therefore, D is P-commutative. Now if ϕ is a MLF on D, then $\phi[(x, y, 0)] = 0$ for every $x, y \in A$ since A has no nonzero MLF's. Thus, the functionals $\phi: D \to C$, defined by $\phi[(x, y, z)] = \phi_B(z)$, where ϕ_B is a MLF on B, are the only MLF's on D. But since B is symmetric and commutative, all the ϕ_B's are Hermitian; thus ϕ is Hermitian. It remains to show that D is not symmetric. Let x be an invertible element of A. Set $y = -(x^{-1})'$ and $z \in B$. Then $(e, e, e) + (x, y, z)^*(x, y, z) = (e + y'x, e + x'y, e + z^*z) = (0, 0, e + z^*z)$ which cannot be invertible in D. Thus, D is not symmetric.

We remark that if A is P-commutative and symmetric, then there is a one-to-one correspondence between the set of all MLF's on A and the set of maximal modular two-sided ideals of A. Indeed, let I be such an ideal. Then A symmetric implies $R = J$; therefore $R \subset I$ and so A/I is commutative. The same argument as used for the analogous commutative result now applies (see [6, p. 192]).

Theorem 5.4. *Let A be symmetric and P-commutative. Then $R = \{x \in A: \nu(x) = 0\}$.*

Proof. Since $R = J$, it follows that $x \in R$ implies $\nu(x) = 0$. Conversely, let $x \in A$, $\nu(x) = 0$. Since A_e is P-commutative and symmetric, and since $\nu_A(x) = \nu_e(x)$ and $R_A = R_e$, we may assume that A has an identity. Let $f \in M$ (if $M = \emptyset$, the theorem is clear). Then f' (usual definition) is a MLF on A/R. But $\nu(x) = 0$ implies $\nu_{A/R}(x) = 0$; hence $0 = f'(x + R) = f(x)$ since A/R is commutative. Therefore, $x \in \bigcap \ker(f)$ $(f \in M)$ implies $x \in R$ (see the paragraph following Theorem 4.3). Q.E.D.

Lemma 5.5. *Suppose A is symmetric, P-commutative, and has an identity.*

Then for every x *in* A, $\sigma(x) = \{f(x): f \in M\}$.

Proof. The containment $\sigma(x) \subset \{f(x): f \in M\}$ follows immediately from Theorem 4.3 and [4, Theorem 4.10]. Now suppose that $\lambda = f(x)$ for some $f \in M$. Since $f'(x + R) = f(x) = \lambda$, it follows that $\lambda \in \sigma_{A/R}(x + R)$. Since $\sigma_{A/R}(x + R) \subset \sigma(x)$, the lemma follows. Q.E.D.

Theorem 5.6. *Let* A *be symmetric and P-commutative. Then for every* $x, y \in A$, $\nu(xy) \le \nu(x)\nu(y)$ *and* $\nu(x + y) \le \nu(x) + \nu(y)$.

Proof. We may assume A has an identity. Then by Lemma 5.5, $\nu(xy) = \sup|f(xy)| \le (\sup|f(x)|)(\sup|f(y)|) = \nu(x)\,\nu(y)$, where the supremums are taken over all f in M. Similarly, $\nu(x + y) = \sup|f(x + y)| \le \sup|f(x)| + \sup|f(y)| = \nu(x) + \nu(y)$. Q.E.D.

From Theorem 5.6 we see that $\nu(\cdot)$ is a continuous algebra seminorm in a symmetric, P-commutative Banach *-algebra.

We next give an example to show that, without the assumption of symmetry, $\nu(\cdot)$ need not be an algebra seminorm.

Example 5.7. Let B be a noncommutative Banach *-algebra with an identity and having two elements x and y satisfying $\nu_B(xy) > \nu_B(x)\nu_B(y)$. We may assume that $\nu_B(y) = \nu_B(x)$. Now form the P-commutative algebra A of Example 3.8 using B. Then A is not symmetric (see Example 5.3). It is easy to check that $\sigma_A[(x, y)] = \sigma_B(x) \cup \sigma_B(y)$. It follows that $\nu_A[(x, y)(y, x)] = \nu_A[(xy, yx)] = \nu_B(xy) > \nu_B(x)\nu_B(y) = \nu_A[(x, y)]\,\nu_A[(y, x)]$.

Theorem 5.8. *Let* A *be a Banach *-algebra with an approximate identity. If* $\nu(x^*x) \le \nu(x)^2$ *for every* $x \in A$, *then* A *is P-commutative.*

Proof. We first note that $R_A = \bigcap \operatorname{Ker}(f)$, where the intersection is over all positive functionals on A. This is proved in [6, p. 259] for algebras with an identity. It is easily extended to the present situation since $R_A = R_e$ and every positive functional is representable. Now let f be an arbitrary positive functional on A and f' its extension to A_e. Then for every $x \in A$, $|f(x)| \le f'(e)\nu(x^*x)^{1/2} \le f'(e)\nu(x)$, and by [1, Corollary 3] $f(xy) = f(yx)$ for every $x, y \in A$. Hence $xy - yx \in \bigcap \operatorname{Ker}(f) = R_A$ for every $x, y \in A$; A is therefore P-commutative. Q.E.D.

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