

CONSTRUCTING ISOTOPIES ON NONCOMPACT 3-MANIFOLDS⁽¹⁾

BY

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ABSTRACT. We consider the question "When are two homeomorphisms of a noncompact 3-manifold onto itself isotopic?" Roughly, the answer is when they are homotopic to each other. More precisely, this paper deals with the question for irreducible 3-manifolds which either have an infinite hierarchy or have such a hierarchy after the removal of a compact set. Manifolds having the first property are called end-irreducible; the others are called eventually end-irreducible. There are two results for each type of manifold depending on whether the homotopy between the two homeomorphisms sends the boundary of the manifold into itself or not.

0. Introduction and acknowledgements. We consider here the problem of when two homeomorphisms of a noncompact 3-manifold are isotopic. Waldhausen [13] has answered this question for compact, irreducible 3-manifolds which admit a hierarchy—roughly the answer is—when they are homotopic. We prove noncompact versions for manifolds which either have an "infinite hierarchy," or which have such a hierarchy after removing a compact subset. For the first class it will not be necessary to assume the homotopy is proper. For the second class of manifolds, in §3 we show it is essential that the homotopy respect the fact that the manifold has a hierarchy only far out, i.e., it must be proper.

The proofs of these theorems are quite long. §§1 and 2 and the main lemma in §3 prove lemmas that are used in all of the results. §3 contains the results in the eventually end-irreducible case and §4 contains the results for end-irreducible manifolds. The reader might find it helpful to read a sketch of the proof in [18]. Also the author has a preprint version with a fuller exposition.

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1. Preliminaries. We adopt the following conventions regarding terminology. By a *manifold* we will mean an orientable triangulable 3-manifold M . A *surface* in ∂M is just some subsurface of ∂M . The results of Bing [3] and Moise [12] show that triangulability assumptions are not a serious restriction. Unless stated otherwise our surfaces and manifolds are connected. Also, unless stated to the contrary, a *surface* F in a manifold M is properly embedded, i.e., $F \cap \partial M \subset \partial F$. If X is a finite polyhedron in a manifold M (surface F) we let $U(X)$ denote a regular neighborhood of X in M (in F) in the sense of J. H. C. Whitehead [15]. Moreover, in general, choose a triangulation in which all subspaces, previously mentioned in an argument, are subcomplexes; construct its second derived, and take the closed star of the object in question. In the presence of a product structure, we will sometimes require that $U(\dots)$ is in some sense compatible with the product structure. This will be indicated in the text. We use $\text{cl}(X)$ to denote the closure of X in Y , where X is any subspace of a space Y . I will denote the unit interval.

If X is a space, an *isotopy* of X is a homotopy $J: X \times I \rightarrow X$ such that $J_t: X \rightarrow X$ defined by $J_t(x) = J(x, t)$ is a homeomorphism for each t in I . When we say "Let J be an *isotopy* of a homeomorphism $b \dots$ ", we will mean that $J_0 = b$. A homotopy $H: X \times I \rightarrow X$ is *rel* a subspace Y of X if $H(y, t) = H(y, t')$ for t and t' in I and y in Y . The same thing is meant by the phrases *fixed on* Y and *constant on* Y .

Let F be a surface (properly embedded) in a manifold M . If we cut M along F , we obtain a manifold \tilde{M} . (M is not necessarily connected.) We will also refer to \tilde{M} as the space resulting from *splitting* M *along* F . \tilde{M} has two copies F_1 and F_2 , of F on its boundary. Denote by $p: \tilde{M} \rightarrow M$ the map which identifies F_1 and F_2 .

Let F be a surface in a manifold M . Since F and M are orientable, F disconnects any regular neighborhood $U(F)$. A homeomorphism b of M such that $b(F) = F$ *interchanges the sides of* F if it maps points close to F in one component of $U(F) - F$ into points in the other component.

As mentioned in the introduction the isotopy results are proved for irreducible manifolds. Such manifolds can be dealt with nicely by using incompressible surfaces. We now define these terms and state some properties (without proof) that will be constantly used. We follow Waldhausen [13] for the definitions and

properties, but extend them to include noncompact surfaces and manifolds.

A system of surfaces in M consists of finitely many pairwise disjoint surfaces in M . Let F be a system of compact surfaces in M . F is *incompressible* in M if both of the following conditions hold.

- (1) If Δ is a disc in M with $\Delta \cap F = \partial\Delta$, then $\partial\Delta$ bounds a disc in F .
- (2) There does not exist a ball B in M such that $B \cap F = \partial B$.

It is known that a system of compact surfaces is incompressible in M if and only if each component is incompressible in M . Also, if F is a surface in M different from the 2-sphere, F is incompressible in M if and only if $\ker(\pi_1(F) \rightarrow \pi_1(M)) = 0$. If F is a boundary component of a manifold M , F is *incompressible* if in addition to (1) and (2), F is not a plane. M is *boundary-irreducible* if ∂M is incompressible. A manifold M is *irreducible* if every 2-sphere bounds a ball in M .

Let M be a manifold and F a system of incompressible compact surfaces in M . Let $U(F)$ be a regular neighborhood of F . Then

- (a) $\text{cl}(M - U(F))$ is irreducible if and only if M is.
- (b) $\ker(\pi_1(M') \rightarrow \pi_1(M)) = 0$ for each component M' of $\text{cl}(M - U(F))$.

Suppose M is an irreducible manifold which is either compact with nonempty boundary or noncompact. Then M is aspherical, i.e., $\pi_j(M) = 0$ for $j \geq 2$.

We will also need the notion of parallel surfaces. First, suppose F and G are compact surfaces in M . By an *ambient isotopy of M carrying F onto G* , we mean an isotopy $J: M \times I \rightarrow M$ of the identity to a homeomorphism J_1 such that $J_1(F) = G$. Now suppose F and G are compact surfaces in M or ∂M . F is *parallel to G* if and only if there is an embedding $e: F \times I \rightarrow M$ such that $F = e(F \times 0)$ and $G = e(\text{cl}(\partial(F \times I) - F \times 0))$. Notice that if F is parallel to G , then G is parallel to F . Also, if F and G are parallel and both are in M , then they are isotopic by an isotopy which is constant on ∂M .

The first proposition permits us to isotope the homeomorphism on new surfaces of a hierarchy without changing the map where it has already been isotoped to the identity.

Proposition 1.1. *Let M be an irreducible manifold. Let C be a compact irreducible manifold contained in M and suppose F is a system of compact incompressible surfaces in C . Let G be a closed incompressible surface in C so that $G \cap F = \emptyset$.*

Further, let h be a homeomorphism of M , such that

- (1) $hF_i = F_i$ and h does not interchange the sides of F_i .
- (2) h is homotopic to the identity by a homotopy H such that $H(G \times I) \subset C$.
- (3) $h(G)$ is incompressible in C .

Then there is an ambient isotopy of C which is fixed on $F \cup \partial C$ carrying $h(G)$ onto G .

Proof. This proof is quite similar to one by Waldhausen [13, Corollary 5.5]. Put $b(G)$ in general position with respect to G by an isotopy which is fixed on $F \cup \partial C$, and change b and H according to this isotopy. Now G intersects $b(G)$ transversally and $G \cap b(G)$ consists of pairwise disjoint simple closed curves.

The rest of the proof exhibits a set of "nice" isotopies of C which first move $b(G)$ completely off G . We will show at that point that $b(G)$ is in fact parallel to G and our conclusion follows.

We apply Proposition 5.4 in [13] to obtain "pieces," \tilde{G} and \widetilde{bG} , of G and $b(G)$ respectively, and an embedding $f: \tilde{G} \times I \rightarrow C$ such that $f(\tilde{G} \times 0) = \widetilde{bG} \subset b(G)$, $f(\text{cl}(\partial(\tilde{G} \times I) - \tilde{G} \times \{0\})) = \tilde{G} \subset G$. Moreover $G \cap \widetilde{bG} = \partial \widetilde{bG}$ and either $bG \cap \tilde{G} = \partial \tilde{G}$ or both \tilde{G} and \widetilde{bG} are discs.

If \tilde{G} and \widetilde{bG} are both discs, then denote by B the ball $f(\tilde{G} \times I)$. One can show that each component of F lies entirely outside this ball. Hence each component of F is contained in the complement of B . If F_i is a component without boundary, the incompressibility of F_i guarantees that it, too, lies outside of B .

Thus we can push \widetilde{bG} out across G by an ambient isotopy of C which is fixed outside a small regular neighborhood of B . This isotopy reduces the number of intersection curves of bG and G .

We now take up a pair $(\tilde{G}, \widetilde{bG})$ with $G \cap \widetilde{bG} = \partial(\widetilde{bG})$ and $bG \cap \tilde{G} = \partial \tilde{G}$ and such that $\tilde{G} \neq G$. Denote by D the set $f(\tilde{G} \times I)$. By [13, Corollary 3.2] we see that any closed component F_i of F must lie outside of D . The same argument, used when \tilde{G} is a disc, shows that each F_i with nonvacuous boundary lies outside of D , too. Therefore, there is an isotopic deformation of bG fixed on $\text{cl}(bG - \widetilde{bG})$ and on $F \cup C$ which takes \widetilde{bG} onto \tilde{G} . Now push off \widetilde{bG} slightly to the other side of G . Again we have reduced the number of intersection curves of $G \cap bG$. Since the number is finite this procedure terminates with $G \cap bG = \emptyset$. We get $f(\tilde{G} \times \{0\}) = G$ and consequently $\widetilde{bG} = b(G)$. Recalling that f is an embedding, $f(\tilde{G} \times I) = D$ is a manifold in C bounded by $G \cup bG$. If F_i is a component of F with nonvacuous boundary, then F_i lies outside of D since $F \cap \partial D = F \cap (G \cup b(G)) = \emptyset$. If F_i is closed and is contained in D , then by [13, Corollary 3.2] F_i is parallel to G .

This cannot occur, for if ρ is a simple arc in the region bounded by G and F_i with an endpoint on each of G and F_i , then $b\rho$ has its endpoints on each of $b(G)$ and F_i . But b does not interchange the sides of F_i .

We have shown that all components of F lie outside of D . This completes the proof.

2. Homotoping H to projection onto a surface. In order to isotope b to the identity we will use certain theorems that say roughly, "If the homotopy H has certain properties, then we can isotope b to a "nicer map." " In particular, we

will see that if the restriction of H to a surface F is projection onto F , then we will be able to construct isotopies which are fixed on F .

A homotopy $H: X \times I \rightarrow X$ is called *projection onto X* or *projection onto the first factor* if $H(x, s) = x$ for every x in X and s in I .

In the first part of this section we answer the question: If h is the identity on ∂M , when and how can we homotope H so that $H|_{\partial M \times I}$ is projection onto ∂M ? (Cf. Propositions 2.1, and 2.2.)

The second and third parts prepare the way for surfaces in M ; Part 2 for closed surfaces F and Part 3 for surfaces with nonvacuous boundary.

Part 1. F is a boundary component. If X and Y are spaces and $f: X \rightarrow Y$ is a map we say f is a *proper map* if for every compact subset C of Y , $f^{-1}(C)$ is compact in X . Given a homotopy $H: X \times I \rightarrow Y$, for each t in I we define the map $H_t: X \rightarrow Y$ by $H_t(x) = H(x, t)$. H is a *proper homotopy* means that H is a proper map not just that H_t is proper for each t .

Proposition 2.1. *Let F be a surface with vacuous boundary, different from a plane, an open annulus or a torus. Let $H: F \times I \rightarrow F$ be a homotopy with $H_0 = H_1 = \text{identity}$. Then there is a homotopy $K: F \times I \times I \rightarrow F \text{ rel } F \times \partial I$ of H to projection onto F . If H is a proper map, then K may be chosen to be proper.*

Before proving Proposition 2.1 let us discuss the three excluded cases. We wish to define the homotopy $K: F \times I \times I \rightarrow F$ of H to projection onto F . K must be fixed on $F \times \partial I$. Thus K is already given on $(F \times I \times \{0\}) \cup (F \times I \times \{1\}) \cup (F \times \partial I \times I)$. Therefore if x is in F , K is defined on $\partial(\{x\} \times I \times I)$. We can extend K to all of $\{x\} \times I \times I$ only if the loop $K|_{\partial(\{x\} \times I \times I)}$ is not null homotopic. If F is an open annulus or a torus, it may happen that $K|_{\partial(\{x\} \times I \times I)}$ is not null homotopic. For example if F is an open annulus and λ is a simple closed curve representing the generator of $\pi_1(F)$, then H could be rotation of the annulus n times around in the direction of λ . Similarly for the torus. For the plane, K would exist for a similar rotation but could not be chosen proper. For these spaces we have the weaker Proposition 2.2 which first "undoes" the rotation by an isotopy which goes n times around in the opposite direction. Precisely, we get

Proposition 2.2. *Let F be a plane, a torus, or an open annulus. Let $H: F \times I \rightarrow F$ be a homotopy (proper homotopy) such that $H_0 = H_1 = \text{identity}$. Then there is an isotopy $J: F \times I \rightarrow F$ of the identity to itself so that " H followed by J " is homotopic (proper homotopic) rel $F \times \partial I$ to projection onto F .*

Lemmas 2.3 and 2.4 prove Proposition 2.1 for closed surfaces.

Lemma 2.3. *Let F be a closed surface different from a torus and $H: F \times I$*

$\rightarrow F$ be a homotopy of the identity to itself. Then for any x in F , the loop $H|_{\{x\} \times I}$ is null homotopic.

Proof. Let α be in $\pi_1(F, x)$. Let f represent α . Denote by γ the homotopy class of $H|_{\{x\} \times I}$. Define $g: I \times I \rightarrow F$ by $g(s, t) = H(f(s), t)$ for s and t in I . The homotopy class of the loop $g|_{\partial(I \times I)}$ is the identity of $\pi_1(F, x)$. But

$$[g|_{\partial(I \times I)}] = \alpha \gamma \alpha^{-1} \gamma^{-1}.$$

Hence $1 = [g|_{\partial(I \times I)}] = \alpha \gamma \alpha^{-1} \gamma^{-1}$. But α was arbitrary. Therefore α lies in the center of $\pi_1(F, x)$. Since F is not a torus, the center of $\pi_1(F, x)$ is trivial. Thus $H|_{\{x\} \times I}$ is null homotopic.

Lemma 2.4. Let F be a closed surface and let $H: F \times I \rightarrow F$ be a homotopy of the identity to itself. If there is a point x_0 in F such that the loop $H|_{\{x_0\} \times I}$ is null homotopic then H is homotopic rel $F \times \partial I$ to projection onto F .

Proof. Let λ be a simple closed curve containing x_0 . Assume λ is not trivial if F is not a 2-sphere. In the case of the 2-sphere the proof is Step 1 followed by Step 4.

Step 1. We assert that there is a homotopy K of H rel $F \times \partial I$ to a map K_1 such that $K_1|_{\lambda \times I}$ is projection onto λ . As we define K we will show where we may choose the range of K . Although this is not necessary when F is compact, we will need this kind of information when we consider the noncompact case.

Let $U(\lambda)$ be a regular neighborhood of λ . We begin by defining K on

$$(F \times \partial I \times I) \cup (F \times I \times \{0\}) \cup (\lambda \times I \times \{1\}) \cup (\text{cl}(F - U(\lambda)) \times I \times I).$$

For any s and t in I , let

$$\begin{aligned} K(x, s, 0) &= H(x, s) && \text{for } x \text{ in } F, \\ K(x, 0, t) &= K(x, 1, t) = x && \text{for } x \text{ in } F, \\ K(x, s, 1) &= x && \text{for } x \text{ in } \lambda, \\ K(x, s, t) &= H(x, s) && \text{for } x \text{ in } \text{cl}(F - U(\lambda)). \end{aligned}$$

We show now that we can define K on $\lambda \times I \times I$. K is already defined on the loop $\partial(x_0 \times I \times I)$. Since $H|_{x_0 \times I}$ is null homotopic the loop $K|_{\partial(x_0 \times I \times I)}$ is null homotopic. Thus we can extend K to all of $\{x_0\} \times I \times I$. The complement of $x_0 \times I \times I$ in $\lambda \times I \times I$ is an open 3-cell. We can extend K over the rest of $U(\lambda) \times I \times I$ by using the homotopy extension theorem.

Step 2. We start afresh; now assume that H is given so that $H|_{\lambda \times I}$ is projection onto λ . We can homotope H to a map which is projection onto the first factor in a chosen regular neighborhood $U'(\lambda)$. (Omit this step for the 2-sphere.)

Applications of the homotopy extension theorem and simple obstruction theory arguments will produce a homotopy K of H to the desired map. One can insure that $K(U'(\lambda) \times I \times I) \subset H(U'(\lambda) \times I)$.

Step 3. (Omit for the 2-sphere.) Let P be a collection of properly embedded disjoint arcs in $\text{cl}(F - U(\lambda))$ so that $F - [\text{cl}(F - U(\lambda)) \cup \{\rho \mid \rho \text{ is in } P\}]$ is an open disc. We assume $H \mid U(\lambda) \times I$ is projection onto $U(\lambda)$ and construct a homotopy $K \text{ rel } F \times \partial I$ of H to a map K_1 so that $K_1 \mid U(\lambda) \cup \{\rho \mid \rho \in P\}$ is projection onto the first factor. Again the argument is simple and again we can choose K so that $K(\rho \times I \times I) \subset H(\rho \times I)$.

Step 4. We now assume H is projection onto the first factor on $A = U(\lambda) \cup \{\rho \mid \rho \in P\}$. $\text{cl}(F - A)$ is a disc with some boundary points identified. We define a homotopy $K': \Delta \times I \times I \rightarrow H(\Delta \times I) \text{ rel } \partial(\Delta \times I)$ of H to projection onto Δ subject to the appropriate restrictions.

The homotopy $K: F \times I \times I$ defined by $K_t \mid A \times I = H$ and $K_t \mid \Delta \times I = K'_t$ is a homotopy $\text{rel } F \times \partial I$ of H to projection onto F . This completes the proof of (2.4).

The preceding two lemmas prove (2.1) for closed surfaces.

Lemmas 2.6 and 2.7 prove Proposition 2.1 for the noncompact case. We will need the following version of the homotopy extension theorem, which is implied by the usual proof.

(2.5) *Let X and Y be complexes and let X' be a subcomplex of X . Suppose K' is a proper map of $X \times \{0\} \cup X' \times I$ into Y . Then there is a proper map $K: X \times I \rightarrow Y$ extending K' . Furthermore K can be chosen so that $K(X \times I) = K'(X \times \{0\} \cup X' \times I)$.*

Let X be a noncompact complex. A collection $\{X_i\}$ for i an integer is an *exhausting sequence* for X if

- (1) X_i is compact for each i .
- (2) $X_i \subset X_{i+1}^\circ$ for each i .
- (3) $\bigcup_{i=1}^\infty X_i = X$.

If X is a surface or a manifold there exists an exhausting sequence for X such that

- (1') X_i is a compact connected manifold (surface) for each i .
- (2') Each component of $X - X_i$ is unbounded for each i . We get such an exhausting sequence by first taking regular neighborhoods of each X'_i of some arbitrary exhausting sequence. Now each X'_i is a manifold (surface) — though not necessarily connected. Let X_1 be any component of X'_1 . Let X_i be the component of X'_i containing X_{i-1} . The reader can check that (X_i) is an exhausting sequence satisfying (1'); we add all bounded components of $X - X_i$ to X_i . One can show this new set is still compact.

If $H: X \times I \rightarrow X$ is any homotopy, by choosing subsequences of X_i we may insure

$$(3') H(X_i \times I) \subset X_{i+1}^\circ.$$

If $H: X \times I \rightarrow X$ is a proper homotopy, then again by choosing subsequences we get

$$(4'a) H(\partial X_i \times I) \subset X_{i+1}^\circ - X_{i-1}.$$

$$(4'b) H^{-1}(X_i) \subset X_{i+1} \times I.$$

The statements of Lemmas 2.6 and 2.7 refer to exhausting sequences satisfying (1'), (2'), (3'), and (4') where applicable.

Lemma 2.6. *Let F be a noncompact surface without boundary and assume F is neither a plane nor an open annulus. Let $H: F \times I \rightarrow F$ be a homotopy of the identity to itself and let $\{F_i\}$ be an exhausting sequence as given above. Moreover, suppose F_1 is not a disc or an annulus. Then for each i and for each boundary component λ of F_i and each point x on λ , the loop $H|_{\{x\} \times I}$ is null homotopic in $H(\lambda \times I)$.*

Proof. This proof borrows ideas from Epstein's paper [6]. We remark first that we can apply results from [6, pp. 91–92] to $\pi_1(H(\lambda \times I), x)$ by noting that $H(\lambda \times I)$ is a strong deformation retract of a regular neighborhood of $H(\lambda \times I)$. Let λ be a boundary component of F_i and let x belong to λ . Let $H|_{\{x\} \times I}$ represent γ in $\pi_1(H(\lambda \times I), x)$. As in Lemma 2.3 define $g_\lambda: I \times I \rightarrow H(\lambda \times I)$ by $g_\lambda(s, t) = H(\lambda(s), t)$. As in 2.3 again, $g|_{\partial(I \times I)}$ represents $[\lambda]\gamma[\lambda]^{-1}\gamma^{-1}$ in $\pi_1(H(\lambda \times I), x)$, so $[\lambda]$ and γ lie in the same cyclic subgroup of $\pi_1(H(\lambda \times I), x)$. We conclude $\gamma = [\lambda]^n$ for some nonnegative integer and show now that $n = 0$, i.e., $H|_{\{x\} \times I}$ is null homotopic.

If F_i has more than one boundary component let β be an element of $\pi_1(F_i, x)$ represented by a simple closed curve μ which is homotopic to another boundary curve of F_i . If F_i has only one boundary component let β be a generator of $\pi_1(F_i, x)$ represented by a meridional simple closed curve μ . Using the function g_μ we get that $\gamma = [\mu]^m$ in $\pi_1(H(\mu \times I), x)$. Using the natural maps $\pi_1(H(\lambda \times I), x) \rightarrow \pi_1(F_{i+1}, x)$ and $\pi_1(H(\lambda \times I), x) \rightarrow \pi_1(F_{i+1}, x)$ we have that, in $\pi_1(F_{i+1}, x)$, $\gamma = [\mu]^m$ and $\gamma = [\lambda]^n$. But $\pi_1(F_i, x) \rightarrow \pi_1(F_{i+1}, x)$ is a monomorphism by (2') and hence $\gamma = [\lambda]^n = [\mu]^m$ holds in $\pi_1(F_i, x)$. But $[\lambda]^n = [\mu]^m$ only if $n = m = 0$.

Lemma 2.7. *Let F be a noncompact surface without boundary. Let $H: F \times I \rightarrow F$ be a homotopy of the identity to itself and let $\{F_i\}$ be an exhausting sequence satisfying (1') – (3') above. If for each i and every component λ of F_i , there is a point x in λ so that the loop $H|_{\{x\} \times I}$ is null homotopic, then H is homotopic rel $F \times \partial I$ to projection onto F .*

If $H: F \times I \rightarrow F$ is a proper homotopy, assume $\{F_i\}$ satisfies (1') - (4'). Suppose there is a function $n(i)$, such that $n(i) < i$ and that $n(i) \rightarrow \infty$ as $i \rightarrow \infty$, so that for each boundary component λ of F_i there is a point x in λ such that the loop $H|_{\{x\} \times I}$ is null homotopic in $F - F_{n(i)}$. Then H is proper homotopic to projection onto F .

Proof. We follow the proof of Lemma 2.4 making the necessary modifications for the fact F is noncompact. We construct the homotopy $K: F \times I \times I \rightarrow F$ of H to projection onto F .

Step 1. Let H be the homotopy (proper homotopy) given in the hypotheses. There is a homotopy (proper homotopy) $K: F \times I \times I \rightarrow F$ rel $F \times \partial I$ of H to a map K_1 so that $K_1|_{\lambda \times I}$ is projection onto λ for each boundary loop of each F_i .

Denote the collection of all boundary loops by Q . Let $\{U(\lambda) | \lambda \in Q\}$ be a collection of pairwise disjoint neighborhoods. We first define K on $[(F \times \partial I \times I)] \cup [(F \times I \times \{0\})] \cup [(\text{cl}(F - \bigcup \{U(\lambda) | \lambda \in Q\}) \times I \times I) \cup [(\{\lambda | \lambda \in Q\} \times I \times \{1\})]$. For s and t in I , define

$$\begin{aligned} K(x, s, 0) &= H(x, s) && \text{for } x \text{ in } F, \\ K(x, 0, t) &= K(x, 1, t) = x && \text{for } x \text{ in } F, \\ K(x, s, 1) &= x && \text{for } x \text{ in } \bigcup \{\lambda | \lambda \in Q\}, \\ K(x, s, t) &= H(x, s) && \text{for } x \text{ in } \text{cl}(F - \bigcup \{U(\lambda) | \lambda \in Q\}). \end{aligned}$$

Repeat the argument given in Step 1 of (2.4) for each λ in Q . This defines the desired K , if we do not assume H is proper. If H is proper, we want K to be proper. To see that we can guarantee this we note that for each λ of F_i , there is an x in λ such that $H|_{\{x\} \times I}$ is null homotopic in $F - F_{n(i)}$. Recalling that the loop $K|_{\partial(\{x\} \times I \times I)}$ is homotopic to $H|_{\{x\} \times I}$ we can extend K to $\{x\} \times I \times I$ such that $K(\{x\} \times I \times I) \subset F - F_{n(i)}$. To extend K over $\lambda \times I \times I$, the reader can check that the obstruction lies in $\pi_2((F - F_{n(i)}) \cup H(\lambda \times I))$, which is trivial. Hence we can extend K over $\lambda \times I \times I$ so that $K(\lambda \times I \times I) \subset (F - F_{n(i)}) \cup H(\lambda \times I)$. Now using (2.5), the proper homotopy extension theorem, K can be extended over $U(\lambda) \times I \times I$ so that $K(U(\lambda) \times I \times I) \subset (F - F_{n(i)}) \cup (H(U(\lambda) \times I))$. It is easily seen that K is a proper map.

We carry out Steps 2, 3 and 4 as in the proof of Lemma 2.4 making the appropriate changes for noncompactness.

Part 2. Closed surfaces.

Proposition 2.8. *Let C be a compact irreducible manifold with nonvacuous boundary. Let F be a closed incompressible surface in C . Assume that if F separates C , then F separates ∂C . Let $h: F \rightarrow C$ be a homeomorphism of F onto itself and let $H: F \times I \rightarrow C$ be a homotopy of h to the inclusion function. Then there is a homotopy of H rel $F \times \partial I$ to a map of $F \times I$ into F .*

Proof. Let $U(F \times \partial I)$ be $F \times ([0, \epsilon] \cup [1 - \epsilon, 1])$ for some $\epsilon < \frac{1}{2}$. Identify points in a regular neighborhood $U(F)$ in C with $F \times [-\epsilon, \epsilon]$ so that x in F corresponds to $(x, 0)$. There is a homotopy H' which is homotopic rel $F \times \partial I$ to H and has the property that $H'^{-1}(F) \cap U'(F \times \partial I) = F \times \partial I$ where $U'(F \times \partial I) = F \times ([0, \epsilon] \cup [1 - \epsilon, 1])$. Since it suffices to show that H' is homotopic rel $F \times \partial I$ to a map of $F \times I$ into F , we will denote H' by H and start anew.

Applying the proposition on p. 60 of [13], we can homotope

$$H|_{\text{cl}(F \times I - U'(F \times \partial I))} \text{ rel } \partial[\text{cl}(F \times I - U'(F \times \partial I))]$$

to a map (which we will continue to denote by H) which is transverse with respect to F and so that $H^{-1}(F) \cap \text{cl}(F \times I - U'(F \times \partial I)) = F_1 \cup \cdots \cup F_m$ where F_j is a system of incompressible surfaces in $F \times I$. We may regard this homotopy as a homotopy of H rel $U'(F \times \partial I)$ so that afterwards $H^{-1}(F) \cap F \times I = F_1 \cup \cdots \cup F_m \cup F \times \partial I$. Each F_i is closed and is parallel to $F \times 0$. By a commutative diagram $(H|_{F_i})_*$ is an isomorphism. Therefore by [13, Lemma 1.4.3], $H|_{F_i}$ is homotopic to a homeomorphism. Let $U(\bigcup_{i=1}^m F_i)$ denote a regular neighborhood of $\bigcup_{i=1}^m F_i$. Using the homotopy given above on each side of the regular neighborhood around F_i and connecting up to a homotopy which is essentially our original one on $\text{cl}(F - U(\bigcup_{i=1}^m F_i))$ we can homotope H rel $F \times \partial I$ to a map which we still call H so that $H|_{F_i}$ is a homeomorphism for each i . Moreover, it is still the case that $H^{-1}(F) = F \times \partial I \cup F_1 \cup \cdots \cup F_m$.

Any two components of $F \times \partial I \cup F_1 \cup \cdots \cup F_m$ bound a domain homeomorphic to $F \times I$. We will denote such a domain by $F \times I'$ where I' is some subinterval of I . Assume we have numbered the F_i so that no F_j lies in the region bounded by $F_i \cup F_{i+1}$. Let \tilde{C} be the manifold (not necessarily connected) obtained from C by splitting along F . For each region bounded by $F_i \cup F_{i+1}$, there is a lifting of $H|_{F \times I'} \rightarrow C$ to a map $\overline{H}|_{F \times I'}: F \times I' \rightarrow \tilde{C}$.

Case 1. Assume C is boundary irreducible.

We have $\ker(H|_{F \times I'}) = 0$. Furthermore because F separates ∂C whenever F separates C , the component of \tilde{C} containing $\overline{(H|_{F \times I'})(F \times I')}$ has more boundary components than just $p^{-1}(F)$. Hence using [13, p. 67] $\overline{H}|_{F \times I'}$ cannot be homotopic to a covering. Therefore each $H|_{F \times I'}$ is homotopic rel $F \times \partial I'$ to a map into $\partial(\tilde{C})$. It follows that each $H|_{F \times I'}$ is homotopic rel $F \times \partial I'$ to a map into F and hence H is homotopic rel $F \times \partial I$ to a map into F .

Case 2. C and hence \tilde{C} does not have irreducible boundary.

There is a system $\{\Delta_i\}$ of properly imbedded discs in \tilde{C} so that $\text{cl}(\tilde{C} - U(\bigcup_i \Delta_i))$ is boundary irreducible and so that $H: F \times I \rightarrow \tilde{C}$ is homotopic rel $F \times \partial I$ to a map into $p(\text{cl}(\tilde{C} - U(\bigcup_i \Delta_i)))$. We are now reduced to the previous case.

Note to Proposition 2.8. Actually it was unnecessary to assume F is in C .

We could have assumed that F was a component of ∂C such that $F \neq \partial C$. We will use this later.

Lemma 2.9. *Let F be a closed surface different from a 2-sphere. Let $H: F \times I \rightarrow F$ be a homotopy of a homeomorphism h to the identity. Then H is homotopic rel $F \times \partial I$ to an isotopy.*

Proof. Define $g: F \times I \rightarrow F \times I$ by $g(x, t) = (H(x, t), t)$. Since $g|_{F \times 0}$ and $g|_{F \times \{1\}}$ are homeomorphisms, by [13, p. 67] there is a homotopy rel $F \times \partial I$ of g to a homeomorphism. Applying [13, p. 69] this last homeomorphism is homotopic rel $F \times \partial I$ to a level-preserving homeomorphism. Compose these homotopies with projection onto the first factor to conclude the proof.

Some of the main results use invariants of proper homotopy type. We make use of the space of ends in the next lemma so we introduce it here. For a more complete exposition see [4].

Let X be a locally-finite complex. Proper maps $b, b': [0, \infty) \rightarrow X$ are said to *determine the same end of X* provided for C compact and t sufficiently large, $b(t)$ and $b'(t)$ are in the same component of $X - C$. This is an equivalence relation and we call the equivalence class of b an *end of X* . The set of ends is denoted by $\pi_0(X)$.

If $f: X \rightarrow Y$ is a proper map, then f induces (by composition) a map $\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$. If f and g are proper homotopic then $\pi_0(f) = \pi_0(g)$.

Lemma 2.10. *Suppose M is a noncompact irreducible manifold and G is any closed surface in M different from the 2-sphere. Let $H: M \times I \rightarrow M$ be a proper homotopy of a homeomorphism h to the identity such that $H(G \times I)$ is contained in G . (In particular $h(G) = G$.) Then h does not interchange the sides of G .*

Proof. If $M - G$ is connected, let λ be a closed curve which intersects G transversally in one point. If h interchanges the sides of G , then h will change the intersection number of G with λ which contradicts the fact that h is homotopic to the identity.

Suppose G separates M . Then $M - G = A \cup B$. Assume B is unbounded. If A is bounded, clearly $h(A) = A$. If A is unbounded, we argue as follows. Since h is proper homotopic to the identity, $\pi_0(h)$ is the identity and hence maps ends of A to themselves. Again $h(A) = A$. Thus in either case h does not interchange the sides of G .

Part 3. Compact surfaces. The next three lemmas are imbedded in proofs from [13].

Lemma 2.11. *Let M be an irreducible manifold and $H: M \times I \rightarrow M$ be a homotopy of a homeomorphism h to the identity. Suppose $H|_{\partial M \times I}$ is projection onto ∂M .*

Let F be a system of compact surfaces in M which are properly imbedded and incompressible in some compact irreducible submanifold C of M . For each component F_i of F assume $b(F_i) = F_i$.

Suppose G is a compact surface with nonvacuous boundary in M such that G is properly imbedded and incompressible in C . Assume also that $G \cap F = \emptyset$. Suppose $H(G \times I)$ is contained in C .

Then there is a homotopy of $H|G \times I \text{ rel } \partial(G \times I)$ to a map of $G \times I$ into $C - U(F)$, where $U(F)$ is some chosen regular neighborhood of F .

Lemma 2.12. Let G be a compact incompressible surface with nonvacuous boundary in an irreducible manifold C . Let b be a homeomorphism of G onto itself and let $H: G \times I \rightarrow C$ be a map with $H_0 = b$, $H_1 = \text{inclusion}$ and $H| \partial G \times I = \text{projection onto } G$. Then there is a homotopy of $H \text{ rel } \partial(G \times I)$ to a map which takes $G \times I$ into G .

Lemma 2.13. Let G be a compact surface with nonvacuous boundary. Let $H: G \times I \rightarrow G$ be a homotopy $\text{rel } \partial G$ of the identity to itself. Then H is homotopic $\text{rel } \partial(G \times I)$ to projection onto G .

Lemma 2.14. Let M be an irreducible manifold and F a compact surface in M with or without boundary. Suppose $U(F)$ and $U'(F)$ are regular neighborhoods of F in M with $U'(F)$ contained in the interior of $U(F)$. Let $H: M \times I \rightarrow M$ be a homotopy of a homeomorphism b to the identity. Suppose $b|U'(F)$ is the identity, $H|F \times I$ is projection onto F , and $H|\partial M \times I$ is projection onto ∂M . Then there is a homotopy $K: M \times I \times I \rightarrow M$ of $H \text{ rel } \partial(M \times I) \cup (F \cup \text{cl}(M - U(F))) \times I$ to a homotopy H' such that $H'|U'(F) \times I$ is projection onto $U'(F)$.

Proof. By setting up the desired end result and using the homotopy extension theorem one can produce the homotopy one needs.

3. The Main Lemma and the isotopy result for eventually end-irreducible manifolds. The isotopy results we have proved are for noncompact irreducible manifolds which are either end-irreducible or eventually end-irreducible. As mentioned in the introduction, end-irreducible is a condition on ends playing a role similar to the condition of incompressible for boundary components. By removing some boundary components of compact manifolds with irreducible boundary, one gets many examples of end-irreducible manifolds. Nontrivial examples will be constructed later in this section.

Since the definition of end-irreducible involves defining the new proper fundamental groups of E. M. Brown and since we use only the facts given in (3.1) below about end-irreducible manifolds the reader is referred to [4] and [5] for definitions.

Let X be a subcomplex of Y . So as to avoid confusion we (in the case Y

is a manifold) use the word *frontier* of X to mean $\text{cl}(X) - X$, and we write $\text{Fr}(X)$. If C is a compact submanifold of a manifold M we say C is *frontier-incompressible* if each component to $\text{Fr}(C)$ is an incompressible surface in M .

(3.1) [5, to appear]. *Let M be an end-irreducible manifold. There exists an exhausting sequence $\{C_n\}$ for M such that*

- (1) C_n is a compact submanifold of M .
- (2) $\text{Fr}(C_n)$ is a system of properly imbedded surfaces in M meeting no compact component of ∂M .
- (3) $\text{Fr}(C_n)$ is incompressible in M .
- (4) C_n is connected.
- (5) All components of $M - C_n$ are unbounded.

Finally for the notion of a *hierarchy* for an irreducible compact manifold D , we follow Waldhausen in [13]. Notice that (3) of (3.1) and the fact that any compact irreducible manifold with nonvacuous boundary has a hierarchy gives us an infinite hierarchy for an irreducible and end-irreducible manifold.

In the proofs presented in the next two sections, a key theorem is

(3.2) [13, p. 76]. *Let C be a compact irreducible manifold. Let F and G be incompressible surfaces in C . Suppose there is a homotopy carrying F onto G which is constant on ∂F . Then there is an ambient isotopy of C carrying F onto G which is constant on ∂C .*

Main Lemma. This proposition forms the heart of the proofs of all the isotopy results which follow. It allows us to reduce the proofs to Alexander's theorem by showing that one can isotope h to the identity on part of a hierarchy without changing the homeomorphism where it has already been isotoped to the identity.

Proposition 3.3. *Let M be a noncompact irreducible manifold and C be a compact submanifold of M . Let D be a compact irreducible manifold in $M - C$ such that $\text{Fr}(U(D))$ is a system of incompressible surfaces in $M - C$.*

Suppose $H: M \times I \rightarrow M$ is a homotopy of a homeomorphism h to the identity. Assume H restricted to $\partial M \times I \cup U(\text{Fr}(D)) \times I$ is projection onto the first factor, for some regular neighborhood $U(\text{Fr}(D))$ of $\text{Fr}(D)$ in $M - C$. Moreover assume $H(D \times I) \subset M - C$.

Suppose $D_j, E_j \subset D$, $U(E_j) \subset D_j$, $D_{j+1} = \text{cl}(D_j - U(E_j))$ for $j = 1, \dots, n$ is a hierarchy for $D_1 = \text{cl}(D - U(\text{Fr}(D)))$.

Then

- (1) *there is an isotopy $J: M \times I \rightarrow M$, fixed on ∂M and on $(M - D) \cup U(\text{Fr}(D))$, with $J_1 = h$ and so that $J_0|_{U(E_j)}$ is the identity for each j .*
- (2) *" J followed by H " is homotopic rel $\partial(M \times I) \cup (M - D) \times I \cup U(\text{Fr}(D)) \times I$ to a map whose restriction to $U(E_j) \times I$ is projection onto $U(E_j)$ for each j .*

Proof. We begin by showing there is an ambient isotopy of M taking E_1 onto $b(E_1)$ which is fixed on $\partial M \cup (M - D) \cup U(\text{Fr}(D))$.

First we show that $H|E_1 \times I$ is homotopic rel $\partial(E_1 \times I)$ to a map into $\text{cl}(D - U(\text{Fr}(D)))$.

Denote by F the system $\text{Fr}(D)$ of incompressible surfaces in $M - C$. $H| \partial(E_1 \times I)$ is trivially transverse with respect to F . Thus by [13, p. 60] there is a homotopy of $H|E_1 \times I$ rel $\partial(E_1 \times I)$ after which $H_{E_1}^{-1}(F)$ is a system of incompressible surfaces in $E_1 \times I$. (H_{E_1} denotes $H|E_1 \times I$.) On the one hand, these surfaces must be closed because $H(\partial(E_1 \times I)) \cap F = \emptyset$. On the other hand, since E_1 has boundary each surface in $H_{E_1}^{-1}(F)$ must have boundary. Hence $H_{E_1}^{-1}(F) = \emptyset$. It follows that $H(E_1 \times I)$ is contained in $D - F$. Now push $H(E_1 \times I)$ out of $U(F)$ and into $\text{cl}(D - U(\text{Fr}(D)))$.

Since $b|U(\text{Fr}(D))$ is the identity and b globally is a homeomorphism, we have $b(E_1)$ is contained in and is incompressible in $\text{cl}(D - U(\text{Fr}(D)))$. Applying (3.2) now to $H_{E_1}: E_1 \times I \rightarrow \text{cl}(D - U(\text{Fr}(D)))$, E_1 and $b(E_1)$, we get an ambient isotopy of $\text{cl}(D - U(\text{Fr}(D)))$, fixed on $\partial(\text{cl}(D - U(\text{Fr}(D))))$, which takes E_1 onto $b(E_1)$. Extending this isotopy to all of M by making it fixed elsewhere gives us the ambient isotopy of M taking E_1 onto $b(E_1)$. We change b and H according to this isotopy and we begin again assuming that in addition to satisfying the hypotheses of the theorem, $b(E_1)$ and $H(E_1 \times I)$ are contained in $\text{cl}(D - U(\text{Fr}(D)))$.

Applying (2.12) with $E_1 = G$, $H|E_1 \times I = H$, and $b|E_1 = b$ we can homotope $H|E_1 \times I$ rel $\partial(E_1 \times I)$ so that afterwards $H_{E_1}(E_1 \times I)$ is contained in E_1 . Thus $b|E_1$ is homotopic in E_1 to the identity. We can now use [6, p. 99] to get an isotopy on $\text{cl}(D - U(\text{Fr}(D)))$ to a map which is the identity on a chosen regular neighborhood $U(E_1)$; moreover this isotopy can be chosen fixed outside a neighborhood $U'(E_1)$ containing $U(E_1)$ in its interior and it is fixed on $\partial(\text{cl}(D - U(\text{Fr}(D))))$. We can trivially extend this isotopy to all of M . Again we change b and H according to this isotopy and we continue, but assume now in addition that $b|U(E_1)$ is the identity where $U(E_1)$ is a regular neighborhood of E_1 in $\text{cl}(D - U(\text{Fr}(D)))$.

By (2.13), $H|E_1 \times I$ is homotopic rel $\partial(E_1 \times I)$ to projection onto E_1 . Construct a homotopy K of $H: M \times I \rightarrow M$ rel $M \times \partial I \cup \text{cl}(M - U(E_1)) \times I$ to a map K_1 (which we will still call H), so that $H|E_1 \times I$ is projection onto E_1 . By (2.14) we get that $H|U(E_1) \times I$ is projection onto $U(E_1)$.

Summarizing, we have isotoped b so that $b|U(E_1)$ is the identity and homotoped H so that $H|U(E_1) \times I$ is projection onto $U(E_1)$. Moreover we have chosen the isotopies and homotopies carefully enough so that the hypotheses of (3.3) still hold. Replacing D by $D_2 = \text{cl}(D - U(E_1))$, we note that all the hypotheses of (3.3) are still satisfied. Hence by induction on the length of the hierarchy for D we are done.

We come now to the main results of this section. The proof uses Waldhausen's isotopy theorem for compact manifolds with nonvacuous boundary. See [13, p. 81].

If M is Euclidean 3-space the correct version of the isotopy result appears in Epstein's paper:

(3.5) [6, p. 96]. *Let $h: R^3 \rightarrow R^3$ be an orientation preserving homeomorphism of Euclidean 3-space. Then there is an isotopy of h to the identity. (This is the same as saying h is proper homotopic to the identity.)*

Our results in this section pertain to eventually end-irreducible manifolds. A noncompact manifold M is *eventually end-irreducible* if $M \neq R^3$ and there exists a compact submanifold D such that $M - D$ is end-irreducible. Removing the boundary from a compact 3-manifold gives one an eventually end-irreducible manifold. A more significant class of manifolds which are all eventually end-irreducible are the uncountably many different contractible open subsets of R^3 constructed by McMillan in [11]. These manifolds are patterned after the one described by Whitehead [14] as a counterexample to his "proof" of the Poincaré conjecture. It can be constructed as follows: Let $f: R^3 \rightarrow R^3$ be a homeomorphism of R^3 such that $f(T_0) = T_1$ where T_0 and T_1 are solid tori and T_0 is linked inside of T_1 . Then $M = \bigcup_i f_i(T_0)$ is a "Whitehead" example. The above function $f: M \rightarrow M$ shows that the assumption in 3.6 is necessary. Since M is contractible f is homotopic to the identity. By the construction given in [17] f cannot be proper homotopic to the identity (and hence it is not isotopic to the identity).

Theorem 3.6. *Let M be an irreducible, eventually end-irreducible manifold. Let $h: M \rightarrow M$ be a homeomorphism which is proper homotopic to the identity via the map $H: M \times I \rightarrow M$. Suppose that $H(\partial M \times I) \subset \partial M$. Then h is isotopic to the identity.*

If $H|_{\partial M \times I}$ is projection onto ∂M , then the isotopy from h to the identity may be chosen constant on ∂M .

Proof. *Step 1. Changing h and H on the boundary.* Since $H|_{\partial M \times I}$ is a proper map, by [6, p. 99] there is an isotopy $J: \partial M \times I \rightarrow \partial M$ of $h|_{\partial M}$ to the identity. By results in §2 we may continue the proof assuming that $H: M \times I \rightarrow M$ is proper homotopic rel $M \times \partial I$ to a map whose restriction to $\partial M \times I$ is projection onto ∂M .

Step 2. Changing H and h on a single component F of $\text{Fr}(C_{2n})$. Choose a compact submanifold C'_0 of M such that $M - C'_0$ is end-irreducible. Let $C_0 = C'_0 \cup h(C'_0)$. Now choose an exhausting sequence $\{C'_i\}$, $i = 1, 2, \dots$, satisfying (3.1) for each component of $M - C_0$. In addition assume the C'_i 's have been chosen so that

- (a) $H(\text{Fr}(C'_i) \times I) \subset C_{i+1}^{\circ'} - C'_i$,
- (b) $H((M - C'_1) \times I) \cap C_0 = \emptyset$,
- (c) $H(C'_i \times I) \subset C_{i+1}^{\circ'}$.

Let $C_i = C'_i \cup C_0$ for $i = 1, 2, \dots$. (a), (b) and (c) above hold when we replace the C'_i 's by the C_i 's.

(1) We denote $\text{cl}(C_{2n+1} - C_{2n-1})$ by S_n . Let F be a component of $\text{Fr } C_{2n}$. We show there is an isotopy on M of $b \text{ rel } \text{cl}(M - S_n) \cup \partial M$ to a homeomorphism (which we continue to denote by b) so that $b(F) = F$.

The C'_i 's were chosen so that $H(F \times I) \subset S_n$. Hence by (3.2) there is an isotopy of S_n , fixed on ∂S_n carrying $b(F)$ onto F . Extending this isotopy so that it is constant outside S_n , we change b via this isotopy so that now $b(F) = F$.

(2) Let F be a closed component of $\text{Fr}(C_{2n})$. Next we show there is an isotopy of b to a homeomorphism which is the identity on F .

We will prove the above by changing H until we are in a position to isotope b . Let H_F denote the map $H|F \times I$. First, there is a homotopy of H_F constant on $F \times \partial I$ after which $H_F(F \times I)$ is contained in F . This follows from (2.8) with $S_n = C$. After this change of H_F , we apply (2.9) to homotope $H_F \text{ rel } F \times \partial I$ to an isotopy $J: F \times I \rightarrow F$ of $b|F$ to the identity. One can now construct a proper homotopy $K: M \times I \times I \rightarrow M \text{ rel } \text{cl}(M - S_n) \times I \cup \partial(M \times I)$ of H to a homotopy $K_1: M \times I \rightarrow M$ so that $K_1|F \times I = J$. Hence we have $K_1: M \times I \rightarrow M$, a homotopy of b to the identity, which agrees with H on ∂M and outside of S_n . Moreover $K_1|F \times I$ is an isotopy of $b|F$ to the identity of F . It follows, now, from (2.10) that b does not interchange the sides of F . Thus given a regular neighborhood $U(F)$, b is isotopic $\text{rel } \text{cl}(M - U(F))$ to a homeomorphism b' so that $b'|F$ is the identity. Moreover b' is homotopic to the identity via a homotopy $H': M|I \rightarrow M$ so that $H'|F \times I$ is projection onto F . We start afresh with b' and H' as our new b and H .

Step 3. Changing b and H on $U(F)$. In order to prepare the way for future isotopies, change b to be the identity in a chosen neighborhood $U(F)$ and H to be projection there.

At this point we will isotope b to be the identity on the other components of $\text{Fr } C_{2n}$.

Step 4. Isotoping b to the identity on all closed components of $\text{Fr } C_{2n}$. Let G be another closed component of $\text{Fr } C_{2n}$. Applying (1.1) with $C = S_n$, we can perform an isotopy of $b \text{ rel } \text{cl}(M - S_n) \cup U'(F)$ so that afterwards $b(G) = G$. We may now apply (2.8) to $H_G: G \times I \rightarrow S_n$ to homotope $H_G \text{ rel } G \times \partial I$ to a map whose image is contained in G . We can now proceed from Step 2(2) using regular neighborhoods $U(G)$ and $U'(G)$ of G in S_n such that $U'(G) \subset U(G)$ and $U(G) \cap U(F) = \emptyset$.

We have a new homeomorphism b which is the identity on $U'(G)$ and a new

homotopy H which on $U'(G) \times I$ is projection onto $U'(G)$. Since $U'(G) \cap (\partial M \cup U'(F)) = \emptyset$ we note that b and H have not been changed on $\partial M \cup U'(F)$. Moreover since $U(G)$ is contained in S_n we have not changed b and H outside of S_n . Thus by induction we can conclude that b is the identity on a regular neighborhood of the union of the closed components of $\text{Fr } C_{2n}$.

Step 5. Isotoping b to the identity on components with boundary. We need to carry out a similar process for components of $\text{Fr } C_{2n}$ which are not closed. These components meet the noncompact boundary components of ∂M . Let G be one such component. Denote by F the system of closed components of $\text{Fr } C_{2n}$ and let $U(F)$ denote a regular neighborhood of F on which H is projection.

(1) We assert there is an isotopy on M fixed on $U(F) \cup \text{cl}(M - S_n) \cup \partial M$ which carries $b(G)$ onto G . By (2.11), with $S_n = C$ and $F =$ all closed components of $\text{Fr } C_{2n}$ there is a homotopy of $H_G \text{ rel } \partial(G \times I)$ so that afterwards $H_G(G \times I)$ is contained in $S_n - U(F)$. Using (3.2) we have an isotopy on $\text{cl}(S_n - U(F))$, fixed on $(\text{cl}(S_n - U(F)))$, which carries $b(G)$ onto G . Extend this isotopy trivially to all of M , and our assertion follows.

(2) Changing b and H by this isotopy we proceed assuming $b(G) = G$. Using a procedure similar to the one used for closed components, there is an isotopy of b fixed on $(M - U(G)) \cup \partial M$ to a homeomorphism which is the identity on $U'(G)$. The results of Part 3, § 2 give us a homotopy of H to a map which is the projection map when restricted to $U'(G)$. It can be done so that H is not changed on $(M - U(G)) \times I \cup \partial(M \times I)$.

It is clear that (2.11) permits us to make an induction on the number of compact surfaces with boundary in $\text{Fr } C_{2n}$. We conclude that, for each positive integer n , H is proper homotopic $\text{rel } \text{cl}(M - S_n) \times I \cup \partial M \times I$ to a map $H'_n: M \times I \rightarrow M$ such that $H'_n|_{U(\text{Fr } C_{2n}) \times I}$ is projection onto the first factor. Moreover $(H'_n)_0$ is isotopic $\text{rel } \text{cl}(M - S_n) \cup M$ to b , and $(H'_n)_1 = \text{identity}$.

Step 6. Fitting together the homotopies and isotopies to bring the proof to an end. There is a well-defined map H' , where $H' = \bigcup_n H'_n \cup H|_{C_0 \times I}$ on $M \times I$. Using the proper homotopy extension theorem for each homotopy of H we have that $H'_n(S_n \times I) \subset H(S_n \times I)$. This and the fact that H is proper show that H' is proper.

Using a similar argument we can show that b is isotopic $\text{rel } \partial M$ to a map which is the identity on $U(\text{Fr } C_{2n})$ for every n .

We continue with the proof assuming $H|_{U(\text{Fr } C_{2n}) \times I}$ is projection onto $U(\text{Fr } C_{2n})$ for each n . By choosing a hierarchy for each shell one can apply Proposition 3.3 and obtain a homotopy of b to a map which is the identity on all but a collection of disjoint open cells of $M - C_2$. Let Δ be the closure of one of the above open cells. $b|_{\partial \Delta}$ is the identity. Therefore by Alexander's theorem,

$b| \Delta$ is isotopic rel $\partial \Delta$ to the identity. Thus we may isotope b rel $\partial \Delta$, for all such cells Δ , to a homeomorphism which is the identity on $\text{cl}(M - C_2)$. Also there is no obstruction to homotoping H so that it will be the projection map when restricted to one of these cells.

At this point $b| \text{cl}(M - C_2) = \text{identity}$ and $H| \text{cl}(M - C_2) \times I = \text{projection}$ onto the first factor. To finish the proof, we note that since we chose the C_i 's so that $H(C_i \times I)$ is contained in C_{i+1} , $H(C_3 \times I)$ is contained in C_3 ; $H| \partial(C_3) \times I$ is projection onto ∂C_3 . Therefore, by [13, Theorem 7.1], $b| C_3$ is isotopic rel ∂C_3 to the identity. Extending this isotopy trivially to all of M we have that b is isotopic to the identity.

Corollary 3.7. *Let M be a noncompact irreducible, eventually end-irreducible manifold with compact, irreducible boundary components. Let $H: M \times I \rightarrow M$ be a proper homotopy of a homeomorphism b to the identity. Then b is isotopic to the identity.*

Proof. See the note following (2.8) to reduce ourselves to (3.6).

4. Isotopy results for end-irreducible manifolds and a twisted I -bundle theorem.

For the next theorems we will not assume that H is proper. As shown in the example given in § 3, we must make some additional hypotheses on the manifold. We assume that M is end-irreducible.

Lemma 4.2 is our replacement of (2.10). For this we need Brown and Tucker's generalization of [13, Theorem 6.1].

(4.1) [5, to appear]. *Let M and N be irreducible, boundary-irreducible, and end-irreducible manifolds. Suppose N is noncompact but that boundary components of M are compact. Finally suppose there is a proper map $f: (M, \partial M) \rightarrow (N, \partial N)$ so that $\pi_1(f)$ is a monomorphism. Then there is a proper homotopy of f (as a map of pairs) to a map f_1 so that either (a) or (b) holds:*

(a) f_1 is a finite sheeted covering map.

(b) *There are closed surfaces F in M and G in N so that M is a vector bundle with zero section F , f_1 is a covering of F onto G , one component of $N - G$ has closure $G \times [0, \infty)$ and $f_1(k, t) = (f_1(k), |t|)$ where $|t|$ is with respect to some Riemannian metric on M . Furthermore, if $f| \partial M$ is a local homeomorphism, then the homotopy may be chosen constant on ∂M .*

Lemma 4.2. *Let M be an irreducible, end-irreducible manifold, and G an incompressible closed surface in M . Let $H: M \times I \rightarrow M$ be a homotopy of a homeomorphism b to the identity. Suppose $b(G) = G$ and $H_G(G \times I) \subset G$. If M is homeomorphic to $G \times R$, where R is the real line, assume b is orientation preserving. Then b does not interchange the sides of G .*

Proof. If $M - G$ is connected the proof is the same as for (2.19).

If $M - G$ is disconnected, let A and B denote the closures of the components of $M - G$. Then $\pi_1(M) = \pi_1(A) *_{\pi_1(G)} \pi_1(B)$. If this is nontrivial, i.e., $\pi_1(G) \rightarrow \pi_1(B)$ is not surjective, and our conclusion is assumed false, then the argument Waldhausen gives in [13, Lemma 7.4, p. 83] shows this leads to a contradiction.

If our representation of $\pi_1(M)$ is trivial and we assume that $b(A) = B$ then $\pi_1(G) \rightarrow \pi_1(A)$ is also an isomorphism. We will show that this implies $M = G \times R$.

We claim that A is homeomorphic to $G \times [0, \infty)$. The inclusion map $i: G \rightarrow A$ induces a homotopy equivalence between G and A . Let $i^{-1}: A \rightarrow G$ be a homotopy inverse. Let $f: A \rightarrow [0, \infty)$ be a proper map so that $f(G) = 0$. Then the map $i^{-1} \times f: A \rightarrow G \times [0, \infty)$ is proper. Moreover $(i^{-1} \times f)_*^{-1}$ is an isomorphism. Thus we apply (4.1) to conclude that $i^{-1} \times f$ is homotopic to a homeomorphism. Hence $A = G \times [0, \infty)$. Thus we get $M = G \times R$, and since b is orientation preserving on all of M , it cannot interchange A and B .

We have two isotopy results for end-irreducible manifolds. They correspond to (3.6) and (3.7); depending on whether or not $H(\partial M \times I)$ is contained in ∂M . Precisely, we show

Theorem 4.3. *Let M be a noncompact irreducible, end-irreducible manifold. Let $H: (M \times I, \partial M \times I) \rightarrow (M, \partial M)$ be a homotopy of pairs of a homeomorphism b to the identity. Assume that if F is a boundary component of M which is a plane or an open annulus $b|_F$ is orientation preserving. Assume further that b is orientation preserving if M is a bundle over a closed surface (possibly nonorientable) with fiber R . Then b is isotopic to the identity. If $H|_{\partial M \times I}$ is projection onto ∂M , the isotopy of b to the identity may be chosen constant on ∂M .*

Theorem 4.4. *Let M be an irreducible, end-irreducible manifold with $\partial M \neq \emptyset$. Suppose M is boundary-irreducible and has no boundary components which are planes. Let $H: M \times I \rightarrow M$ be a homotopy of a homeomorphism b to the identity such that for each boundary component F , $H|_{F \times I}$ is a proper map. If M is a bundle over a noncompact surface (possibly nonorientable) with fiber I , assume b is orientation preserving. Then b is isotopic to the identity.*

The assumption that $H|_{F \times I}$ is proper is essential; for let M be a solid torus with two disjoint longitudinal curves removed from its boundary. ∂M consists of two open annuli. The rotation of the torus which takes one annulus onto the next is homotopic to the identity. Clearly since it interchanges boundary components it cannot be isotopic to the identity.

Proof of Theorem 4.3 when all boundary components are compact. The first step is exactly the same as Step 1 of Theorem 3.6. We can go on assuming b and H are fixed up on ∂M ; b has been isotoped to a homeomorphism which is the

identity on M and H has been changed so that when restricted to ∂M it is projection onto the first factor.

Choose an exhausting sequence $\{C_i\}$ for M satisfying (3.1). In addition, choose $\{C_i\}$ so that $H(C_i \times I)$ is contained in C_{i+1} . We will define our isotopy of b to the identity by an inductive procedure. That is, we will isotope b to the identity on $C_{i-1} \text{ rel } \partial M$. Then we will construct an isotopy which is fixed on C_{i-1} and ∂M and changes $b|_{C_i}$ to the identity. Hence all these put together give a well defined isotopy of b to the identity.

Let us assume $b|_{U(C_{i-1})}$ is the identity and $H|_{U(C_{i-1}) \times I}$ is projection onto the first factor. We now fix b and H on the shell $\text{cl}(U(C_i) - U(C_{i-1}))$. Let $C_0 = \emptyset$.

(1) Let F_1, \dots, F_k be the components of $\text{Fr } C_i$. By (3.1), each F_i is closed and incompressible in M and hence certainly in C_{i+1} . We apply (1.1) with $C = C_{i+1}$, the system $F = \text{Fr } U(C_{i-1})$ and $G = F_1$. (Use (4.2) to see that b does not interchange the sides of the components of F .) We obtain an isotopy of C_{i+1} , which is fixed on $\partial C_{i+1} \cup \text{Fr } U(C_{i-1})$ carrying $b(F_1)$ onto F_1 . Since $\text{Fr } U(C_{i-1})$ separates C_{i+1} , and one of the components of the complement is $U(C_{i-1})$ and the isotopy is fixed on $\partial U(C_{i-1}) = \text{Fr } U(C_{i-1})$, we actually have an isotopy on $\text{cl}(C_{i+1} - U(C_{i-1})) \text{ rel the boundary}$ which takes $b(F_1)$ onto F_1 . We extend this isotopy trivially to all of M . Changing b and H by this isotopy we continue under the assumption that $b(F_1) = F_1$.

(2) We would like to homotope $H_{F_1}: F_1 \times I \rightarrow C_{i+1} \text{ rel } F_1 \times \partial I$ to a map into F_1 .

Case 1. C_i has more boundary than F_1 . We apply (2.8) to get our result; for if $\partial C \neq F_1$, then if F_1 separates C_{i+1} it also separates ∂C_{i+1} .

Case 2. ∂C_i is connected. Discard it from our exhausting sequence. If one can find a subsequence of $\{C_i\}$ so that each member has nonconnected boundary we apply (2.8) to homotope H_{F_1} and continue with the proof. If ∂C_i is connected for each i , but still we can find a homotopy of $H|_{\partial C_i \times I} \text{ rel } \partial C_i \times \partial I$ to a map into ∂C_i , we again have that (2) is true and we go on. We consider at the end of this proof the case in which ∂C_i is connected for each i and there is no homotopy of $H|_{\partial C_i \times I}$ to a map into ∂C_i . Thus except for the last mentioned case we now have a homotopy of $H_{F_1} \text{ rel } F_1 \times I$ to a map into F_1 .

The rest of the proof is now quite similar to the proof of Theorem (3.6). This proves (4.3) except for the case in which ∂C_i is connected for each i and $H|_{\partial C_i \times I}$ is not homotopic $\text{rel } \partial C_i \times I$ to a map into ∂C_i . This can occur only when $\partial M = \emptyset$. In fact this happens only when M is a bundle over a nonorientable closed surface with fiber R and b is orientation-reversing. That is, this cannot occur under the hypotheses.

We first show that C_i is homeomorphic to a bundle over a nonorientable closed

surface with fiber I . Since $H(\partial C_i \times I)$ is contained in C_{i+1} , and both ∂C_i and $b(\partial C_i)$ are incompressible in C_{i+1} , by (3.2) there is an isotopy of M fixed outside of C_{i+1} carrying $b(\partial C_i)$ onto ∂C_i . Change b and H according to this isotopy so that now $b(\partial C_i) = \partial C_i$. Let F be a homeomorphic copy of ∂C_i , and define the map $H': F \times I \rightarrow C_{i+1}$ as the restriction $H|_{\partial C_i \times I}$. Given a regular neighborhood of $F \times \partial I$ in $F \times I$ there is a homotopy of H' rel $F \times \partial I$ so that afterwards $H'^{-1}(\partial C_i \cap U(F \times \partial I)) = F \times \partial I$. Follow (2.8) exactly up to Case 1. Summarizing, we have homotoped H' rel $F \times \partial I$ so that $H'^{-1}(\partial C_i) = F \times \partial I \cup F_1 \cup \dots \cup F_m$, where each F_i is an incompressible surface in $F \times I$ which is parallel to $F \times 0$. In addition, at this point assume H' has been homotoped so that the number of F_i 's is minimal. Let us denote by $\widetilde{C_{i+1}}$ the result of splitting C_{i+1} along ∂C_i . Lift $H'|_{F \times I}$ to $\widetilde{C_{i+1}}$. We continue independently of (2.8) now. We apply [13, Theorem 6.1] to each of the $H'|_{F \times I'}$. If each one was homotopic rel $F \times \partial I'$ to a map into $\widetilde{\partial C_{i+1}}$ this would imply $H': F \times I \rightarrow C_{i+1}$ was homotopic to a map into ∂C_i . We assumed at the start that this did not occur. Hence some $H'|_{F \times I'}$ is homotopic to a covering map. (In fact, since the number of F_i 's is assumed minimal, we have that all of the $H'|_{F \times I'}$ s are homotopic to coverings). The covering maps go onto the component of $\widetilde{C_{i+1}}$ homeomorphic to C_i . Since $H'|_{F \times I'}$ is a homeomorphism on each component of $F \times \partial I'$, it is homotopic to a 2-1 covering. By [13, p. 70] we can conclude that C_i is a bundle with fiber I over a nonorientable closed surface.

Later in the proof we will need that all of $H'|_{F \times I}$ is homotopic rel $F \times \partial I$ to a 2-1 covering. It suffices to show that if m is the minimum number of F_i 's then $m = 0$. If this were not so, two adjacent liftings of $H'|_{F \times I'}$ are homotopic to coverings. Denote them by $H_1: F \times I_1 \rightarrow C_i$ and $H_2: F \times I_2 \rightarrow C_i$. Then H_1 followed by H_2 is homotopic to a map into ∂C_i . This contradicts our minimality assumption.

Next we show that M is homeomorphic to a bundle with fiber R over a nonorientable closed surface. In fact if B_i is the bundle with fiber R associated with the bundle C_i , then M is homeomorphic to B_i by a homeomorphism which "extends" $H: \partial C_i \times I \rightarrow C_i$.

Let λ be a loop on ∂C_i , and let ρ be a path in $\text{cl}(C_{i+1} - C_i)$ from ∂C_{i+1} to the base point of λ . Then twice around $(\rho\lambda\rho^{-1})$ is homotopic to a loop in ∂C_{i+1} since $\pi_1(\partial C_{i+1})$ has index 2 in $\pi_1(C_{i+1})$. But using the fact that ∂C_i is incompressible in C_{i+1} and Van Kampen's theorem one can show that $\pi_1(\text{cl}(C_{i+1} - C_i)) \rightarrow \pi_1(C_{i+1})$ is a monomorphism. Therefore λ is homotopic in $\text{cl}(C_{i+1} - C_i)$ to a loop in ∂C_{i+1} . By [13, Lemma 5.1] $\text{cl}(C_{i+1} - C_i)$ is homeomorphic to $\partial C_i \times I$. Similarly for each shell. Hence M is homeomorphic to an R -bundle over the same nonorientable closed surfaces as C_i . The argument given

on p. 83 of [13] shows that if b is orientation preserving, we reach a contradiction.

Proof of (4.3) when some component of ∂M is noncompact. $b|_{\partial M}$ is isotopic to the identity. Proceeding as in the previous theorem of this section we may assume $b|_{\partial M}$ is the identity and that H is projection onto ∂M .

Now that H is fixed up on the boundary, the only difference remaining in the noncompact case is that if $\{C_n\}$ is an exhausting sequence satisfying (3.1), $\text{Fr } C_n$ may contain components which are not closed. (2.11) takes care of such surfaces.

Proof of Theorem 4.4 when all components of ∂M are compact. Using [13, Theorem 6.1] one can reduce this to (4.3). For the proof of (4.4) when we allow noncompact boundary components we need a noncompact version of [13, Proposition 4.1].

The twisted-bundle result. We will say $f: (M, \partial M) \rightarrow (N, \partial N)$ contracts as a map of pairs if f is homotopic by a homotopy of pairs to a map $g: (M, \partial M) \rightarrow (\partial N, \partial N)$. If t is a homeomorphism of M onto itself we say $N \subset M$ is t -invariant or invariant under t if $t(N) = N$.

Proposition 4.5. *Let F be a noncompact surface without boundary, different from a plane. Let M be a manifold with connected boundary. Suppose $f: F \times I \rightarrow M$ is a 2-1 covering map which is proper. Then M is a bundle with fiber I over a nonorientable noncompact surface.*

Proof. From the hypotheses of the theorem it is easy to see that $f|_{F \times 0}$ and $f|_{F \times 1}$ are homeomorphisms onto the boundary of M . Thus we can regard F as the boundary of M and identify points of F so that $(x, 0)$ corresponds to x in F via $f|_{F \times 0}$. We denote $f|_{F \times i}$, $i = 0$ and 1 , by f_i and regard them as homeomorphisms of F onto itself. f_0 is the identity function. At the beginning of Step 1 we refer to f_1 . The above function is the one we mean.

Let $t: F \times I \rightarrow F \times I$ denote the covering transformation. We will construct a fibering of $F \times I$ which is invariant under t in order to prove the result. We handle the case where F is an open cylinder at the end.

The bulk of the proof consists in producing an exhausting sequence for $F \times I$, which is invariant under t and whose frontier components consist of annuli which are pairwise interchanged under t . After that, the proof uses the techniques of the compact version in [13].

Step 1. We begin by constructing an exhausting sequence $\{G_n\}$ for F which consists of compact connected surfaces which are invariant under f_1 and such that $F - G_n$ has only unbounded components. Let $\{F_n\}$ be an exhausting sequence for F consisting of compact connected surfaces. Since F is not an open annulus assume F_1 is not an annulus or disc. Small deformations of the F_i 's will insure that ∂F_i and $f_1(\partial(F_i))$ intersect transversally. Hence $F_i \cup f_1(F_i)$ is a surface. Moreover, if n is sufficiently large, $F_n \cup f_1(F_n)$ is connected.

By eliminating the first few F_i 's, $\{F_i \cup f_1 F_i\}$ is an exhausting sequence of connected surfaces which are invariant under f_1 . If for some n , $F - (F_n \cup f_1(F_n))$ has some components with compact closure we add them to $F_n \cup f_1(F_n)$. Since there are only finitely many boundary components of $F_n \cup f_1 F_n$ this set is still compact. Let us denote the resulting exhausting sequence by $\{G_n\}$.

Step 2. We now modify $G_1 \times I$ so that the resulting set which we will denote by C_1 is a t -invariant set whose frontier components are annuli which are interchanged by t . C_1 will be the first member of the exhausting sequence we want.

(1) Let λ be a boundary loop of G_1 . We can deform $\lambda \times I$ so that afterwards $\lambda \times I \cap t(\lambda \times I)$ consists only of simple closed double curves and double arcs.

(2) Denote $\lambda \times I$ by A_λ . Intersections of the frontiers occur only when one annulus is an A_λ and one is a tA_μ . We wish to alter the A_λ so that $t(A_\lambda) = A_\mu$. We first modify the A_λ so that whenever $A_\lambda \cap tA_\mu$ is nonvacuous $A_\lambda \cap tA_\mu = A_\lambda$ or ∂A_λ .

It is not difficult to first change $G_1 \times I \cup tG_1 \times I$ so that the intersections of the resulting annuli are only noncontractible intersection loops or only arcs which intersect both $F \times 0$ and $F \times 1$. Clearly the intersection arcs occur only for pairs $A_\lambda, tA_{f_1^{-1}(\lambda)}$ for some λ in G_1 . The same is true of the intersection loops since F is not an open annulus.

(a) $A_\lambda \cap tA_\mu$ consists of simple closed curves which are parallel to $\lambda \times 0$.

(a-1) $\lambda = \mu$. Take a small regular neighborhood $U(A_\lambda \cup t(A_\lambda))$ which is invariant under t . $\text{Fr}(U(A_\lambda \cup tA_\lambda))$ contains two incompressible annuli A' and A'' which intersect both $F \times 0$ and $F \times 1$. Each frontier component is either a torus or an annulus. If it is an annulus with both boundary loops on $F \times 0$ say, then it would disconnect $\lambda \times 0$ from $\lambda \times 1$. This would be possible only when F is a torus. But F is noncompact. Therefore our assertion about A' and A'' is true. If $t(A') = A'$, then $f(A')$ (f is the covering map) would be a 2-sided Moebius band in M . Since M is orientable this cannot occur. On the other hand if $t(A') = A''$, this would contradict the fact that G_1 is invariant under f_1 . Thus we have shown that (a-1) cannot occur; i.e., $\lambda = \mu$. Note that this same argument also shows that $t(A_\lambda) \neq A_\lambda$. We shall keep this in mind for a later argument.

(a-2) $\lambda \neq \mu$. In this case since we have shown that at most one pair of annuli intersect; $A_\lambda \cup tA_\mu$ and $tA_\lambda \cup A_\mu$ are disjoint sets. Let $U(A_\lambda \cup tA_\mu)$ be a regular neighborhood of $A_\lambda \cup tA_\mu$ so that $t(U(A_\lambda \cup tA_\mu)) \cap U(A_\lambda \cup tA_\mu) = \emptyset$. As in (a-1), $\text{Fr } U(A_\lambda \cup tA_\mu)$ contains two incompressible annuli A' and A'' each one intersecting both $F \times 0$ and $F \times 1$. Since $t(U(A_\lambda \cup tA_\mu)) \cap U(A_\lambda \cup tA_\mu) = \emptyset$, $tA' \cap A' = tA'' \cap A'' = \emptyset$. Replace A with one of the A'' 's modified slightly so that $\partial A' = \lambda \times 0 \cup \lambda \times 1$.

(b) $A_\lambda \cap tA_\mu$ consists only of arcs, each of which intersects both $F \times 0$ and

$F \times I$. First of all it is not possible that $A_\lambda \cap tA_\mu$ consists of just one arc ρ ; for since A_λ intersects tA_μ transversally along $\text{int } \rho$ this would imply that λ has intersection number 1 with itself. But F is orientable, so this is impossible. Therefore two consecutive arcs ρ_1, ρ_2 on A_λ cut off discs D and D' on A_λ and tA_μ respectively. It is not hard now to eliminate at least ρ_1 and ρ_2 .

Thus we have modified the frontier components A_λ of $G_1 \times I$ so that for each boundary loop λ , $tA_\lambda = A_\mu$ and $\lambda \neq \mu$. We call this modified set C_1 . It qualifies as our first term of an exhausting sequence $\{C_i\}$ for $F \times I$ so that each C_i is a connected manifold which is invariant under t and whose boundary components are annuli which are interchanged by t .

By [13, p. 68] there is an isotopy of $F \times I$ constant on $F \times \partial I$ which makes all the components of $\text{Fr } C_1$ vertical. Instead let us use the inverse of the isotopy to change the product structure on $F \times I$.

Step 3. We now provide an inductive procedure for constructing the C_i 's. Having constructed C_i , first choose m sufficiently large so that $[\text{Fr}(G_m \times I) \cup \text{Fr}(t(G_m \times I))] \cap (C_i \cup G_{i+1} \times I) = \emptyset$. Now change $G_m \times I \cup t(G_m \times I)$ as we changed $G_1 \times I \cup t(G_1 \times I)$ to construct C_{i+1} . We use [13, p. 68] to "straighten up" $\text{Fr } C_{i+1}$ in $\text{cl}(F \times I - C_i)$. In fact, we can regard the isotopies as taking place in shells of the form $\text{cl}(G_{n(i+1)} - G_{n(i)}) \times I$, which contain the frontier of only one of the C_i 's. Thus we can combine them to get a global isotopy of $F \times I$ so that for each n $\text{Fr } C_n$ is vertical. Moreover, $\{C_i\}$ exhausts $F \times I$, since at each stage of the construction we chose C_i so that it contained our original $G_i \times I$.

Step 4. We begin to construct the fibering of $F \times I$ which is invariant under t . We first do it on regular neighborhoods of the frontiers of C_n .

Fix n . Let A be a component of $\text{Fr } C_n$. We make $t|A$ fiber preserving by deforming $M \times I$ near $t(A)$ by an isotopy which takes $t(A)$'s "natural" fibering to the one which is defined by $t: A \rightarrow tA$. Using this isotopy we can get vertical neighborhoods $U(A)$ and $U(tA)$ such that $t(U(A)) = U(tA)$ and such that $t|U(A)$ is fiber preserving.

Step 5. We make t fiber preserving on the rest of $F \times I$, exactly as Waldhausen suggests in [13, Proposition 4.1].

We are now finished except when F is an open annulus. A similar but much simplified proof will work.

An outline of the proof of (4.4) when some component is not compact. If $H| \partial M \times I \rightarrow M$ is homotopic rel $\partial M \times \partial I$ to a map into ∂M , then by (4.3) the result follows. If however there exists a noncompact component F of ∂M such that $H|F \times I$ is not homotopic to a map into F , then we show that either

- (a) $H|F \times I$ is homotopic to a homeomorphism of $F \times I$ onto M , or
- (b) $H|F \times I$ is homotopic to a 2-1 covering of $F \times I$ onto M .

Case (a) occurs if we assume $b(F) \neq F$, while (b) occurs if $b(F) = F$. In either case we get a contradiction.

Proof. We begin by assuming there is a noncompact component F for which $H: F \times I \rightarrow M$ is not homotopic to a map into F .

Part 1. We show H is homotopic to a covering of $F \times I$ onto M . Essentially we are mimicking the proof of [13, Theorem 6.1].

Choose an exhausting sequence $\{C_n\}$ for M satisfying (3.1).

(1) There is a proper homotopy of H to a map which we continue to denote by H so that $H^{-1}(C_n)$ is a compact manifold with incompressible frontier components. One can do this using (2.13) and the fact that H is proper.

Let $\{D_n\}$ be an exhausting sequence for $F \times I$ with each $\{D_n\}$ a component of $H^{-1}(C_n)$.

(2) We show that for every n , and every component G of $\text{Fr } D_n$, $H|_G: \rightarrow \text{Fr } C_n$ is homotopic rel ∂G to a covering. For if we argue as in [13, p. 78] one can conclude that H is homotopic to a map into F rel $F \times \partial I$. We are assuming $H|_{F \times I}$ is not homotopic to a map into F ; so (2) follows. Thus H is proper homotopic to a map which is a covering when restricted to $\text{Fr } D_n$ for each n . Furthermore since $H(D_n \cap (F \times I)) \cap H(\text{int Fr } D_n) = \emptyset$, $H|_{\partial D_n}$ is a local homeomorphism.

(3) Next we prove $H: F \times I \rightarrow M$ is proper homotopic rel $F \times \partial I$ to a covering.

Choose hierarchies for $S_0 = C_1$, $S_n = \text{cl}(C_{n+1} - C_n)$, $n = 1, 2, \dots$; let $S_n^1 = S_n$, $F_j \subset S_n^j$, $U(F_j) \subset S_n^j$, $S_n^{j+1} = \text{cl}(S_n^j - U(F_j))$, $j = 1, \dots, k_n$.

(a) Fix n ; we will show that $H|_{H^{-1}(\partial C_n \cup \bigcup_{j=1}^{k_n} U(F_j)) \cap D_n} \rightarrow S_n$ is homotopic to a local homeomorphism. By the above it is true for $j = 0$. Suppose we have proved it for all $j < l$. By [13, p. 60] there is a homotopy of $H: \text{cl}(D_{n+1} - D_n) \rightarrow S_n = \text{cl}(C_{n+1} - C_n)$ rel $\partial(\text{cl}(D_{n+1} - D_n))$ so that afterwards H is transverse with respect to F_l and so that $H^{-1}(F_l)$ is a system of incompressible surfaces in $\text{cl}(D_{n+1} - D_n)$. Then $\ker(H_*) = 0$; it follows that if G is any component of $H^{-1}(F_l)$ either $H|_G: \rightarrow F_l$ is homotopic to a covering – in which case (3) follows – or we have one of two things occurring:

(i) G is a disc and hence so is F_l and the covering map $H|_{\partial G}$ is not a homeomorphism, or

(ii) by [13, Lemma 1.4.3], G is an annulus and $H|_G: (G, \partial G) \rightarrow (F_l, \partial F_l)$ contracts to $(\partial F_l, \partial F_l)$. Note that here $H(\partial G)$ is contained in one boundary curve of F_l .

Both (i) and (ii) above allow us to choose a simple arc ρ which is properly embedded in $F \times I$ so that $H(\partial \rho)$ is one point and $H|_\rho: (\rho, \partial \rho) \rightarrow (F, H(\partial \rho))$ contracts. This implies that $H|_{F \times I} \rightarrow M$ is homotopic to a map into F . We have assumed this is not so. Thus we conclude that H is proper homotopic

rel $\partial(F \times I)$ to a map which is a covering on all but a collection of 3-cells of $F \times I$. Hence for each 3-cell C , H restricted to C is homotopic rel ∂C to a homeomorphism. Since H is proper and M is not R^3 , H is actually proper homotopic to a covering on all of $F \times I$; i.e., H is surjective.

Thus we conclude the proof of Part 1, that is, if F is not a plane, and $H: F \times I \rightarrow M$ does not contract then H is homotopic rel $F \times \partial I$ to a homeomorphism or a 2-1 covering of $F \times I$ onto M .

Part 2. We show that the conclusion of Part 1 leads us to a contradiction.

(a) Suppose H is 1-1, then $M = F \times I$. In this case we are assuming b is orientation-preserving. Using $H: F \times I \rightarrow M$, identify F with $F \times 0$. Then $b(F \times 0) = F \times 1$. $b|_{F \times 0}$ followed by projection of M onto $F \times 0$ is a homeomorphism b' of $F \times 0$ onto itself which is proper homotopic to the identity. By (3.3) $b': F \times 0 \rightarrow F \times 0$ is isotopic to the identity. Therefore $b': F \times 0 \rightarrow F \times 0$ is orientation-preserving; but then so is $b: F \times 0 \rightarrow F \times 1$. Since b interchanges $F \times 0$ and $F \times 1$, $b: M \rightarrow M$ is orientation-reversing.

(b) If H is 2-1, by (4.5) M is a bundle with fiber I ; hence we have b is orientation-preserving. See [13, p. 83] to see that leads to a contradiction.

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