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# MAXIMAL REGULAR RIGHT IDEAL SPACE OF A PRIMITIVE RING. II

BY

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ABSTRACT. If $R$ is a ring, let $X(R)$ be the set of maximal regular right ideals of $R$. For each nonempty subset $E$ of $R$, define the *hull* of $E$ to be the set $\{I \in X(R) \mid E \subseteq I\}$ and the *support of* $E$ to be the complement of the hull of $E$. Topologize $X(R)$ by taking the supports of right ideals of $R$ as a subbase. If $R$ is a right primitive ring, then $X(R)$ is homeomorphic to an open subset of a compact space $X(R^{\#})$ of a right primitive ring $R^{\#}$, and $X(R)$ is a discrete space if and only if $X(R)$ is a compact Hausdorff space if and only if either $R$ is a finite ring or a division ring. Call a closed subset $F$ of $X(R)$ a *line* if $F$ is the hull of $I \cap J$ for some two distinct elements $I$ and $J$ in $X(R)$. If $R$ is a semisimple ring, then every line contains an infinite number of points if and only if either $R$ is a division ring or $R$ is a dense ring of linear transfomations of a vector space of dimension two or more over an infinite division ring such that every pair of simple (right) $R$-modules are isomorphic.

**Introduction.** Let $R$ be a ring. A right ideal $I \subseteq R$ is called regular if there exists an $e \in R$ such that, for all $a \in R$, $a - ea \in I$. Let $X(R)$ be the set of maximal regular right ideals of $R$. If $E$ is a nonempty subset of $R$, the set of maximal regular right ideals of $R$ which contain $E$ is called the *hull* of $E$ and the complement of the hull of $E$ with respect to $X(R)$ is called the *support* of $E$. We topologize $X(R)$ by taking the set of supports of right ideals of $R$ as a subbase. Then the topological space $X(R)$ is called the *maximal regular right ideal space* of $R$ [3]. If $1 \in R$, $X(R)$ is a compact space [3, 1.7]; however, in general, it is not a compact space. Recall that a topological space $X$ is *irreducible* [4] if $X \neq \emptyset$ and $X$ is not the union of two proper closed subsets. $X$ is *reducible* if it is not irreducible. In [3], we have shown that if $R$ is a (right) primitive ring, then $X(R)$ is reducible if and only if $R$ is a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field, and $X(R)$ is a Hausdorff space if and only if either $R$ is a division ring or $X(R)$ is reducible and $R$ modulo its socle is a radical ring. In this paper we continue our previous work [3]. Our main results are as follows:

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If $R$ is a ring and $S$ is an ideal of $R$ then $X(S)$ is homeomorphic to an open subset of $X(R)$. If $R$ is a primitive ring then $X(R)$ is embeddable as an open subset of a compact space $X(R^{\#})$ for some primitive ring $R^{\#}$. As a consequence of this fact, we will be able to show that if $R$ is a primitive ring then $X(R)$ is a compact Hausdorff space if and only if $X(R)$ is a discrete space if and only if either $R$ is a division ring or a finite ring. For any subset $Y$ of $X(R)$, define $j(y) = \bigcap \{I \in X(R) \mid I \in Y\}$.

Let us call a closed subset $F$ of $X(R)$ a *line* if $F$ is the hull of $j(\{x\}) \cap j(\{y\})$ for some distinct points $x, y$. Call a closed subset $F$ of $X(R)$ a *hyperplane* provided that $j(F) \neq J(R)$, the Jacobson radical, and if $F'$ is a closed subset such that $F' \supseteq F$ and $F' \neq F$ then $j(F') = J(R)$. If $R$ is a primitive ring, then $R$ has nonzero socle if and only if $X(R)$ has a *hyperplane*, and $R$ is simple artinian if and only if (i) $X(R)$ is compact, (ii) $X(R)$ has no line which contains exactly two points, and (iii) $X(R)$ has a hyperplane. For a semisimple ring $R$, $X(R)$ contains at least two points and every line of $X(R)$ contains an infinite number of points if and only if $R$ is a dense ring of linear transformations of a vector space of dimension two or more over an infinite division ring such that every pair of simple (right) $R$-modules are isomorphic. Throughout this paper, by a *primitive ring* we always mean a *right* primitive ring and every module is a *right* module.

1. **Preliminaries.**

1.1. **Definition.** If $E$ is a subset of $R$, we define the *support* of $E$ to be the set of maximal regular right ideals of $R$ which do not contain the set $E$. It will be denoted by $\operatorname{supp}(E)$.

1.2. **Definition.** If $R$ is a ring, let $X(R)$ be the set of maximal regular right ideals of $R$. We give a topology to $X(R)$ which is generated by the subbasis consisting of all supports of the right ideals of $R$. We will call $X(R)$ together with this topology the *maximal regular right ideal space* of $R$. It will simply be denoted by $X(R)$.

1.3. **Definition.** If $E$ is a subset of $R$, the *hull* of $E$ is the set $X(R)\setminus\operatorname{supp}(E)$. It will be denoted by $h(E)$. If $E = \{e\}$ is a singleton set, we let $h(e) = h(E)$.

1.4. **Definition.** A nonempty topological space $X$ is *irreducible* if $X$ is not the union of two proper closed subsets. $X$ is *reducible* if it is not irreducible. A subset $Y$ of a space $X$ is irreducible if $Y$ is irreducible in the induced topology.

1.5. **Definition.** If $x$ is an element of $X(R)$ for some ring $R$, then $x$ is also a right ideal of $R$. To make a distinction, we write $j(x)$ for the right ideal $x$. If $Y$ is a subset of $X(R)$, we define $j(Y) = \bigcap \{j(x) \mid x \in Y\}$. If $Y = \{y\}$ a singleton set, we let $j(y) = j(Y)$.

1.6. **Definition.** Let $R$ be a ring and let $V$ be a right $R$-module. For each $v \in V$, we define $v^{\perp} = \{r \in R \mid vr = 0\}$. If $E$ is a subset of $V$, we define $E^{\perp} = \bigcap \{v^{\perp} \mid v \in E\}$.

1.7. **Proposition.** *Let $R$ be a ring and $I$ be a maximal regular right ideal of $R$. If $A$, $B$ are right ideals of $R$ such that $AB \subseteq I$, then either $A \subseteq I$ or $B \subseteq I$.*

**Proof.** Suppose that $A \not\subseteq I$ and $B \not\subseteq I$. Then $A + I = R$ and $RB = (A + I)B \subseteq I$. Let $e$ be a left identity modulo $I$. Since $eb \in I$, for every $b \in B$, $B \subseteq I$. This is a contradiction.

1.8. **Corollary.** *Let $R$ be a ring and $I$ be a maximal regular right ideal of $R$. If $a \in R$ such that $aR \subseteq I$, then $a \in I$.*

**Proof.** Let $A = \{a \in R \mid aR \subseteq I\}$. Then $A$ is a right ideal of $R$ and $AR \subseteq I$. Hence by 1.7, $A \subseteq I$ and $a \in I$.

1.9. **Corollary.** *Let $R$ be a ring and $S$ be an ideal of $R$. If $m$ is a maximal regular right ideal of $S$ then $m$ is a right ideal of $R$.*

**Proof.** Let $a$ be an element of $m$ and let $r$ be an element of $R$. Then $(ar)S = a(rS) \subseteq m$. Hence by 1.8, $ar \in m$.

1.10. **Proposition.** *Let $S$ be an ideal of a ring $R$. Let $m$ be a maximal regular right ideal of $S$ and let $f$ be a left identity modulo $m$ in $S$. Define $I(m, f) = \{r \in R \mid fr \in m\}$. Then $I(m, f)$ is a maximal regular right ideal of $R$ such that $I(m, f) \cap S = m$. If $e$ is another left identity modulo $m$ in $S$, then $I(m, e) = I(m, f)$.*

**Proof.** By 1.9, $m$ is a right ideal of $R$. Hence $I(m, f)$ is a right ideal of $R$. Since $f(fr - r) = f(fr) - fr \in m$ for every $r \in R$, $f$ is a left identity modulo $I(m, f)$. Clearly $m \subseteq I(m, f)$. Since $fs - s \in m$ for every $s \in S$, $m = I(m, f) \cap S$. If $I(m, f)$ were not maximal, then there would exist a maximal right ideal, say $M$ of $R$, such that $I(m, f) \subseteq M$ but $I(m, f) \neq M$. Let $a \in M$ such that $a \notin I(m, f)$. Then $fa \notin m$ and $fa \in M$. Hence $m \subseteq M \cap S$ but $m \neq M \cap S$. Since $M \cap S$ is a right ideal of $S$ and $m$ is maximal, $M \cap S = S$ and $f \in M$. Thus $R = M$. This is a contradiction. In order to prove that $I(m, f) = I(m, e)$, we observe that $fs - s \in m$ and $s - es \in m$ for every $s \in S$. This implies that $fs - es = (f - e)s \in m$ for every $s \in S$. Hence, by 1.8, $f - e \in m$ and $fr \in m$ if and only if $er \in m$ for every $r \in R$.

1.11. **Proposition.** *Let $R$ be a ring and $S$ be an ideal of $R$. If $I_1$, $I_2$ are maximal regular right ideals of $R$ such that $S \not\subseteq I_i$, $i = 1, 2$, then $S \cap I_i$ is a maximal regular right ideal of $S$ for each $i$, $i = 1, 2$, and if $S \cap I_1 = S \cap I_2$, then $I_1 = I_2$.*

**Proof.** Let $e_i$ be a left identity modulo $I_i$ for each $i$ where $i = 1, 2$. Since $S \not\subseteq I_i$, $S + I_i = R$ and $e_i = s_i + a_i$ for some $s_i \in S$ and $a_i \in I_i$. Then $s_i$ is a left identity modulo $I_i \cap S$ in $S$. Let $m$ be a maximal regular right ideal of $S$ such that $I_i \cap S \subseteq m$. Clearly, $s_i$ is a left identity modulo $m$. Hence by 1.10, $m = I(m, s_i) \cap S$ and $I_i \cap S \subseteq I(m, s_i) \cap S$. Since $S$ is a two-sided ideal of $R$, $I_i S \subseteq I_i \cap S \subseteq I(m, s_i)$. Since $S \not\subseteq I(m, s_i)$, $I_i \subseteq I(m, s_i)$ by 1.7 and hence $I_i = I(m, s_i)$. Thus $I_i \cap S = m$ and if $S \cap I_1 = S \cap I_2$ then, by 1.10, $I_1 = I_2$.

1.12. **Theorem.** *Let $R$ be a ring and $S$ be an ideal of $R$. Then* $\operatorname{supp}(S)$ *is homeomorphic to $X(S)$.*

**Proof.** Define a function $\phi$ from $\operatorname{supp}(S)$ into $X(S)$ by $\phi(I) = I \cap S$. Then $\phi$ is a bijection by 1.10 and 1.11. Let $A$ be a right ideal of $S$. Then $\phi^{-1}(\operatorname{supp}(A)) = \{I \in \operatorname{supp}(S) \mid A \not\subseteq I \cap S\}$. But this set is equal to $\operatorname{supp}(AR)$ in $X(R)$ which is contained in $\operatorname{supp}(S)$. Thus $\phi$ is a continuous mapping. Since $S$ is a two-sided ideal, if $B$ is a right ideal of $R$ then $\operatorname{supp}(S) \cap \operatorname{supp}(B) = \operatorname{supp}(BS)$ (refer 1.7) and $\phi(\operatorname{supp}(BS)) = \{I \cap S \mid I \in X(R) \text{ and } BS \not\subseteq I\}$. This is clearly an open set in $X(S)$ and $\phi$ is an open mapping.

1.13. **Proposition.** *If $R$ is a primitive ring, then there is a primitive ring $R^*$ with 1 such that $R$ is an ideal of $R^*$ and $X(R)$ is homeomorphic to an open subset of $X(R^*)$.*

**Proof.** Let $E = \mathbf{Z} \times R$, the usual extension ring of $R$, Let $I$ be a maximal right ideal of $E$ such that $R \not\subseteq I$ and $R/(I \cap R)$ is a faithful simple $R$-module. If $E/I$ were not faithful, let $S$ be the largest ideal of $E$ which is contained in $I$. Then $S \cap R = 0$ since $R/(I \cap R)$ is faithful. Let $R^* = E/S$. Then $R^*$ is a primitive ring with 1 and $R$ is isomorphic to an ideal of $R^*$. Hence by 1.12, $X(R)$ is homeomorphic to an open subset of $X(R^*)$.

1.14. **Definition.** A ring $R$ is called *(von Neumann) regular* if and only if for every element $a$ in $R$ there is $b$ in $R$ such that $aba = a$ (refer [6]).

1.15. **Remark.** If $R$ is (von Neumann) regular, then $a \in aR$ for every $a \in R$ even though $R$ may not contain a unit element and $aR = eR$ for some idempotent $e \in R$. von Neumann proved that if $R$ is a regular ring with 1, then for any idempotent $e \in R$ and an element $b \in R$, $eR + (1-e)bR = gR$ for some idempotent $g \in R$ (refer [6]). Since $eR + (1-e)bR = eR + bR$, this means that any finitely generated right ideal of $R$ is principal. With a slight modification of von Neumann's proof for the above assertion, one can also conclude that if $R$ is a regular ring (not necessarily with 1) then for any idempotent $e \in R$ and an element $b \in R$, $eR + bR = eR + \{br - ebr \mid r \in R\} = gR$ for some idempotent $g$ in $R$.

1.16. **Theorem.** *If $R$ is a regular ring, then $X(R)$ is compact if and only if $1 \in R$.*

**Proof.** If $1 \in R$ then $X(R)$ is compact for every ring $R$ by [3, 1.7]. Assume that $R$ is a regular ring and $X(R)$ is a compact space. Let $E$ be the set of idempotents in $R$. Then $X(R) = \bigcup_{e_\alpha \in E} \operatorname{supp}(e_\alpha R)$. Since $X(R)$ is compact, there exist $e_{\alpha_1}, e_{\alpha_2}, \cdots, e_{\alpha_n}$, a finite number of idempotents in $R$, such that $X(R) = \bigcup_{i=1}^n \operatorname{supp}(e_{\alpha_i} R)$. Hence $\emptyset = \bigcap_{i=1}^n h(e_{\alpha_i} R) = h(\Sigma_{i=1}^n e_{\alpha_i} R)$. Let $e$ be an idempotent such that $\Sigma_{i=1}^n e_{\alpha_i} R = eR$. If $eR \neq R$, then the right ideal $\{r - er \mid r \in R\}$ is not zero. Let $b \in R$ be such that $b - eb \neq 0$. Then by 1.15, there is an idempotent $g \in R$ such that $gR = eR + (b - eb)R$. Hence $ge = e$ and $g - e \neq 0$. Since $R$ is a semisimple ring, there is a simple right $R$-module $M$ such that $M(g - e) \neq 0$. Let $m \in M(g - e)$ such that $m \neq 0$. Then $meR = 0$ and $eR \subseteq m^\perp$, which is maximal regular right ideal. This means that $h(eR) \neq \emptyset$, which is a contradiction. Thus $eR = R$ and $ex = x$ for every $x \in R$. Since $(x - xe)R = 0$ for every $x \in R$, $e = 1$.

1.17. **Theorem.** *If $R$ is a primitive ring, then $X(R)$ is a compact Hausdorff space if and only if $R$ is either a division ring or a finite ring.*

**Proof.** If $R$ is either a division ring or a finite ring, then certainly $X(R)$ is a compact Hausdorff space. Suppose that $X(R)$ is a compact Hausdorff space. By [3, 2.5], if $S$ is the socle of $R$, then $R/S$ is a radical ring. Since $X(S)$ is homeomorphic to $\operatorname{supp}(S)$ and $X(R) = \operatorname{supp}(S)$, $X(S)$ is a Hausdorff compact space. Since $S$ is a regular ring, by 1.16, $1 \in S$ and $R = S$. Thus by [3, 2.7] either $R$ is a division ring or a finite ring.

2. **Irreducible closed sets.** If $R$ is a dense ring of linear transformations of a vector space, then an element $a \in R$ generates a minimal right ideal if and only if the rank of $a$ is 1 (refer [1, p. 76]). It is interesting to note that the following assertion is also true:

2.1. **Proposition.** *If $R$ is a dense ring of linear transformations of a left vector space $V$ over a division ring $D$ and $A$ is a nonzero right ideal of $R$ such that every nonzero element of $A$ is of rank 1, then $A$ is a minimal right ideal.*

**Proof.** Let $a$, $b$ be two nonzero elements of $A$. Then $V = \ker a \oplus Du = \ker b \oplus Dw$ for some nonzero vectors $u$ and $w$ since the ranks of $a$ and $b$ are one. If either $\ker a \subseteq \ker b$ or $\ker b \subseteq \ker a$ then $\ker a = \ker b$. Hence $ua \neq 0$, $ub \neq 0$ and $uar = ub$ for some $r \in R$. Thus $V(ar - b) = 0$ and $aR = bR$ and, therefore, $A$ is a minimal right ideal. So suppose now that there exist $v_1 \in (\ker a)\setminus(\ker b)$, $v_2 \in (\ker b)\setminus(\ker a)$. There is $r \in R$ such that $v_2 ar$ and $v_1 b$ are linearly independent if the dimension of the space is greater than 1. Since $a$, $b$ are elements of $A$, $ar + b \in A$. However, the rank of $ar + b$ is greater than or equal to 2 since $v_1(ar + b) = v_1 b$ and $v_2(ar + b) = v_2 ar$. This is a contradiction.

2.2. **Proposition.** *Let $R$ be a dense ring of linear transformations of a vector space $V$ over a division ring $D$ and let $A$ be a nonzero right ideal of $R$. Then $h(a) = h(A)$ for every nonzero $a \in A$ if and only if $A$ is a minimal right ideal.*

**Proof.** Since $h(a) = h(aR)$ for every $a \in R$, if $A$ is a minimal right ideal, then certainly $h(a) = h(A)$. Assume now that $h(a) = h(A)$ for every nonzero $a \in A$. In order to prove that $A$ is minimal, it suffices to show that the rank of nonzero $a \in A$ is 1. If the rank of $a$ is not 1, then there exist vectors $v_1, v_2$ in $V$ such that $v_1a$, $v_2a$ are linearly independent. Hence there is $r \in R$ such that $v_1ar = 0$ and $v_2ar \neq 0$. By hypothesis, $h(A) = h(a) = h(ar)$. Since $v_1^{\perp} \in X(R)$ and $v_1^{\perp} \in h(ar) = h(a)$, $v_1a = 0$. This is a contradiction.

2.3. **Definition.** If $R$ is a ring and $A$ is a right ideal of $R$, the radical of $A$ is $\sqrt{A} = \bigcap\{I \mid I \in X(R) \text{ and } I \supseteq A\}$.

2.4. **Proposition.** *Let $R$ be a dense ring of linear transformations of a left vector space $V$ over a division ring $D$. If $A$ is a minimal right ideal of $R$, then $\sqrt{A} = A$.*

**Proof.** Suppose $A \neq \sqrt{A}$. Since $A \subseteq \sqrt{A}$ always, if $A \neq \sqrt{A}$ then there is $b \in \sqrt{A}$ such that $b \notin A$. Since $R$ is semisimple, $X(R) \neq h(b)$ and $h(b) \supseteq h(\sqrt{A}) = h(A)$. Let $a$ be a nonzero element of $A$. Then $h(A) = h(a)$ since $A = aR$. Hence $h(b) \supseteq h(a)$. Let $\ker a = \{v \in V \mid va = 0\}$ and $\ker b = \{v \in V \mid vb = 0\}$. Then $\ker a \subseteq \ker b$ since $h(b) \supseteq h(a)$. Since the rank of $a$ is 1, $V = \ker a \oplus Dw$ for some nonzero vector $w$ in $V$. Since $\ker a \subseteq \ker b$ and $b \neq 0$, $wb \neq 0$. Let $r \in R$ such that $war = wb$. Then $V(ar - b) = 0$ and $ar = b$. Hence $b \in A$ and this is a contradiction.

2.5. **Definition.** Let $R$ be a ring. A closed subset $F$ of $X(R)$ is said to be a hyperplane of $X(R)$ provided that $j(F) \neq J(R)$, the Jacobson radical, and if $F'$ is a closed subset of $X(R)$ such that $F' \supseteq F$ and $F' \neq F$ then $j(F') = J(R)$.

2.6. **Theorem.** *Let $R$ be a primitive ring. If $A$ is a minimal right ideal of $R$, then $h(A)$ is a hyperplane. If $F$ is a hyperplance of $X(R)$ then $j(F)$ is a minimal right ideal of $R$.*

**Proof.** Let $A$ be a minimal right ideal of $R$ and let $F = h(A)$. Let $F'$ be a closed subset of $X(R)$ such that $F' \supseteq F$ and $j(F') \neq 0$. Let $B = j(F')$. Then $0 \neq B = j(F') \subseteq j(F) = \sqrt{A} = A$. By minimality of $A$, $B = A$, so $F' \subseteq h(A) = F$ and $F = F'$. Conversely, assume that $F$ is a hyperplane. Let $A = j(F)$. If $a$ is a nonzero element of $A$, then $j(h(a)) \neq 0$ and $h(a) \supseteq h(A) \supseteq F$. It follows that $h(a) = h(A) = F$. Hence by 2.2, $A$ is a minimal right ideal of $R$.

2.7. **Corollary.** *Let $R$ be a primitive ring. Let $\Sigma_1$ be the set of minimal right ideals of $R$. Let $\mathfrak{F} = \{F \mid F$ is a closed subset of $X(R)$ such that $j(F) \neq 0\}$.*

*Let $\Sigma_2$ be the set of maximal elements of $\mathfrak{F}$. Then there is a bijection from $\Sigma_1$ onto $\Sigma_2$.*

**Proof.** The mapping $A \to h(A)$ for $A \in \Sigma_1$ is a bijection from $\Sigma_1$ onto $\Sigma_2$.

2.8. **Theorem.** *If $R$ is a primitive ring such that $X(R)$ is irreducible and $A$ is a minimal right ideal of $R$, then either $h(A) = \emptyset$ or $h(A)$ is a maximal proper irreducible closed subset of $X(R)$.*

**Proof.** If $A$ is a minimal right ideal of a primitive ring $R$ then there is an idempotent $e \in R$ such that $A = eR$, and if $h(A) \neq \emptyset$ then $e^l = \{r \in R \mid re = 0\}$ is a nonzero left ideal of $R$. Hence $V_0 = e^l \cap A \neq 0$ and $V_0$ is a subspace of the vector space $eR$ over the division ring $eRe$. Since $R$ is a dense ring of linear transformations of the space $eR$ and $X(R)$ is irreducible, by [3, 2.4] $eRe$ is an infinite field. Since $A$ is minimal, by 2.2 $h(a) = h(A)$ if $a$ is a nonzero element of $A$. Let $\ker a = \{x \in A \mid xa = 0\}$. Then $V_0 = \ker a$. Since the rank of $a$ is 1, there is a vector $u$ such that $eR = V_0 \oplus eReu$. Hence $V_0^{\perp} \cap u^{\perp} = 0$ and $V_0^{\perp}$ is a minimal right ideal of $R$ since $u^{\perp}$ is a maximal right ideal of $R$. Since $V_0^{\perp} \supseteq aR$ and $V_0^{\perp}$ is a minimal right ideal, $V_0^{\perp} = aR$. Now suppose that $h(A)$ is reducible. Then by [3, 1.11] there exist right ideals $A_1, A_2, \cdots, A_n$ such that $h(A) \neq h(A_i)$ for every $i$ and $h(A) = \bigcup_{i=1}^n h(A_i)$. Let $W_i = \{v \in eR \mid vA_i = 0\}$ for every $i$. Then $W_i$ is a subspace of $V_0$ and $V_0 = \bigcup_{i=1}^n W_i$. If $V_0 = W_i$ for some $i$, then $V_0^{\perp} = W_i^{\perp} = aR$, and $h(A) = h(aR) = h(W_i^{\perp})$. Since $A_i \subseteq W_i^{\perp}$, $h(A_i) \supseteq h(W_i^{\perp}) = h(A)$. This is a contradiction.

2.9. **Lemma.** *Let $R$ be a simple ring with nonzero socle. If $X(R)$ is a discrete space then either $R$ is a division ring or $R$ is a finite ring.*

**Proof.** Suppose that $R$ is neither a division ring nor a finite ring. By [3, 2.6], $R$ is isomorphic to a dense ring of linear transformations of finite rank of a vector space $V$ over a finite field $D$ and hence $1 \notin R$. Let $p$ be the characteristic of $D$ then $pr = 0$ for every $r \in R$. Let $\mathbf{Z}/(p)$ be the field of integers modulo $p$. Then $R$ is an algebra over $\mathbf{Z}/(p)$. Let $R^* = \mathbf{Z}/(p) \times R$ be the usual extension ring of $R$ with 1 in which $R$ is an ideal. Let $m$ be a maximal regular right ideal of $R$ and let $e$ be a left identity modulo $m$ in $R$. Then $I(m, e)$ is a maximal right ideal of $R^*$ by 1.10. Let $S$ be an ideal of $R^*$ such that $S \subseteq I(m, e)$. If $S$ were a nonzero ideal then $S \cap R = 0$ since $m$ contains no nonzero ideal of $R$. Hence $SR = 0$. Let $(f, s_0)$ be a nonzero element of $S$ for some $f \in \mathbf{Z}/(p)$ and $s_0 \in R$. Then $f \neq 0$ and $(f, s_0)(0, r) = (0, 0)$ for every $r \in R$. Hence $-(f^{-1}s_0)r = r$ for every $r \in R$. Since $R$ is a simple ring, $-(f^{-1}s_0) = 1$ and $R$ must be a finite ring. Therefore, $S = 0$ and $R^*/I(m, e)$ is a faithful simple $R^*$-module. Thus $R^*$ is a primitive ring with 1. Observe $h(R) = \{R\}$ in $X(R^*)$ and

$X(R^*) = \operatorname{supp}(R) \cup \{R\}$. Since $X(R)$ is homeomorphic to $\operatorname{supp}(R)$ in $X(R^*)$ by 1.12 and $X(R)$ is a discrete space, every point of $\operatorname{supp}(R)$ is open in $X(R^*)$. Since the point $R$ in $X(R^*)$ can be separated from any point of $\operatorname{supp}(R)$ by open sets, this means that $X(R^*)$ is a Hausdorff space. Since $R^*$ is a primitive ring with 1, by [3, 2.7] $R^*$ must be a finite ring and therefore $R$ must be also a finite ring. This is a contradiction.

2.10. **Theorem.** *Let $R$ be a primitive ring. Then $X(R)$ is a discrete space if and only if either $R$ is a division ring or a finite ring.*

**Proof.** Since $X(R)$ is a $T_1$-space, if $R$ is a finite ring, then certainly $X(R)$ is a discrete space. Suppose $X(R)$ is a discrete space such that $R$ is not a division ring. Then $X(R)$ is a Hausdorff space and, by [3, 2.5], $R$ is a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field and if $S$ is the socle of $R$ then $R/S$ is a radical ring. If $S$ is a finite ring, then $R$ must be also a finite ring since $X(R) = \operatorname{supp}(S) \cup h(S) = \operatorname{supp}(S)$ and since a primitive ring which is not a division ring with a finite number of maximal regular right ideals is a finite ring. Now observe that $S$ satisfies the hypothesis of 2.9. Hence if $X(R)$ is a discrete space, then so is $X(S)$ and, by 2.9, $S$ is a finite ring.

2.11. **Example.** Let $\mathbf{Z}$ be the ring of integers and let $R$ be the ring of all row-finite infinite matrices over $\mathbf{Z}/(2)$ of the following form:

$$\begin{pmatrix} A_n & & & & \\ & 0 & & * & \\ & & 0 & & \\ & & & 0 & \\ & 0 & & & \ddots \end{pmatrix}$$

where $A_n$ is an $n \times n$ matrix for some positive integer $n$. Then $R$ is a primitive ring with nonzero socle and if $S$ is the socle of $R$ then $R/S$ is a radical ring. Hence $X(R)$ is a Hausdorff space by [3, 2.5] and it is not a discrete space by 2.10.

3. **Rings in which every proper right ideal is incompressible.** Let $I$, $J$ be two distinct maximal regular right ideals of a ring $R$. If $K$, $M$ are distinct maximal regular right ideals of $R$ such that $\{K, M\} \subseteq h(I \cap J)$, then $(K \cap M)/(I \cap J)$ is a submodule of the right $R$-module $R/(I \cap J)$ which is isomorphic to a direct sum of two simple modules $I/(I \cap J)$ and $J/(I \cap J)$. Hence if $(K \cap M)/(I \cap J)$ is not the zero submodule then either $(K \cap M)/(I \cap J) = R/(I \cap J)$ or $(K \cap M)/(I \cap J)$ is isomorphic to $I/(I \cap J)$ or to $J/(I \cap J)$. In any case, this

means that either $R/(K \cap M)$ is simple or $R/(K \cap M) = 0$. This, of course, is absurd. Thus $K \cap M = I \cap J$ and $h(K \cap M) = h(I \cap J)$.

3.1. **Definition.** If $x, y$ are two distinct points in $X(R)$, a *line* containing $x$ and $y$ is $l(x, y) = h(j(x) \cap j(y))$.

3.2. **Proposition.** *Given two distinct points in $X(R)$, there is one and only one line containing these two points.*

**Proof.** Let $x, y$ be two distinct points in $X(R)$. If $z, w$ are distinct points in $X(R)$ such that $\{z, w\} \subseteq l(x, y)$, then $l(z, w) = l(x, y)$ since $j(z) \cap j(w) = j(x) \cap j(y)$.

3.3. **Proposition.** *If $x, y$ are two distinct points in $X(R)$ such that $R/j(x) \cong R/j(y)$ as $R$-modules, then for every $w \in l(x, y)$, $R/j(w) \cong R/j(x)$.*

**Proof.** If $w \in l(x, y)$, then $j(w)/(j(x) \cap j(y))$ is a nonzero submodule of $R/(j(x) \cap j(y))$. Hence $j(w)/(j(x) \cap j(y))$ is either isomorphic to $R/j(x)$ or to $R/j(y)$.

3.4. **Lemma.** *Let $R$ be a ring and $M$ be a simple $R$-module. Let $D = \mathrm{Hom}_R(M, M)$. If $m_1, m_2$ are nonzero elements of $M$ such that $m_1^\perp \neq m_2^\perp$ then $h(j(m_1^\perp) \cap j(m_2^\perp))$ contains at least three elements.*

**Proof.** Since $M$ is a left vector space over a division ring $D$, if $m$ is a nonzero element of $Dm_1 \oplus Dm_2$ then $m^\perp \in h(j(m_1^\perp) \cap j(m_2^\perp))$, and if $m = m_1 + m_2$ then $m^\perp \neq m_1^\perp$ and $m^\perp \neq m_2^\perp$.

3.5. **Theorem.** *Let $R$ be a ring. Then there is exactly one isomorphic class of simple $R$-modules if and only if $X(R)$ has no line which contains exactly two points.*

**Proof.** Assume that $R$ has a unique isomorphic class of simple modules. If there is a two-point line, say $l(x, y)$, for some two distinct points $x$ and $y$ in $X(R)$, then $h(j(x) \cap j(y)) = \{x, y\}$. Let $M = R/j(x)$. Then $x = m_1^\perp$ and $y = m_2^\perp$ for some $m_1, m_2$ in $M$ and $M^\perp = J(R)$ since all simple $R$-modules are isomorphic. By 3.4, $l(x, y)$ contains at least 3 points. This is a contradiction. Conversely, suppose that $R/I_1 \cong M_1 \ncong M_2 \cong R/I_2$ are simple modules where $I_1, I_2 \in X(R)$. If there exists maximal regular right ideal $K$ such that $K \supseteq I_1 \cap I_2$ and $I_j \neq K$ for $j = 1, 2$, then by the remark at the beginning of §3, $R/I_1 = (K + I_1)/I_1 \cong K/(I_1 \cap K) = K/(I_2 \cap K) \cong (K + I_2)/I_2 = R/I_2$, a contradiction. Thus the line containing $I_1$ and $I_2$ contains exactly two points.

3.6. **Proposition.** *If $R$ is a ring with 1 and if there is exactly one isomorphic class of simple $R$-modules then $R/J(R)$ is a simple ring. There is a primitive* *ring with nonzero socle in which every pair of simple modules are isomorphic, but the ring is not simple.*

**Proof.** If $R/J(R)$ is not a simple ring, then there is an ideal $S$ in $R$ such that $S \supseteq J(R)$ but $S \neq J(R)$ and $R \neq S$. Let $I$ be a maximal right ideal of $R$ such that $I \supseteq S$. Then $R/I$ is a simple $R$-module and $(R/I)^{\perp} \supseteq S$. Since every pair of simple $R$-modules are isomorphic, if $M$ is a simple $R$-module, then $M^{\perp} \supseteq S$. Hence $J(R) \supseteq S$. This is a contradiction. Now, let $R$ be the primitive ring in Example 2.11. If $S$ is the socle of $R$ then $R/S$ is a radical ring. Hence if $x \in X(R)$, $R/j(x)$ is isomorphic to a minimal right ideal of $R$. Thus every pair of simple $R$-modules are isomorphic. However, $R$ is not a simple ring.

3.7. **Theorem.** *Let $R$ be a primitive ring. Then $R$ is simple artinian if and only if* (i) *$X(R)$ is compact,* (ii) *$X(R)$ has no two-point line and* (iii) *$X(R)$ has a hyperplane.*

**Proof.** Assume (i), (ii) and (iii). Then $R$ has nonzero socle by 2.6. Let $S$ be the socle of $R$. Since $X(R)$ has no two-point line, by 3.5 all simple $R$-modules are isomorphic. Hence every simple module is isomorphic to a minimal right ideal. Therefore, $R/S$ is a radical ring. Thus by 1.12, $X(S)$ is compact and by 1.16, $1 \in S$. Since the unity of the ring $S$ is also the unity of $R$, $R = S$ and $R$ is a simple artinian ring. Conversely, if $R$ is a simple artinian ring then $1 \in R$. Hence $X(R)$ is compact by [3, 1.7]. Since every pair of $R$-modules in a simple artinian ring are isomorphic, by 3.5, (ii) is true. Finally (iii) is true by 2.6.

3.8. **Theorem.** *Let $R$ be a semisimple ring. Then the following two statements are equivalent:*

(i) *There exist at least two points in $X(R)$ and every line contains an infinite number of points.*

(ii) *$R$ is a dense ring of linear transformations of a vector space of dimension two or more over an infinite division ring such that every pair of simple $R$-modules are isomorphic.*

**Proof.** Assume (i). Then by 3.5 there is exactly one isomorphic class of simple $R$-modules. Hence if $M_1$, $M_2$ are two simple $R$-modules then $M_1^{\perp} = M_2^{\perp}$. Thus $M_1^{\perp} = J(R) = 0$. Therefore, every simple $R$-module is faithful and, in particular, $R$ is a primitive ring. Let $M$ be a simple $R$-module and let $D$ be the endomorphism ring of $R$-module $M$. Since $X(R)$ contains at least two points, the dimension of the vector space $M$ over $D$ is at least two. Let $m_1$, $m_2$ be two linearly independent vectors in $M$. Then $m_1^{\perp} \neq m_2^{\perp}$ and the line $l(m_1^{\perp}, m_2^{\perp})$ contains an infinite number of points by hypothesis. If $x \in l(m_1^{\perp}, m_2^{\perp})$ then $x = m^{\perp}$ for some $m \in M$ since $M \cong R/(j(x))$ and hence $m \in Dm_1 \oplus Dm_2$ by [1, Lemma, p. 27].

Conversely, if $m$ is a nonzero element of $Dm_1 \oplus Dm_2$, then $m^\perp \in l(m_1^\perp, m_2^\perp)$. Thus $D$ must be an infinite division ring. Assume, now, (ii). Let $R$ be a dense ring of linear transformations of a vector space $M$ over an infinite division ring $D$ of dimension two or more, then $X(R)$ contains at least two points. Let $x$ and $y$ be two distinct points in $X(R)$. Then $x = m_1^\perp$ and $y = m_2^\perp$ for some $m_1, m_2$ in $M$ since $M \cong R/j(x) \cong R/j(y)$. Hence the line $l(x, y) = h(m_1^\perp \cap m_2^\perp)$. Let $W$ be the two-dimensional subspace of $M$, which is spanned by $m_1$ and $m_2$. Observe that $z \in l(x, y)$ if and only if $z = w^\perp$ for some $0 \neq w \in W$. Since $D$ is infinite and $(m_1 + \alpha m_2)^\perp \neq (m_1 + \beta m_2)^\perp$ for $\alpha \neq \beta$ in $D$, $W$ contains infinitely many vectors. Thus $l(x, y)$ contains an infinite number of points.

3.9. **Theorem.** *Let $R$ be an arbitrary ring. Then every line in $X(R)$ contains exactly two points if and only if every maximal regular right ideal is also a left ideal.*

**Proof.** Assume that every line in $X(R)$ contains exactly two points. Let $I$ be a maximal regular right ideal of $R$. Let $N(I) = \{r \in R \mid rI \subseteq I\}$. Then by [1, Theorem 1, p. 25], $N(I)/I = \operatorname{Hom}_R(R/I, R/I)$ and $R/I$ is a left vector space over the division ring $N(I)/I$. If the dimension of the vector space $R/I$ over $N(I)/I$ is greater than one, then by 3.4 there will be a line which contains more than two points. Hence $R/I$ is one dimensional. Let $e$ be a left identity modulo $I$. Then for every $a \in R$ there is $b \in N(I)$ such that $be - a \in I$. Since $be \in N(I)$, $a \in N(I)$. Thus $I$ is a left ideal. Conversely, if every maximal regular right ideal of $R$ is also a left ideal, then $IJ \subseteq I \cap J$ for every pair of maximal regular right ideals $I$ and $J$. Hence if $K \in h(I \cap J)$ then $IJ \subseteq K$ and either $I = K$ or $J = K$ by 1.7. Thus $h(I \cap J) = \{I, J\}$.

3.10. **Proposition.** *Let $R$ be an arbitrary ring. Assume that every proper right ideal of $R$ is an intersection of maximal regular right ideals. Then for every right ideal $A$, $AR = A$.*

**Proof.** If $R^2 \neq R$, then by hypothesis, $R^2$ must be an intersection of maximal regular right ideals. This is of course impossible in view of 1.7. Therefore, $R^2 = R$. If $A$ is a proper right ideal of $R$ then $AR = \sqrt{AR}$ and $A = \sqrt{A}$. If there is a maximal regular right ideal $I$, such that $AR \subseteq I$, then by 1.7, $A \subseteq I$ so $\sqrt{AR} = \sqrt{A}$. Hence $AR = A$.

3.11. **Definition.** Let $R$ be an arbitrary ring. An $R$-module $M$ is called *injective* if and only if for every pair $A$, $B$ of $R$-modules with $A \subseteq B$, each homomorphism of $A$ into $M$ can be extended to one of $B$ into $M$. If $R$ is a ring with 1 and if $M$ is a unital $R$-module, then it is well known that $M$ has a minimal injective extension which is also a maximal essential extension of $M$. Such minimal injective extension is unique up to an isomorphism. It has been shown by

R. E. Johnson [2, 7.1] that these results are also true for any ring $R$ and any $R$-module $M$. For each $R$-module $M$, we denote by $\hat{M}$ the minimal injective extension of $M$.

3.12. **Proposition.** *Let $R$ be a ring. If every simple $R$-module is injective, then for any simple module $M$ the $R$-homorphism $\phi$: $M \longrightarrow \operatorname{Hom}_R(R, M)$ where $\phi(m)(r) = mr$ for every $m \in M$ and $r \in R$ is an isomorphism.*

**Proof.** Clearly, $\phi$ is a monomorphism. Let $f \in \operatorname{Hom}_R(R, M)$. Let $R^{\#}$ be an extension ring of $R$ with 1. Then there is $\bar{f} \in \operatorname{Hom}_R(R^{\#}, M)$ such that $\bar{f}\,|_R = f$ since $M$ is injective. Let $\bar{f}(1) = m_0$. Then $f(r) = m_0 r$ for every $r \in R$. Therefore, $f = \phi(m_0)$ and $\phi$ is an isomorphism.

3.13. **Proposition.** *Let $R$ be a ring such that $R^2 = R$. If every simple $R$-module is injective then every maximal right ideal of $R$ is regular.*

**Proof.** Let $I$ be a maximal right ideal of $R$. Since $R^2 = R$, $R/I$ is a simple $R$-module. Hence, by hypothesis, $R/I$ is injective and, by 3.12, $\phi(R/I) = \operatorname{Hom}_R(R, R/I)$ where $\phi(r + I)(r') = rr' + I$ for $r, r' \in R$. Define $g$: $R \longrightarrow R/I$ such that $g(r) = r + I$ for each $r \in R$. Then $g \in \operatorname{Hom}_R(R, R/I)$ and $g = \phi(a + I)$ for some $a \in R$. Thus $g(r) = ar + I$ for every $r \in R$ and hence $ar - r \in I$ for every $r \in R$.

3.14. **Proposition.** *Let $R$ be a ring such that every maximal right ideal of $R$ is regular. If $M$ is a simple $R$-module such that $\hat{M}R = M$, then $\hat{M} = M$.*

**Proof.** Consider $\phi$: $\hat{M} \longrightarrow \operatorname{Hom}_R(R, \hat{M}R)$ as in 3.12. If the ker $\phi$ were not zero then $\phi(M) = 0$ and $MR = 0$ since $\hat{M}$ is an essential extension of the simple module $M$. $\phi$ is a monomorphism. Let $f \in \operatorname{Hom}_R(R, \hat{M}R)$. Since $\hat{M}R \subseteq \hat{M}$, there is $\bar{f} \in \operatorname{Hom}_R(R^{\#}, \hat{M})$ such that $\bar{f}(r) = f(r)$ for every $r \in R$, where $R^{\#}$ is an extension ring of $R$ with 1. Let $\bar{f}(1) = \hat{m}$ where $\hat{m} \in \hat{M}$. Then $f = \phi(\hat{m})$ and $\phi(\hat{M}) = \operatorname{Hom}_R(r, \hat{M}R)$. On the other hand, if $f \neq 0$ then $R/\ker f \cong \hat{M}R = M$. Hence $\ker f$ is a maximal right ideal of $R$ and therefore, by hypothesis, it is regular. Let $e$ be a left identity modulo $\ker f$. Then $f(er) = f(r) = f(e)r$ for every $r \in R$ and $f = \phi(f(e))$. Hence $\phi(M) = \operatorname{Hom}_R(R, \hat{M}R)$. Thus $\phi(M) = \phi(\hat{M})$ and $M = \hat{M}$.

3.15. **Remark.** Villamayor [5] proved that if $R$ is a ring with 1 then every simple $R$-module is injective if and only if every proper right ideal of $R$ is the intersection of maximal right ideals.

3.16. **Theorem.** *Let $R$ be an arbitrary ring. Then the following statements are equivalent*:

(i) *Every proper right ideal of $R$ is an intersection of maximal regular right ideals.*

(ii) *If* $A$ *is a right ideal of* $R$ *then* $AR = A$ *and every simple R-module is injective.*

**Proof.** Assume (i). Then by 3.10, $AR = A$ for every right ideal $A$ of $R$. Let $M$ be a simple $R$-module and let $0 \neq \hat{m} \in \hat{M}$. Then $(\hat{M})^{\perp} \neq R$. For if $(\hat{m})^{\perp} = R$ then the submodule $N = \{\hat{m} \in \hat{M} \mid \hat{m}R = 0\}$ must contain $M$ since $\hat{M}$ is an essential extension of $M$ and $M$ is simple. Hence $MR = 0$. This is, of course, impossible. Thus $(\hat{m})^{\perp} \neq R$ and $(\hat{m})^{\perp} = \bigcap_{\alpha \in \Lambda} \{I_\alpha \mid I_\alpha$ is a maximal regular right ideal$\}$ for some index set $\Lambda$. Let $M_\alpha = R/I_\alpha$. Let $p_\alpha$ be the $\alpha$th projection of $\Pi_{\beta \in \Lambda} M_\beta \longrightarrow M_\alpha$ and let $\mu$: $\hat{m}R \longrightarrow \Pi_{\alpha \in \Lambda} M_\alpha$ be a map such that $p_\alpha \circ \mu(\hat{m}r) = r + I_\alpha$ for every $r \in R$. Then $\mu$ is a monomorphism. If there exists $\alpha \in \Lambda$ such that $p_\alpha \circ \mu$ is a monomorphism then $\hat{m}R$ is simple and $\hat{m}R = M$ since $M$ is the only simple submodule of $\hat{M}$. Now, if, for each $\alpha \in \Lambda$, $p_\alpha \circ \mu$ is not a monomorphism, then $p_\alpha \circ \mu(M) = 0$ for every $\alpha \in \Lambda$ and $\mu(M) = 0$. Since $\mu$ is a monomorphism, this means that $M = 0$. This is impossible. Thus $\hat{M}R = M$ and, by 3.14, $\hat{M} = M$ and (ii) holds. Now assume (ii). Let $A$ be a right ideal of $R$ such that $A \neq R$. Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be the family of all maximal right ideals of $R$ which contain $A$. Suppose there exists $b \in \bigcap_{\alpha \in \Lambda} I_\alpha$ such that $b \notin A$. Let $A_0$ be a right ideal of $R$ maximal with respect to the property that $A \subseteq A_0$ but $b \notin A_0$. Then the submodule of $R/A_0$ generated by $b + A_0$ is equal to $J/A_0$ for some right ideal $J$ which contains $A_0$ properly. Since $JR = J$, $J/A_0$ must be a simple $R$-module. Hence it is injective by hypothesis. Let $\phi$ be a homomorphism from $R/A_0$ onto $J/A_0$ such that $\phi|_{J/A_0} = 1_{J/A_0}$, the identity map on $J/A_0$. Such $\phi$ exists since $J/A_0$ is injective. Then $R/A_0 = J/A_0 \oplus \ker \phi$. If $\ker \phi \neq 0$ then $\phi(b + A_0) = 0$. This is impossible. Thus $R/A_0 = J/A_0$ and $A_0$ is a maximal right ideal of $R$. Hence $b \in A_0$. This is a contradiction. Thus, $A$ is an intersection of maximal right ideals. Since every maximal right ideal of $R$ is regular by 3.13, $A$ is an intersection of maximal regular right ideals.

3.17. **Definition.** A right ideal $C$ of a ring $R$ is said to be *incompressible* provided that if $\{A_1, A_2, \cdots, A_n\}$ is a finite set of right ideals such that $A_i \not\subseteq C$ for every $i$, then there is a maximal regular right ideal $I$ such that $C \subseteq I$ and $A_i \not\subseteq I$ for every $i$.

3.18. **Remark.** If $R$ is a commutative ring, then every prime ideal $P$ such that $R/P$ is semisimple is incompressible. For if $\{A_1, A_2, \cdots, A_n\}$ is a finite set of ideals such that $A_i \not\subseteq P$ for every $i$, then $A_1A_2 \cdots A_n \not\subseteq P$. Hence there is a maximal ideal $I$ such that $P \subseteq I$ but $A_1A_2 \cdots A_n \not\subseteq I$. Obviously in any ring every maximal regular right ideal is incompressible.

3.19. **Proposition.** *If* $C$ *is an incompressible right ideal of a ring* $R$, *then* $\sqrt{C} = C$.

**Proof.** If $\sqrt{C} \neq C$, then there exists a maximal regular right ideal $I$ such that $C \subseteq I$ but $\sqrt{C} \not\subseteq I$. This is a contradiction.

3.20. **Proposition.** *Let $R$ be a ring. Then a right ideal $C$ is incompressible if and only if $C = \sqrt{C}$ and $h(C)$ is an irreducible closed subset of $X(R)$.*

**Proof.** Assume that $C$ is incompressible. Then $C = \sqrt{C}$ by 3.19. If $h(C)$ were reducible, then there exist right ideals $A_1, A_2, \cdots, A_n$ such that $h(A_i) \neq h(C)$ and $h(C) = \bigcup_{i=1}^n h(A_i)$. Since $h(A_i) \subseteq h(C)$ and $h(A_i) \neq h(C)$, $A_i \not\subseteq C$ for every $i$. Hence there is $I \in X(R)$ such that $C \subseteq I$ and $A_i \not\subseteq I$ for every $i$. This means that $h(C) \not\subseteq \bigcup_{i=1}^n h(A_i)$, which is a contradiction. Conversely, assume that $C = \sqrt{C}$ and $h(C)$ is irreducible. Let $\{A_1, A_2, \cdots, A_n\}$ be a finite set of right ideals such that $A_i \not\subseteq C$ for every $i$. If there is no maximal regular right ideal $I$ such that $C \subseteq I$ and $A_i \not\subseteq I$ for every $i$, then $h(C) \subseteq \bigcup_{i=1}^n h(A_i)$. Since $h(C)$ is irreducible, $h(C) \subseteq h(A_i)$ for some $i$. Hence $C \supseteq \sqrt{A_i} \supseteq A_i$. This is impossible.

3.21. **Corollary.** *Let $R$ be a ring such that $R$ is not a radical ring. Then $X(R)$ is irreducible if and only if $J(R)$, the Jacobson radical, is incompressible.*

3.22. **Example.** Let $R$ be a primitive ring with a minimal right ideal $A$. If $A = eR$ for some idempotent $e \neq 1$ in $R$ such that $eRe$ is an infinite division ring then $eR$ is incompressible. Because, in this case, $X(R)$ is irreducible by [3, 2.4] and $h(eR)$ is irreducible by 2.8.

3.23. **Theorem.** *Let $R$ be a ring. Then every proper right ideal of $R$ is incompressible if and only if* (i) *$A = AR$ for every right ideal $A$ of $R$, and* (ii) *either $R$ is a division ring or $R$ is a dense ring of linear transformations of a left vector space $V$ of dimension two or more over an infinite division ring $D$ such that $V$ is injective as an $R$-module and every simple $R$-module is isomorphic to $V$.*

**Proof.** Assume that every proper right ideal of $R$ is incompressible. Then for every right ideal $A$ of $R$, $A = AR$ by 3.19 and 3.10. Suppose that $R$ is not a division ring. Let $x$ and $y$ be two distinct points in $X(R)$. Since $j(x) \cap j(y)$ is incompressible, $h(j(x) \cap j(y))$ is irreducible by 3.20. Hence $l(x, y)$ must contain an infinite number of points. Thus by 3.5, every pair of simple $R$-modules are isomorphic. Since zero ideal is incompressible, $R$ is semisimple and, since every pair of simple $R$-modules are isomorphic, every simple $R$-module is faithful. Hence $R$ must be a primitive ring and $X(R)$ is irreducible. Thus by [3, 2.4], $R$ must be a dense ring of linear transformations of a vector space of dimension two or more over an infinite division ring. If every proper right ideal of $R$ is incompressible, then every proper right ideal of $R$ is an intersection of maximal regular right ideals by 3.19. Thus by 3.16, every simple $R$-module is injective. Conversely, assume (i) and (ii). Then by 3.16, if $A$ is a right ideal such that $A \neq R$ then $\sqrt{A} = A$. Since every simple $R$-module is isomorphic to $V$, if $x \in X(R)$ then $x = v^{\perp}$ for some $0 \neq v \in V$. Hence there is subspace $W$ of $V$ such that $A = W^{\perp}$. If $h(A)$ were not irreducible then there exist right ideals $A_1, A_2, \cdots, A_n$ such that $h(A) = \bigcup_{i=1}^{n} h(A_i)$ and $h(A) \neq h(A_i)$ for every $i$. Let $W_i = \{w \in W \mid wA_i = 0\}$ for each $i$. Then $W_i$ is a subspace of $W$ and $W = \bigcup_{i=1}^{n} W_i$. Note that $W_i$ is a proper subspace of $W$ since $h(A_i) \neq h(A)$. Hence by [3, 2.1] the division ring $D$ must be a finite field. This is impossible. Thus $A$ is incompressible by 3.20.

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