TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 180, June 1973

A CHARACTERISTIC ZERO NON-NOETHERIAN FACTORIAL RING OF DIMENSION THREE[1]

BY

JOHN DAVID

ABSTRACT. This paper shows the previously unknown existence of a finite dimensional non-Noetherian factorial ring in characteristic zero. The example, called "J", contains a field of characteristic zero and is contained in a pure transcendental extension of degree three of that field. J is seen to be an ascending union of polynomial rings and degree functions are introduced on each of the polynomial rings. These are the basic facts that enable it to be seen that two extensions of J are Krull. One of these extensions is a simple one and the other is a localization of J at a prime ideal P. In the case of the latter extension, it is necessary to show that the intersection of the powers of P is zero. As J is the intersection of these two extensions, a theorem of Nagata is all that is needed to show then that J is factorial. It is easily proved that J is non-Noetherian once it is known to be factorial.

0. Notation and introduction.

(a) *Notation.* (i) k is any field of characteristic zero.

(ii) $x, y, \alpha_0, \alpha_1, \cdots, \alpha_k, \cdots$ are algebraically independent variables over k.

(iii) s is the set map from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ given by

$$s(1) = 2, \qquad s(2) = 3, \qquad s(n) = s(n-2) \cdot \cdots \cdot s(2)s(1) \cdot n, \qquad n \geq 3.$$

(iv) For $N = 2, 3, \cdots, I'_N$ is the following ideal in $k[x, y, \alpha_0, \alpha_1, \cdots, \alpha_N]$:

$$I'_N = (x\alpha_1 - y,\ x\alpha_2 + \alpha_1^{s(2)} - \alpha_0,\ x\alpha_3 + \alpha_2^{s(3)} - \alpha_1,\ x\alpha_4 + \alpha_3^{s(4)} - \alpha_2, \cdots,$$
$$x\alpha_{i-1} + \alpha_{i-2}^{s(i-1)} - \alpha_{i-3}, \cdots, x\alpha_N + \alpha_{N-1}^{s(N)} - \alpha_{N-2}),$$

$$I'_1 = (x\alpha_1 - y).$$

Presented to the Society, January 26, 1973; received by the editors July 19, 1972.

AMS (MOS) subject classifications (1970). Primary 13F15, 13E05; Secondary 13A15, 13F20, 13C15, 13H99.

Key words and phrases. Factorial, Krull dimension, Noetherian, polynomial ring, degree function.

([1]) This work forms part of the author's doctoral dissertation at the University of Rochester. He wishes to thank Warren May for his kind help.

For $N = 1, 2, \cdots$, $R_0 = k[x, y, \alpha_0]$, $R_N = k[x, y, \alpha_0, \cdots, \alpha_N]/I'_N$.

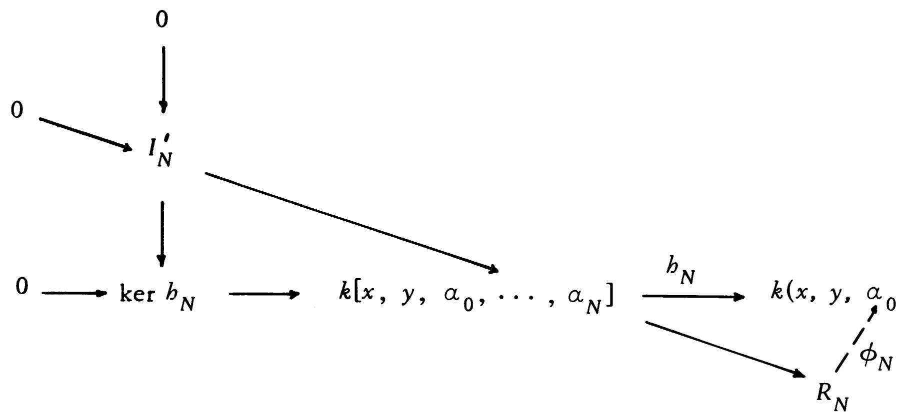
(v) For $N = 1, 2, \cdots$ we define a map

$$h_N \colon k[x, y, \alpha_0, \alpha_1, \cdots, \alpha_N] \to k(x, y, \alpha_0)$$

as follows:

$$h_N | k[x, y, \alpha_0] = \mathrm{id}_{k[x, y, \alpha_0]}, \qquad h_N(\alpha_1) = y/x.$$

For $i = 2, 3, \cdots$, $h_N(\alpha_i) = (-h_N^{s(i)}(\alpha_{i-1}) + h_N(\alpha_{i-2}))/x$. We now have the following exact diagram:



and so we get the map $\phi_N = \bar{h}_N$.

(vi) For $N = 1, 2, \cdots$ define

$$J_N = \phi_N(R_N), \qquad J_{-1} = J_0 = k[x, y, \alpha_0], \qquad J = \bigcup_{N=0}^{\infty} J_N.$$

(b) *The major result concerning J.*

Theorem 9. *J is a non-Noetherian factorial ring of dimension three in characteristic zero.*

General method of proof. We first show that ϕ_N is an isomorphism and J_N is a polynomial ring in three variables. We use this isomorphism freely to make the calculations easier. We show that x is a prime element in J and that $J = J_{(x)} \cap J[1/x]$. Next we show $J[1/x]$ is Krull; indeed it is a quotient ring of a polynomial ring in three variables. We then complete the major task, to show that $J_{(x)}$ is a rank one discrete valuation ring. We accomplish this by proving $\bigcap (x)^i = (0)$. We introduce degree functions on J_N, called

$$\overset{(N)}{\deg}_x, \quad \overset{(N)}{\deg}_{\beta_N}, \quad \overset{(N)}{\deg}_{\beta_{N-1}}.$$

Note that J_N is a polynomial ring in three variables x, β_N, β_{N-1}. We look at expansions of an arbitrary element α of J in J_N for N large enough and show that if there exists N such that $s(N+1) > \deg_{\beta_N}^{(N)} \alpha$ then $x^{(\deg_x^{(N)}(\alpha)+1)}$ does not divide α in J. Also we show that there does indeed exist such an N for each α of J, by looking again at expansions of α in J_m for m sufficiently large, and noting the rate of growth of s. Consequently we get $\bigcap (x)^i = 0$. It may be noted here that characteristic zero is essential for these arguments.

That J is Krull follows immediately. That J is three dimensional and non-Noetherian involves some simple combinatorial arguments.

Remark. The author wishes that J be called an "Ericka ring", and that the class of rings into which J logically falls be called the class of "Ericka rings". This class is those rings, commutative domains with identity, derived by adding some variables to a field ringwise and then "dividing out" infinitely often, as in the case of J.

(c) *Two theorems.*

Theorem 1 (Nagata [1, p. 21]). *Let A be a Krull ring, S a multiplicatively closed set in A, $0 \notin S$, generated by prime elements of A. Then if $S^{-1}A$ is factorial, so is A.*

Theorem 2 (Samuel [2, p. 28]). *Let A be factorial, $a, b \in A$ with $(a, b) = 1$. Let X be a variable over A. Then $aX - b$ is prime in $A[X]$.*

1. The maps ϕ_N are coherent injections, x is prime in J, and J is a three dimensional domain.

Lemma 1.1. *$\bar{x}$ is prime in R_i for all i.*

Proof. It suffices to show that $Q = (\alpha_1^{s(2)} - \alpha_0, \alpha_2^{s(3)} - \alpha_1, \cdots, \alpha_{N-1}^{s(N)} - \alpha_{N-2})$ is prime in $k[\alpha_0, \alpha_1, \cdots, \alpha_{N-1}]$. We will show something stronger:

$$k[\alpha_0, \cdots, \alpha_{N-1}]/Q \cong k[\alpha_{N-1}]$$

in a natural way, and we will use induction on this.

$k[\alpha_0, \alpha_1]/(\alpha_1^{s(2)} - \alpha_0)$ is a domain, by Eisenstein, and thus

$$\phi\colon k[\alpha_0, \alpha_1]/(\alpha_1^{s(2)} - \alpha_0) \to k[\alpha_1]$$

induced by $\alpha_0 \mapsto \alpha_1^{s(2)}$, $\alpha_1 \mapsto \alpha_1$, and id_k is an isomorphism.

Assume $\phi\colon k[\alpha_0, \alpha_1, \cdots, \alpha_m]/(\alpha_1^{s(2)} - \alpha_0, \cdots, \alpha_m^{s(m+1)} - \alpha_{m-1}) \to k[\alpha_m]$ induced by

$$\alpha_p \mapsto \alpha_m^{\prod_{i=1}^{m-p} s(i+p+1)}, \qquad 0 \le p \le m-1,$$

and $\mathrm{id}_{k[\alpha_m]}$ is an isomorphism. Then

$$k[\alpha_0, \cdots, \alpha_{m+1}]/(\alpha_1^{s(2)} - \alpha_0, \cdots, \alpha_{m+1}^{s(m+2)} - \alpha_m)$$

$$\cong_{(1)} (k[\alpha_0, \cdots, \alpha_m]/(\alpha_1^{s(2)} - \alpha_0, \cdots, \alpha_m^{s(m+1)} - \alpha_{m-1}))[\alpha_{m+1}]/(\alpha_{m+1}^{s(m+2)} - \bar{\alpha}_m)$$

$$\cong_{(2)} k[\alpha_m][\alpha_{m+1}]/(\alpha_{m+1}^{s(m+2)} - \alpha_m) \equiv V_{m+1},$$

where the composition of the two isomorphisms is the map ϕ':

ϕ' is natural, i.e.

$$\phi'(\bar{\alpha}_p) = \bar{\alpha}_m^{\prod_{i=1}^{m-p} s(p+i+1)}, \qquad 0 \le p \le m-1,$$

$$\phi'(\bar{\alpha}_m) = \bar{\alpha}_m, \quad \phi'(\bar{\alpha}_{m+1}) = \bar{\alpha}_{m+1} \quad \text{and} \quad \phi'|_k = \text{id}.$$

By Eisenstein, V_{m+1} is a domain. From dimension theory we have $V_{m+1} \cong_{(3)} k[\alpha_{m+1}]$ induced by

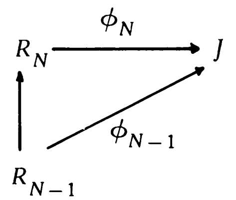
$$\alpha_m \mapsto \alpha_{m+1}^{s(m+2)}, \qquad \alpha_{m+1} \mapsto \alpha_{m+1}, \qquad \text{id}_k.$$

Thus the composition of all these isomorphisms (1), (2), (3), gives us the natural isomorphism to satisfy the induction assumption, and to prove the lemma.

Lemma 1.2. *$\bar{x} \nmid -\bar{\alpha}_{N-1}^{s(N)} + \bar{\alpha}_{N-2}$ in R_{N-1}, $N \ge 2$.*

Proof. Suppose there exist h_i, $q \in k[x, y, \alpha_0, \cdots, \alpha_{N-1}]$ such that $\alpha_{N-1}^{s(N)} - \alpha_{N-2} = qx + h_1(x\alpha_1 - y) + \cdots + h_{N-1}(x\alpha_{N-1} + \alpha_{N-2}^{s(N-1)} - \alpha_{N-3})$. Since $s(N-1) > 1$, the α_{N-2} term on the left cannot be cancelled and the equation is impossible.

Proposition 1. *Let $\beta_0 = \alpha_0$, $\beta_1 = y/x$, $\beta_2 = (-\beta_1^{s(2)} + \beta_0)/x$, $\beta_i = (-\beta_{i-1}^{s(i)} + \beta_{i-2})/x$ for $i \ge 3$. Then $\phi_N(\bar{\alpha}_i) = \beta_i$ and $\phi_N: R_N \to J_{N-1}[\beta_N]$ is an isomorphism and R_N is factorial of dimension 3. Thus it follows that the ϕ_N are coherent in the sense that $J_N \subseteq J_{N+1}$ and the following diagram commutes:*



where $R_{N-1} \to R_N$ is the natural map.

Proof. By induction on N:

$N = 0$: Clear.

$N = 1$: $\phi_1: k[x, y, \alpha_0, \alpha_1]/(x\alpha_1 - y) \to k[x, y, \alpha_0, y/x] = J_0[\beta_1]$ is an isomorphism by Theorem 2.

Since $k[x, y, \alpha_0, y/x] = J_0[\beta_1]$ is clearly a polynomial ring in three variables, it is factorial of dimension 3. Thus R_1 is also.

Now we assume $N > 1$ and that the proposition is true for $N-1$. First we show that $\phi_N: R_N \to J_{N-1}[\beta_N]$ is an isomorphism: We have $\phi_{N-1}: R_{N-1} \to J_{N-1}$ an isomorphism and $\overline{\phi}_{N-1}: R_{N-1}[X]/(\overline{x}X + \overline{\alpha}_{N-1}^{s(N)} - \overline{\alpha}_{N-2}) \to J_{N-1}[X]/(xX + \beta_{N-1}^{s(N)} - \beta_{N-2})$ is an isomorphism. Since R_{N-1} is assumed factorial, by Theorem 2 and Lemmas 1.1 and 1.2, $l: R_{N-1}[X]/(\overline{x}X + \overline{\alpha}_{N-1}^{s(N)} - \overline{\alpha}_{N-2}) \to R_N$ is an isomorphism where

$$l(\overline{\alpha}_i) = \overline{\alpha}_i, \quad i < N, \qquad l|_k = \text{id},$$
$$l(\overline{x}) = \overline{x}, \qquad l(y) = \overline{y}, \qquad l(X) = \overline{\alpha}_N.$$

Thus the map $\epsilon: R_N \to J_{N-1}[X]/(xX + \beta_{N-1}^{s(N)} - \beta_{N-2})$ where $\epsilon = \overline{\phi}_{N-1} \circ l^{-1}$ is an isomorphism. Since by induction J_{N-1} is factorial and $\phi_{N-1}: R_{N-1} \to J_{N-1}$ is an isomorphism, from Lemmas 1.1, 1.2 and Theorem 2, we can conclude that

$$\sigma: J_{N-1}[X]/(xX + \beta_{N-1}^{s(N)} - \beta_{N-2}) \to J_{N-1}[\beta_N]$$

is an isomorphism where

$$\sigma|_{J_{N-1}} = \text{id} \qquad \text{and} \qquad \psi[X] = \beta_N.$$

Therefore $\phi_N = \sigma \circ \epsilon: R_N \to J_{N-1}[\beta_N]$ is an isomorphism.

To complete Proposition 2, we must show that R_N is a three dimensional factorial ring. To show R_N factorial, we observe that

$$\psi: k[x, y, \alpha_0, \cdots, \alpha_N] \to k[x, \alpha_{N-1}, \alpha_N]$$

by (inductively)

$$\psi(\alpha_{N-2}) = x\alpha_N + \alpha_{N-1}^{s(N)}$$
$$\cdots$$
$$\psi(\alpha_{N-K}) = x\psi(\alpha_{N-K+2}) + \psi(\alpha_{N-K+1}^{s(N-K+2)})$$
$$\cdots$$
$$\psi|_{k[x, \alpha_N, \alpha_{N-1}]} = \text{id}$$

is such that $\ker \psi \supseteq I'_N$. So we get the following exact sequence

$$R_N \xrightarrow{\overline{\psi}} k[x, \alpha_{N-1}, \alpha_N] \to 0.$$

As shown above, R_N is a domain; if we can now show it to have dimension less than or equal to three, then $\overline{\psi}$ will be an isomorphism and R_N factorial of dimension 3.

By induction $R_N \cong J_{N-1}[\beta_N] \cong R_{N-1}[X]/(\overline{x}X + \overline{\alpha}_{N-1}^{s(N)} - \overline{\alpha}_{N-2})$ and R_{N-1} is a three dimensional domain. R_N must be of dimension three, since all the R_i are affine rings. Hence R_N is factorial of dimension three.

Remark 1.1. J_N is a polynomial ring in three variables. This follows from the fact that $\overline{\psi}$ is an isomorphism (Proposition 1).

Corollary 1.1. *x is prime in J.*

Proof. By Proposition 1 and Lemma 1.1, we have x prime in J_N for all N since $\phi_N(\overline{x}) = x$. Thus x is prime in $J = \bigcup J_N$.

Corollary 1.2. *J is a three dimensional domain.*

Proof. Since $J \subseteq k(x, y, \alpha_0)$, it is a domain. Now suppose there exists a chain of prime ideals of length greater than 3 in J, say $P_i \subsetneq P_{i+1}$, $i = 1, 2, 3, 4$. Since for all N, J_N is three dimensional, it must be false that $J_N \cap P_i \subsetneq P_{i+1} \cap J_N$, for all i and all N. By the "pigeon-hole" principle, there exists j such that $P_j \cap J_N = P_{j+1} \cap J_N$, infinitely often. This implies that $P_j = P_{j+1}$, since $J = \bigcup J_K$ and $J_K \subseteq J_{K+1}$, a contradiction.

Therefore $\dim J < 4$. That $\dim J = 3$ follows from the proof of the next lemma and the fact that $k[x, 1/x, y/x, \alpha_0/x]$ is three dimensional.

2. J **is factorial.**

Lemma 2.1. *$J[1/x]$ is factorial.*

Proof. We first calculate $J_N[1/x]$:

$$\begin{aligned} J_N[1/x] &= k[x, y, \alpha_0, \beta_1, \cdots, \beta_N][1/x] \\ &= k[x, 1/x, y/x, \alpha_0/x, ((y/x)^{s(2)} - (\alpha_0/x)x) \cdot 1/x, \\ &\qquad ((((y/x)^{s(2)} - (\alpha_0/x)x)1/x)^{s(3)} - y/x) \cdot 1/x, \cdots] \\ &= k[x, 1/x, y/x, \alpha_0/x]. \end{aligned}$$

Since this is a ring of quotients of a polynomial ring, a factorial ring, it too is factorial. Thus since it is seen that $J_N[1/x] = J_M[1/x]$ for all N, M, $J[1/x] = J_N[1/x]$ for all N. Thus $J[1/x]$ is a factorial ring.

Remark 2.1. $J[1/x]$ is Krull. This follows from Lemma 2.1.

Lemma 2.2. $J = J[1/x] \cap J_{(x)}$. (We are using Corollary 1.1.)

Proof. Since J is contained in both $J[1/x]$ and $J_{(x)}$ we will show that $J[1/x] \cap J_{(x)} \subseteq J$. Let $\alpha \in J[1/x] \cap J_{(x)}$; write

$$\alpha = r_0 + r_1/x + \cdots + r_N/x^N, \qquad r_i \in J,$$

as a minimal expression in N. Assume $N \geq 1$. Since $\alpha \in J_{(x)}$, there exists $s \in J$ such that $s\alpha \in J$ and $x \nmid s$. Since $s(\alpha x^{N-1} - r_0 x^{N-1} - r_1 x^{N-2} - \cdots - r_{N-1}) = r_N \cdot s/x$, $r_N \cdot s/x \in J$.

Since x is a prime not dividing s, x divides r_N in J, and the expression for α is not minimal—a contradiction. Hence the lemma.

Lemma 2.3. *$J_{(x)}$ is a rank one Dvr.*

Proof. The following equivalences hold: $J_{(x)}$ is rank 1 Dvr $\Leftrightarrow J_{(x)}$ is 1-dim $\Leftrightarrow \bigcap (x)^i = 0$.

Now we must show that $\bigcap (x)^i = 0$; this will be done in two parts:

Part 1. Let $0 \neq \alpha \in J$. Then there exists N such that $\alpha \in J_N$ and $\alpha = \Sigma c_{ijk} x^i \beta_N^j \beta_{N-1}^k$. Since J_n is the polynomial ring $k[x, \beta_n, \beta_{n-1}]$, by Remark 1.1, for $\alpha' \in J_m$:

> $\deg_x^{(m)} \alpha'$, $\deg_{\beta_K}^{(m)} \alpha'$, $K = m, m-1$, are well defined
>
> where $\deg_x^{(m)} \alpha'$ is the degree of α' considered as
>
> a polynomial in the variable x in J_m; and
>
> $\deg_{\beta_K}^{(m)} \alpha'$ is the degree of α' considered as a polynomial in the variable β_K in J_m, $K = m, m-1$.

We shall show in this part that if $s(N+1) > \deg_{\beta_N}^{(N)} \alpha$, then

$$x^{(\deg_x^{(N)} \alpha)+1}$$

does not divide α in J_m, for all $m \geq N$. Now let $t_1 = \deg_{\beta_N}^{(N)} \alpha$, $t_2 = \deg_{\beta_{N-1}}^{(N)} \alpha$, $t_3 = \deg_x^{(N)} \alpha$, and assume $s(N+1) > t_1$. Simply, we must show that α expressed as a polynomial in J_m, $m \geq N$, has a nonzero term $d_m x^{a_m} \beta_m^{b_m} \beta_{m-1}^{c_m}$ where $d_m \neq 0 \in k$ and $a_m \leq t_3$—i.e., the sub $\deg_x^{(m)} \alpha$ is less than or equal to t_3.

Clearly x^{t_3+1} does not divide α in J_N. We have α as it appears in J_{N+1}:

$$\begin{aligned} \alpha &= \sum c_{ijk} x^i \beta_N^j (x\beta_{N+1} + \beta_N^{s(N+1)})^k \\ &= \sum_{0 \leq w \leq k; i,j,k} c_{ijk} x^{i+w} \beta_{N+1}^w \beta_N^{j+s(N+1)(k-w)} d_{ijkw} \end{aligned}$$

where $0 \neq d_{ijkw} \in k$ since k is of characteristic 0. There exist i_0, j_0 such that $c_{i_0, j_0, t_2} \neq 0$ and $j_0 \geq \max\{j \mid \exists\ i \text{ s.t. } c_{ijt_2} \neq 0\}$. It is easy to see that (j_0, t_2) is the unique element of the set $\{(j, k) \mid \exists\ i \text{ s.t. } c_{ijk} \neq 0 \text{ and } j + s(N+1)k \text{ is maximum}\}$, because $s(N+1) > t_1$. Now if we express α as a polynomial in β_N in J_{N+1}, it will look like

$$\alpha = \beta_N^{j_0 + s(N+1)t_2} \cdot h_0 + f_0, \qquad 0 \neq h_0, f_0 \in J_{N+1},$$

where $\deg_{\beta_N}^{(N+1)} f_0 < j_0 + s(N+1)t_2$ and $x^{t_3+1} \nmid h_0$ in J_{N+1}. Therefore $x^{t_3+1} \nmid \alpha$ in J_{N+1}.

We now write α as it appears in J_{N+2}:

$$\alpha = \sum_{\substack{0\le w\le k \\ 0\le v\le j+s(N+1)(k-w)}} c_{ijk} x^{i+w+v} \cdot \beta_{N+2}^{v} \cdot \beta_{N+1}^{w+s(N+2)(j+s(N+1)(k-w)-v)} \cdot d_{ijkwv},$$

where $0 \neq d_{ijkwv} \in k$.

Clearly,

$$\overset{(N+2)}{\deg}_{\beta_{N+1}} \alpha = s(N+2)(j_0 + s(N+1)t_2)$$

and

$$\alpha = \beta_{N+1}^{s(N+2)(j_0+s(N+1)t_2)} \circ b_1 + f_1, \qquad 0 \neq b_1,\ f_1 \in J_{N+2},$$

where

$$\overset{(N+2)}{\deg}_{\beta_{N+1}} f_1 < s(N+2)(j_0 + s(N+1)t_2)$$

and

$$x^{t_3+1} \nmid b_1 \quad \text{in } J_{N+2}.$$

Therefore $x^{t_3+1} \nmid \alpha$ in J_{N+2}.

In general, we see that, in J_{N+r},

$$(*) \quad \alpha = \sum_{\substack{0\le w_i\le s_i;\ s_i\ge 0 \\ w_r\le g_{r-1}(w_1,\cdots,w_{r-1},s(N+1),\cdots,s(N+r-1),j,k)}} c_{ijk}$$

$$\cdot\, x^{i+\Sigma_{\delta=1}^{r} w_\delta} \cdot \beta_{N+r}^{w_r} \cdot \beta_{N+r-1}^{g_r(w_1,\cdots,w_r,s(N+1),\cdots,s(N+r),j,k)} \cdot d_{ijk,w_1,\cdots,w_r};$$

(a) where for fixed j, k, $g_r(0,\cdots,0,s(N+1),\cdots,s(N+r),j,k) > g_r(a_1,\cdots,a_r,s(N+1),\cdots,s(N+r),j,k)$ if there exists q such that $a_q > 0$, and

(b)

$$g_r(0,\cdots,0,s(N+1),\cdots,s(N+r),j_0,t_2)$$
$$> \max\{g_r(0,\cdots,0,s(N+1),\cdots,s(N+r),j,k)$$
$$\text{s.t. either } k \neq t_2 \text{ or } j \neq j_0, \text{ and } \exists i \text{ s.t. } c_{ijk} \neq 0\}$$

since $s(N+1) > t_1$. Putting (a) and (b) together we see that

$$\alpha = \beta_{N+r-1}^{P(r)} \cdot b_{r-1} + f_{r-1}, \qquad 0 \neq b_{r-1}, \qquad f_{r-1} \in J_{N+r},$$

where

$$\overset{(N+r)}{\deg}_{\beta_{N+r-1}} \alpha = g_r(0,\ldots,0,\ s(N+1),\ldots,s(N+r),\ j_0,\ t_2) = P(r),$$

$$\overset{(N+r)}{\deg}_{\beta_{N+r-1}} f_{r-1} < P(r), \text{ and } \quad x^{t_3+1} \nmid b_{r-1} \quad \text{in } J_{N+r}.$$

Thus $x^{t_3+1} \nmid \alpha$ in J_{N+r} for all r.

This completes Part 1 of Lemma 2.3.

Remark 2.2. A bound on the value of g_r can now be established: $s(N+r)\cdots s(N+2)(t_1 + s(N+1)t_2)$; this can be seen by using (a) and (b) in Part 1, above.

Part 2. Let $0 \neq \alpha \in J$ such that $\alpha \in J_N$, and write

$$\alpha = \sum c_{ijk} x^i \beta_N^j \beta_{N-1}^k.$$

We shall show that there exists m such that $s(m) > \deg^{(m-1)}_{\beta_{m-1}} \alpha$. Let $r > 0$. We will rewrite α in terms of β_{N+r}, β_{N+r-1}, and x in J_{N+r}. We note that a bound for $\deg^{(N+r)}_{\beta_{N+r}} \alpha$ is bounded by a bound for $g_{r-1}(w_1, \cdots, w_{r-1}, s(N+1), \cdots, s(N+r-1), j, k)$. This follows from (*) of Part 1. By Remark 2.2, we see that such a bound is

$$s(N+r-1)\cdots s(N+2)\left(\deg^{(N)}_{\beta_N} \alpha + \left(\deg^{(N)}_{\beta_{N-1}} \alpha\right) s(N+1)\right).$$

So if we let $m = \deg^{(N)}_{\beta_N} \alpha + (\deg^{(N)}_{\beta_{N-1}} \alpha) s(N+1) + 1$ and let

$$r = \sup \begin{cases} m - N - 1 \\ 0 \end{cases}$$

in the above, then $s(N+1) > \deg^{(N)}_{\beta_N} \alpha$, if $r = 0$ as $s(q) > q$ for all q, or $s(m) = s(N+r+1) > \deg^{(N+r)}_{\beta_{N+r}} \alpha = \deg^{(m-1)}_{\beta_{m-1}} \alpha$ if $r \neq 0$.

This completes Part 2 of Lemma 2.3.

Putting both parts together, we see that if $0 \neq \alpha \in J$, there exists C such that $x^C \nmid \alpha$ in J_N for all N. Thus $\alpha \notin \bigcap (x)^i$ since $J = \bigcup J_P$, $J_P \subsetneq J_{P+1}$, and therefore $\bigcap (x)^i = 0$. The proof of Lemma 2.3 is complete.

Proposition 2.1. *J is a Krull ring.*

Proof. The proposition follows from Lemmas 2.2, 2.3 and Remark 2.1.

Proposition 2.2. *J is factorial.*

Proof. By Corollary 1.1, $S = \{x^K \mid K \geq 0\}$ is a multiplicative system generated by a prime element. From Lemma 2.1 we have $S^{-1}J$ factorial. So by Proposition 2.1 and Theorem 1, J is factorial.

3. J is non-Noetherian.

Lemma 3.1. *Let $n \geq 1$. Then $x \nmid \beta_n$ in J.*

Proof. Suppose so. Since $\beta_n - x\beta_{n+2} = \beta_{n+1}^{s(n+2)}$, by Corollary 1.1, $x \mid \beta_{n+1}$ in J. Since $\beta_{n+p} - x\beta_{n+p+2} = \beta_{n+p+1}^{s(n+p+1)}$, $p \geq 0$, we may deduce in a similar fashion

$$x \mid \beta_m \quad \text{in } J \text{ for } m \geq n. \tag{1}$$

We claim this implies $x^p \mid \beta_n$ for all p in J, which is impossible in view of Lemma 2.3.

Let $p > 0$.

$$\beta_n = x\beta_{n+2} + \beta_{n+1}^{s(n+2)}$$

$$= x(x\beta_{n+4} + \beta_{n+3}^{s(n+4)}) + (x\beta_{n+3} + \beta_{n+2}^{s(n+3)})^{s(n+2)}$$

$$= x^2(x\beta_{n+6} + \beta_{n+5}^{s(n+6)}) + x(x\beta_{n+5} + \beta_{n+4}^{s(n+5)})^{s(n+4)}$$

$$+ (x(x\beta_{n+5} + \beta_{n+4}^{s(n+5)}) + (x\beta_{n+4} + \beta_{n+3}^{s(n+4)})^{s(n+3)})^{s(n+2)} \ldots .$$

In view of (1), this implies the claim.

Remark 3.1. Let $s \in J$. We define the constant term of s to be that of its representation in any J_n it belongs to. This is well defined in view of the particular embedding of J_n in J_{n+1}.

Lemma 3.2. *Let $s \in J$ with nonzero constant term. Let $n > 0$. Then*

$$s\beta_{n+1} \notin (\beta_n, \beta_{n-1}, x)J_r, \qquad r \geq n+1.$$

Proof. Induction on r: Since $(\beta_n, \beta_{n-1}, x)J_{n+1} = (\beta_n, x)J_{n+1}$, and J_{n+1} is a polynomial ring, the statement easily follows for $r = n+1$.

Since $(\beta_n, x)J_{n+2} = (\beta_{n+1}^{s(n+2)}, x)J_{n+2}$, and J_{n+2} is a polynomial ring, the statement again follows for $r = n+2$.

Suppose there exists $r > n+2$ such that $s\beta_{n+1} \in (\beta_{n+1}^{s(n+2)}, x)J_r$. Then there exist $b, c \in J_r$ such that $s\beta_{n+1} + b\beta_{n+1}^{s(n+2)} = cx$. Thus $\beta_{n+1}(s + b\beta_{n+1}^{s(n+2)-1}) = cx$. Since x is prime by Corollary 1.1, by Lemma 3.1 it must divide $s + b\beta_{n+1}^{s(n+2)-1}$. Since J_r is a polynomial ring in the variables x, β_{r-1}, β_r, and β_{n+1} is a function of those variables without constant term, this is impossible.

This concludes Lemma 3.2.

Proposition 3.1. *Let S be a multiplicative system in J such that $S \cap M = \emptyset$ where $M = (x, y, \alpha_0, \beta_1, \cdots, \beta_k, \cdots)J$. Then $S^{-1}J$ is non-Noetherian.*

Proof. Claim: $S^{-1}M$ is not finitely generated. Suppose it were. Then there exists n such that a set of generators of $S^{-1}M$ lie in J_n. Since $M \cap J_n \subseteq (x, \beta_n, \beta_{n-1})S^{-1}J$, it follows that $S^{-1}M = (x, \beta_n, \beta_{n-1})S^{-1}J$. This says that there exists $s \in S$ such that $s\beta_{n+1} \in (x, \beta_n, \beta_{n-1})J$. Since $S \cap M = \emptyset$, and $J = \bigcup J_m$, $J_m \subset J_{m+1}$, in view of Lemma 3.2 we have a contradiction.

This completes Proposition 3.1.

4. Conclusions.

Theorem 3. *J is a three dimensional non-Noetherian factorial ring, in characteristic zero.*

Proof. This follows from Propositions 2.2 and 3.1 and Corollary 1.2.

Corollary 4.1. *There exists a three dimensional non-Noetherian quasi-local factorial ring in characteristic zero.*

Proof. Let $P = (\alpha_0, \beta_1, \cdots, \beta_k, \cdots)J$. Since $J/P \simeq k[x]$ a domain, P is prime. By the arguments of Proposition 3.1, P cannot be finitely generated, much less be principal. Since $P \subsetneq M$, we deduce that M has height three. In view of Theorem 3, and Proposition 3.1, $(J \setminus M)^{-1}J$ is our example.

REFERENCES

1. Pierre Samuel, *Lectures on unique factorization domains*, Lectures on Math., no. 30, Tata Institute of Fundamental Research, Bombay, India, 1964. MR 35 #5428.

2. Ibid.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627

Current address: P. O. Box 43, Tuxedo Park, New York 10987