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HEEGAARD SPLITTINGS OF HOMOLOGY 3-SPHERES [1]

BY

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ABSTRACT. We investigate properties of Heegaard splittings of closed 3-manifolds which are known for simply-connected manifolds and which might provide the basis for a general test for simple-connectivity. Our results are negative: each property considered is shown to hold in a wider class of manifolds.

1. **Introduction.** The problem of determining whether or not a given manifold is simply connected appears frequently in the study of 3-manifolds. It would be extremely useful to have a general method for deciding this question. We investigate several properties of Heegaard splittings which might provide the basis for such a simple-connectivity test. Our results are generally negative: properties which have been known for simply-connected manifolds actually hold in a wider class of manifolds.

The author wishes to thank the referee for many helpful suggestions.

2. **Definitions and notation.** In this paper all spaces and maps are assumed to be piecewise linear; any subspace which appears is assumed polyhedral.

If M is a manifold, we use $\mathring{M}$ to denote the interior of M and ∂M for the boundary of M. A submanifold $S \subset M$ is said to be *properly embedded* if $S \cap \partial M = \partial S$. A regular neighborhood of a subspace $P \subset M$ is denoted by $N(P)$. Here regular neighborhood is used in the sense of [9, Chapter 3].

We are concerned with connected orientable 3-manifolds. A closed 3-manifold with trivial fundamental group is called a *homotopy* 3-*sphere*. A *homology* 3-*sphere* is a closed 3-manifold with trivial first homology group.

A 3-manifold is a *handlebody of genus* n if it is homeomorphic to a regular neighborhood in E^3 of a finite connected graph of Euler characteristic $1-n$. If H is a handlebody of genus n, then there are disjoint discs $D_1, \cdots, D_n$, properly embedded in H, such that $H - \mathring{N}(\bigcup D_i)$ is a 3-cell. Any such system of discs is called a *cutting* for H. The boundaries $m_i = \partial D_i$ of the cutting discs are called *meridians* for H.

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A system $\{l_1, \cdots, l_n\}$ of disjoint simple closed curves on ∂H, whose union does not separate ∂H, is called *independent.* If $\{m_1, \cdots, m_n\}$ and $\{l_1, \cdots, l_n\}$ are independent systems on ∂H satisfying

(1) $l_i \cap m_i$ is a single (crossing) point, and

(2) $l_i \cap m_j = \emptyset$ if $i \neq j$,

we say that $\{m_1, \cdots, m_n\}$ and $\{l_1, \cdots, l_n\}$ are *transverse.* If the m_i are also meridians for H, the l_i are called *canonical longitudes* for H and $\{m_1, \cdots, m_n, l_1, \cdots, l_n\}$ is called a *meridian-longitude system* for H.

It is known that any closed (connected orientable) 3-manifold can be decomposed as the union of two handlebodies of the same genus, which intersect in their common boundary [8, §63]:

$$M = H \cup H', \qquad \partial H = H \cap H' = \partial H'.$$

Such a decomposition is called a *Heegaard splitting* of M. Alternately, we may view M as obtained from the two handlebodies by identifying their boundaries under a homeomorphism $\phi: \partial H' \to \partial H$. Since the resulting manifold is determined up to homeomorphism by the images $w_i = \phi(m_i')$ of a system $\{m_1', \cdots, m_n'\}$ of meridians for H', the handlebody H, with the *Heegaard curves* $w_1, \cdots, w_n$ on ∂H, completely describes M; we say $\{H; w_1, \cdots, w_n\}$ is a *Heegaard diagram* of M.

3. **Embedding Heegaard diagrams.** The following result, due essentially to Moise (see [2, Theorem 8] and the references cited there), does provide a necessary and sufficient condition for simple-connectivity:

Let $\{H; w_1, \cdots, w_n\}$ be a Heegaard diagram for the closed 3-manifold M. M is simply connected if and only if there is an embedding $\phi: H \to S^3$ with the $\phi(w_i)$ bounding disjoint surfaces in $S^3 - \phi(\mathring{H})$.

However, it is unlikely that this condition could ever be easily checked. If one is after a usable criterion it is natural to consider slightly weaker embedding properties. For example, Haken has described an algorithm which completely decides the following question [2, §2C]:

Given a Heegaard diagram $\{H; w_1, \cdots, w_n\}$ for a closed 3-manifold M, is there an embedding $\phi: H \to S^3$ with each $\phi(w_i)$ homologous to zero in $S^3 - \phi(\mathring{H})$?

A necessary and sufficient condition for the existence of such an embedding was found. Checking it in any particular case involves an easy computation in terms of the intersection numbers of the Heegaard curves with the curves of a meridian-longitude system for H. Because of Moise's theorem one would expect that this algorithm would provide at least a partial test for simple-connectivity. If $\Pi_1(M)$ were trivial then certainly such an embedding would exist; presumably, if

M were not simply-connected, it might not. It turns out, however, that *all* homology 3-spheres have this embedding property:

Theorem 1. *Let* $\{H; w_1, \cdots, w_n\}$ *be a Heegaard diagram for a closed 3-manifold* M. *Then there is an embedding* $\phi: H \to S^3$ *with each* $\phi(w_i)$ *homologous to zero in* $S^3 - \phi(\mathring{H})$ *if and only if* $H_1(M) = (0)$.

Proof. To prove that $H_1(M) = (0)$ implies the embedding property we use the "meridian-longitude theorem" of Papakyriakopoulos [7, Theorem 32.3]. This provides a cutting $\{D_1, \cdots, D_n\}$ for H and transverse longitudes $l_1, \cdots, l_n$ on ∂H, with each l_i homologous to zero in $M - \mathring{H}$.

We also need the following:

Lemma. *Suppose* M *is a homology 3-sphere,* $H \subset M$ *a genus* n *handlebody, and* $x_1, \cdots, x_n$ *are independent simple closed curves on* ∂H *each homologous to zero in* $M - \mathring{H}$. *Let* $\bar{x}_i \in H_1(\partial H)$ *denote the class of* x_i *(with some fixed orientation),* α *denote the inclusion* $\partial H \subset M - \mathring{H}$, *and* $[\bar{x}_1, \cdots, \bar{x}_n]$ *denote the subgroup of* $H_1(\partial H)$ *generated by* $\bar{x}_1, \cdots, \bar{x}_n$. *Then* $[\bar{x}_1, \cdots, \bar{x}_n] = \ker(\alpha_*)$.

By assumption $[\bar{x}_1, \cdots, \bar{x}_n] \subset \ker(\alpha_*)$; we prove the opposite inclusion. Choose disjoint simple closed curves $y_1, \cdots, y_n$ on ∂H such that $\{\bar{x}_1, \cdots, \bar{x}_n, \bar{y}_1, \cdots, \bar{y}_n\}$ is a basis for $H_1(\partial H)$. Since α_* is onto and $\alpha_*(\bar{x}_i) = 0$ for each i, $\alpha_*(\bar{y}_1), \cdots, \alpha_*(\bar{y}_n)$ generate $H_1(M - \mathring{H}) = Z \oplus \cdots \oplus Z$ (n summands). Any set of n generators for this group is a basis. Hence if $x = \sum a_i \bar{x}_i + \sum b_i \bar{y}_i$ is in $\ker(\alpha_*)$, so that $0 = \alpha_*(x) = \sum b_i \alpha_*(\bar{y}_i)$, then we must have each $b_i = 0$. Thus $\ker(\alpha_*) \subset [\bar{x}_1, \cdots, \bar{x}_n]$ and the lemma is proved.

Here we apply the lemma to the systems $\{w_1, \cdots, w_n\}$ and $\{l_1, \cdots, l_n\}$ to obtain

$$[\bar{w}_1, \cdots, \bar{w}_n] = \ker(\alpha_*) = [\bar{l}_1, \cdots, \bar{l}_n] \subset H_1(\partial H). \tag{1}$$

Now, there *is* a homeomorphism $\phi: H \to H'$ of H onto a "standard" genus n handlebody $H' \subset S^3$, taking the D_j to cutting discs D'_j for H' and taking the l_j to curves bounding discs in $S^3 - \mathring{H}'$. Then, if α' denotes the inclusion $\partial H' \subset S^3 - \mathring{H}'$, we have $[\phi_*(\bar{l}_1), \cdots, \phi_*(\bar{l}_n)] = \ker(\alpha'_*)$. But, because of (1) $[\phi_*(\bar{w}_1), \cdots, \phi_*(\bar{w}_n)] = [\phi_*(\bar{l}_1), \cdots, \phi_*(\bar{l}_n)]$. In particular, each $\phi(w_j)$ is homologous to zero in $S^3 - \mathring{H}'$, as desired.

We use the lemma again to prove the converse: Suppose that there is an embedding $\phi(H) = H' \subset S^3$ with each $\phi(w_i)$ homologous to zero in $S^3 - \mathring{H}'$. From the exactness of the Mayer-Vietoris sequence

$$0 \to H_1(\partial H') \xrightarrow{(\alpha'_*, \beta'_*)} H_1(S^3 - \mathring{H}') \oplus H_1(H') \to 0$$

we see that $H_1(\partial H') = \ker(\alpha'_*) \oplus \ker(\beta'_*)$. As above, this means, for any cutting $\{D_1, \cdots, D_n\}$ of H,

$$H_1(\partial H') = [\phi_*(\bar{w}_1), \cdots, \phi_*(\bar{w}_n)] \oplus [\phi_*(\bar{m}_1), \cdots, \phi_*(\bar{m}_n)]$$

(where $m_i = \partial D_i$). This is preserved by $\phi^{-1}|_{\partial H'}$; hence, in M,

$$H_1(\partial H) = [w_1, \cdots, w_n] \oplus [m_1, \cdots, m_n] = \ker(\alpha_*) \oplus \ker(\beta_*).$$

It follows (using the Mayer-Vietoris sequence in M analogous to the one above) that $H_1(M) = (0)$.

At this point a natural way to proceed is to try to use more of the strength of Moise's theorem. If we do not demand too much, we may still get an effective test. We examine one possibility in the remainder of this section.

Let $\{H; w_1, \cdots, w_n\}$ be a Heegaard diagram of a closed 3-manifold M and let $\psi : H \to S^3$ be an embedding. Denote the image of ψ by H' and let α denote the inclusion $\partial H' \subset S^3 - \mathring{H}'$. Each of the simple closed curves $\psi(w_i)$ determines a conjugacy class $C(\psi w_i) \subset \Pi_1(\partial H')$. Let G denote $\Pi_1(S^3 - \mathring{H}')$ and G_p the pth term of the lower central series of G, defined inductively by

$$G_1 = G, \quad G_p = [G, G_{p-1}].$$

Definition. We say that ψ is a *p-embedding* of $\{H; w_1, \cdots, w_n\}$ if, for each i, we have

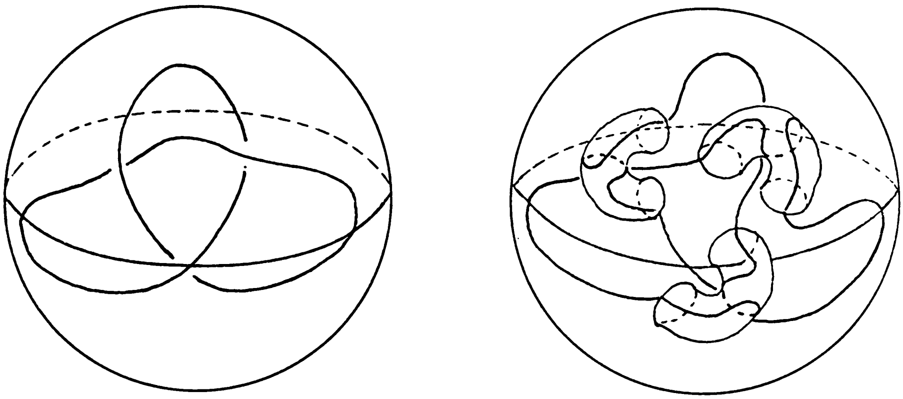
$$\alpha_*(C(\psi w_i)) \subset G_p.$$

Thus we have a sequence of successively stronger embedding properties of Heegaard diagrams. Note that Theorem 1 says that any Heegaard diagram of a homology 3-sphere admits a 2-embedding into S^3. Moise's theorem implies that the diagram of a simply connected manifold admits a p-embedding into S^3 for any p. Beyond this we have only the following partial result:

Theorem 2. *For each integer $p > 2$ there exists a Heegaard diagram of a non-simply-connected homology 3-sphere M_p which admits a p-embedding into S^3.*

We construct M_p by removing and replacing a regular neighborhood $M(k)$ of a knotted simple closed curve k in S^3. The resulting manifold depends only on the image of the meridian of $M(k)$ under the attaching homeomorphism $\partial M(k) \to \partial(S^3 - \mathring{M}(k))$. If this *surgery curve* is not homotopic to the original meridian, this process is called an *elementary surgery* (*along* k) on S^3 [1]. We first describe a method for constructing a Heegaard diagram for any homology 3-sphere obtained by elementary surgery on S^3.

Heegaard diagrams for manifolds obtained by elementary surgery. Compare [3], [4]. Suppose the homology 3-sphere M is obtained from S^3 by an elementary surgery along the knot $k \subset S^3$. We first place k on the boundary of an unknotted handlebody $H \subset S^3$, so that it is homologous to zero in the complement of H. We can do this for any given knot type, for example, as illustrated in Figure 1.



(a) (b)

Figure 1

In (a) we have k projected onto the boundary of a 3-cell; in (b) we add handles to accomodate the over-crossings and "twists" about the handles to make k homologous to zero in the complement.

Now let A be the union of a disc D and an interval I as shown in Figure 2. We can embed $A \times S^1$ into H so that $A \times S^1 \cap \partial H = \{a\} \times S^1 = k$. The centerline k^* of $D \times S^1$ has the same knot type as k does, so we can obtain M by removing $D \times S^1$ and replacing it differently. In terms of the (somehow oriented) coordinate circles $m = \partial D \times \{0\}$, $l = \{b\} \times S^1$ the surgery curve must have the form $m + ql$ $(q \in Z)$, because $H_1(M)$ was assumed to be trivial.

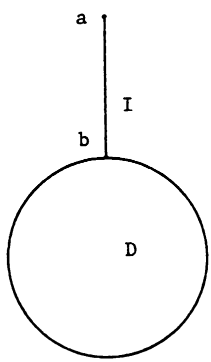


Figure 2

But there is a homeomorphism ϕ of $H - (\overset{\circ}{D} \times S^1)$ onto itself, taking any such simple closed curve onto m. We obtain ϕ by simply cutting $H - (\overset{\circ}{D} \times S^1)$ along the annulus $I \times S^1$, "untwisting" the q longitudinal windings of the given curve, and reattaching.

Finally, let $l_1, \cdots, l_n$ be the boundaries of the discs of a cutting for $S^3 - \overset{\circ}{H}$ and let $w_1, \cdots, w_n$ be the images of these curves under ϕ. Then $\{H; w_1, \cdots, w_n\}$ is the Heegaard diagram of a closed 3-manifold, say M'. Since $\phi(l_i) = w_i$, ϕ extends to a homeomorphism ϕ' of $S^3 - (\overset{\circ}{D} \times S^1)$ onto $M' - (\overset{\circ}{D} \times S^1)$. But ϕ' carries the surgery curve onto the original meridian; hence there is a further extension to a homeomorphism of M onto M'. It follows that $\{H; w_1, \cdots, w_n\}$ is the desired Heegaard diagram of M.

Construction of M_p. We construct a Heegaard diagram for M_p and the desired p-embedding simultaneously.

($p = 3$) We start with a trefoil knot k_2 on an unknotted handlebody of genus 3, as shown in Figure 3(a). If we add another handle to provide for the link, we can place the "double" k_3 of k_2 on the resulting handlebody H_4 (see Figure 3(b)). If we read along k_3 from the point $*$ in the direction indicated by the arrow, then, in terms of the generators a, b, c, d of $G = \Pi_1(S^3 - \overset{\circ}{H}_4)$ shown, we obtain the following expression for k_3:

$$w = ab^{-1}ca^{-1}bc^{-1} \cdot d \cdot cb^{-1}ac^{-1}ba^{-1} \cdot d^{-1} = [[a, b^{-1}][b^{-1}a, c], d] \in G_3.$$

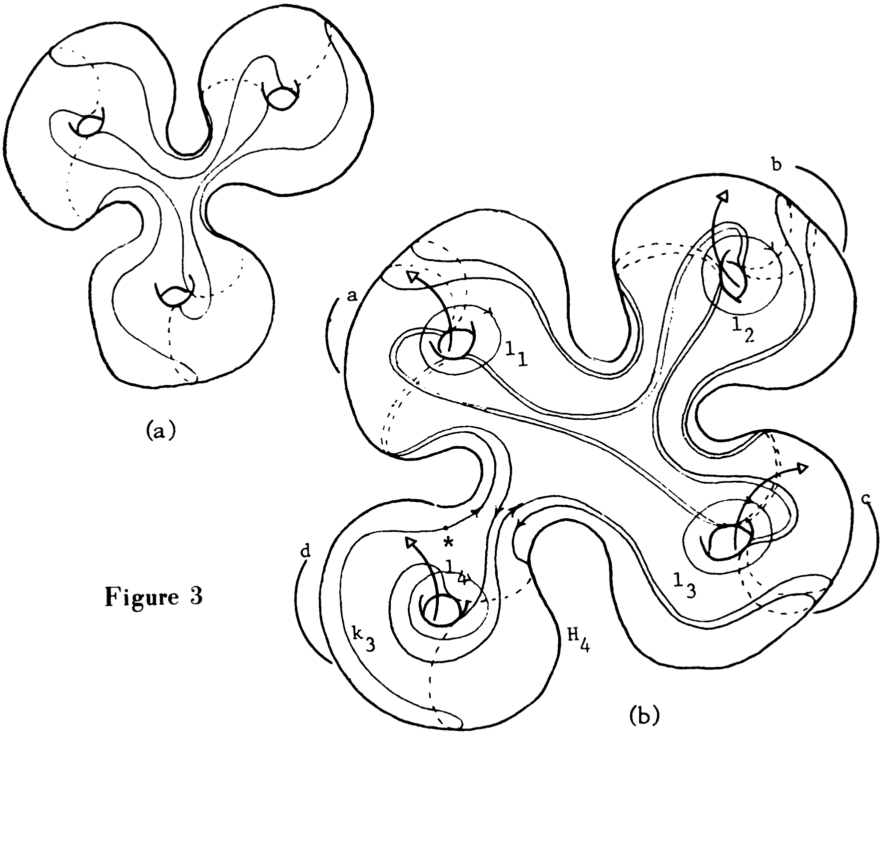


Figure 3

M_3 is defined to be the result of a 1-1 surgery along k_3 (that is, the surgery curve is a 1-1 curve in terms of the coordinate circles $(m, 1)$ of our general construction above). The Heegaard curves $w_1, \cdots, w_4$ for the diagram obtained are the images under the twisting homeomorphism of the canonical longitudes $l_1, \cdots, l_4$ shown in Figure 3(b). Hence, we consider H_4 as it is already embedded in S^3. Expressed in terms of a, b, c, d, the w_i are products of conjugates of w and w^{-1}. Thus, if α denotes the inclusion $\partial H_4 \subset S^3 - \mathring{H}_4$, we have $\alpha_*(C(w_i)) \subset G_3$ for each i, as desired.

($p > 3$) To get M_4 we simply repeat the above construction with k_3 in place of k_2. M_p is then defined by the obvious induction.

Finally, we note that none of the M_p's is simply connected, since each was constructed by elementary surgery along an iterated double of the trefoil, and all doubled knots are known [5] to have *property* P (a knot $k \subseteq S^3$ is said to have *property* P if no simply-connected manifold can be obtained by elementary surgery along k).

If we define *(p)-embedding* analogously, replacing G_p with the pth derived subgroup $G^{(p)}$ of G:

$$G^{(0)} = G, \qquad G^{(p)} = [G^{(p-1)}, G^{(p-1)}],$$

we have the corresponding result:

For each $p > 0$ there is a non-simply-connected homology 3-sphere $M_{(p)}$ with a Heegaard diagram which admits a (p)-embedding into S^3.

The construction is similar to the one above. We start with the trefoil of Figure 3(a); instead of doubling we compose two identical copies as illustrated in Figure 4. The resulting simple closed curve is $k_{(2)}$. If G denotes the fundamental group of the complement of the genus-6 handlebody, embedded as shown, then $C(k_{(2)}) \subset G^{(2)}$. It follows that any Heegaard diagram obtained by applying our general construction to $k_{(2)}$ is automatically (2)-embedded. Hence we can take $M_{(2)}$ to be the result of a 1-1 surgery along $k_{(2)}$.

We obtain $k_{(p)}$ and $M_{(p)}$ by repeating this with $k_{(p-1)}$ in place of the trefoil knot.

Again each $M_{(p)}$ is non-simply-connected because each $k_{(p)}$ can be shown to belong to a class of knots which are known to have property P [5].

4. Generalization of a theorem of Papakyriakopoulos.

Theorem 3. *Suppose M is a homology 3-sphere with Heegaard diagram $\{H; w_1, \cdots, w_n\}$. Then there is a meridian-longitude system $\{m_1, \cdots, m_n, l_1, \cdots, l_n\}$ for H, with each l_i homologous to w_i on ∂H.*

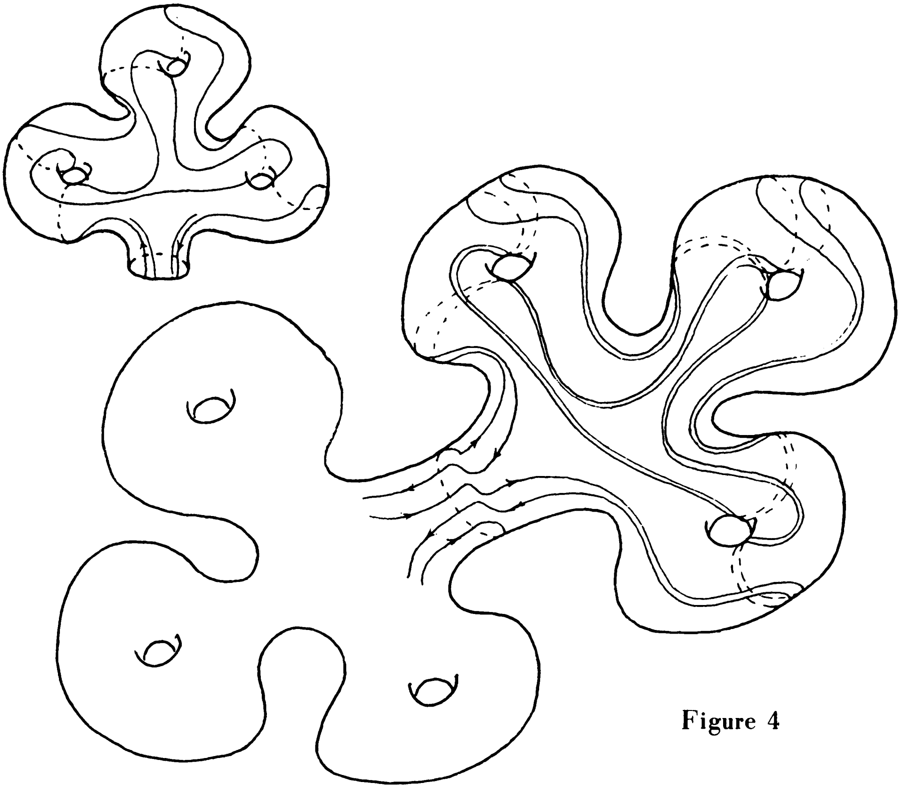


Figure 4

This was proved by Papakyriakopoulos [7, Theorem 35.5] in the case that M is simply connected. The referee has informed the author that the generalization to homology 3-spheres has also been proved by F. Waldhausen (1965, unpublished).

For the proof we need the following two results:

(A) *Any automorphism of the free group* $F = F(a_1, \cdots, a_n)$ *of rank* n *is a composition of automorphisms of the following three types*:

(I) $\quad a_1 \to a_1 a_2 \quad a_i \to a_i \qquad (i \neq 1),$

(II) $\quad a_1 \to a_j \quad a_j \to a_1 \quad a_i \to a_i \qquad (i \neq 1, j),$

(III) $\quad a_1 \to a_1^{-1} \quad a_i \to a_i \qquad (i \neq 1).$

(B) *Any basis* $\{\bar{b}_1, \cdots, \bar{b}_n\}$ *of* F/F_2 *is the image, under the natural projection of a set of free generators* $\{b_1, \cdots, b_n\}$ *of* F.

(A) is a version of Nielsen's Theorem [6, Theorem N1, p. 163] and (B) follows from [6, Corollary 3.5.1].

Proof of Theorem 3. By Papakyriakopoulos' "meridian-longitude theorem", there is a meridian-longitude system $\{m'_1, \cdots, m'_n, l'_1, \cdots, l'_n\}$ for $H \subset M$ with each $l_i \sim 0$ in $M - \mathring{H}$. There is an embedding ϕ: $H \to S^3$ so that the complement of $\phi(H) = H'$ in S^3 is a handlebody, and taking the given meridian-longitude system onto the "standard" system for H'. By the lemma of the preceding section, each $\phi(w_i)$ is homologous to zero in $S^3 - \mathring{H}'$. If we can find a new meridian-longitude system $\{m_1, \cdots, m_n, l_1, \cdots, l_n\}$ for $H' \subset S^3$ with $l_i \sim \phi(w_i)$ on $\partial H'$, then the preimage under ϕ of this system is the one required for $H \subset M$.

So, we assume that $H \subset S^3$, that $\{m_1', \cdots, m_n', l_1', \cdots, l_n'\}$ is the "standard" meridian-longitude system, and that $w_i \sim 0$ in $S^3 - \mathring{H}$. We show, in fact, that there is a meridian-longitude system $\{m_1, \cdots, m_n, l_1, \cdots, l_n\}$ for H with

(1) l_i *homotopic* to zero ($l_i \simeq 0$) in $S^3 - \mathring{H}$, and

(2) $l_i \sim w_i$ on ∂H.

We first establish some notation. Let α and β denote the inclusions $\partial H \subset S^3 - \mathring{H}$, $\partial H \subset H$, respectively. The m_i' bound disjoint properly embedded discs $D_i \subset H$, which form a cutting for H and therefore determine a basis $\{a_1, \cdots, a_n\}$ for $\Pi_1(H)$, which we now fix for the remainder of the argument. We identify $\Pi_1(H)$ with $F(a_1, \cdots, a_n)$.

A simple closed curve s on ∂H determines a conjugacy class $C(s)$ in $\Pi_1(H)$; as words in the a_i, the elements of this class are exactly the conjugates of any given "reading" of s with respect to the m_i'. Also, s determines elements $\bar{s}$ of $H_1(\partial H)$ and $\beta_*(\bar{s})$ of $H_1(H)$. In terms of the generators $\bar{a}_1, \cdots, \bar{a}_n$ of $H_1(H)$ corresponding to $a_1, \cdots, a_n$, $\beta_*(\bar{s})$ is just the abelianized reading of s.

Finally, if g is any element of $\Pi_1(H)$ (that is, any word in the a_i), let $\bar{g}$ denote the corresponding element of $H_1(H)$ (under the natural homomorphism: $a_i \to \bar{a}_i$).

Now, $\{\beta_*(\bar{w}_1), \cdots, \beta_*(\bar{w}_n)\}$ is a basis for $H_1(H)$. By (B) there are generators $g_1, \cdots, g_n$ of $\Pi_1(H)$ with

$$\bar{g}_i = \beta_*(\bar{w}_i) \qquad (i = 1, \cdots, n).$$

We claim that

(*) *For any set* $\{g_1, \cdots, g_n\}$ *of generators of* $\Pi_1(H)$, *there are canonical longitudes* $l_1, \cdots, l_n$ *on* ∂H *satisfying*

(1) $l_i \simeq 0$ *in* $S^3 - \mathring{H}$, *and*

(2) $\beta_*(\bar{l}_i) = \bar{g}_i$.

By (A), the g_i may be obtained from $a_1, \cdots, a_n$ by a finite sequence of Nielsen transformations. Hence, by induction on the length of this sequence, it is enough to prove (*) for the result of a single Nielsen transformation.

Hence, suppose that $b_1, \cdots, b_n$ are generators of $\Pi_1(H)$, that we have canonical longitudes $k_1, \cdots, k_n$ on ∂H satisfying

(1) $l_i \simeq 0$ in $S^3 - \mathring{H}$, and

(2) $\beta_*(\bar{k}_i) = \bar{b}_i$,

and that $g_1, \cdots, g_n$ are obtained from $b_1, \cdots, b_n$ by a single Nielsen transformation

Case 1°. ($g_1 = b_1 b_2, g_2 = b_2, \cdots, g_n = b_n$).

By assumption there are disjoint properly embedded discs $E_1, \cdots, E_n$ in H, transverse to $k_1, \cdots, k_n$. Since $k_1 \cup \cdots \cup k_n \cup \partial E_1 \cup \cdots \cup \partial E_n$ does not separate ∂H, there is an arc γ from k_1 to k_2 whose interior misses this set. A small regular neighborhood in ∂H of $k_1 \cup \gamma \cup k_2$ has three boundary components, two of them isotopic to k_1 resp. k_2. Since we can join either side of k_2 with a given side of k_1, it can be arranged that the remaining boundary component (appropriately oriented) has a reading, with respect to the original m_i', which is a product of a conjugate of the reading of k_1 and a conjugate of the reading of k_2. Denote this oriented simple closed curve by l_1. Then the system

$$l_1, l_2 = k_2, \cdots, l_n = k_n$$

satisfies 2. It remains to be seen that the l_i are canonical longitudes for H, homotopic to zero in the complement.

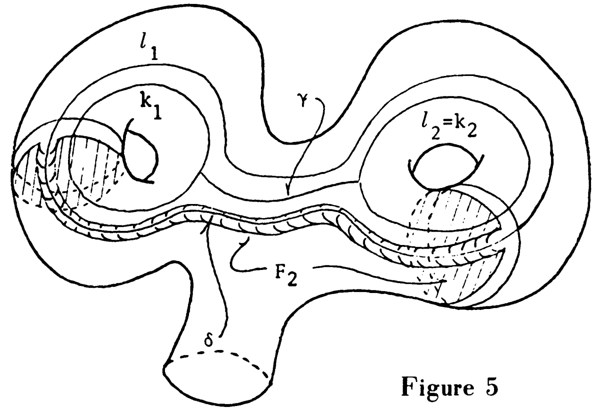


Figure 5

For the former we construct a cutting $F_1, \cdots, F_n$ for H, transverse to the l_i, as follows. Let δ be one of the subarcs of l_1 joining E_1 to E_2. Let N be a small regular neighborhood in H of $E_1 \cup \delta \cup E_2$. The closure of one of the components of $\partial N \cap \mathring{H}$ is a disc, F_2, disjoint from l_1 and transverse to l_2 (see Figure 5). The system

$$\{F_1 = E_1, F_2, F_3 = E_3, \cdots, F_n = E_n\}$$

is then the required cutting.

For the latter, note that since k_1 and k_2 both bound discs in $S^3 - \mathring{H}$, so does l_1. The remaining longitudes were assumed to be trivial in $S^3 - \mathring{H}$.

Cases 2° *and* 3°. For the Nielsen transformation of type II we simply relabel the given longitudes; for the one of type III just change the orientation of k_1.

The theorem now follows: Since the l_i are canonical longitudes, there is a transverse meridian system $\{m_1, \cdots, m_n\}$. We have arranged that

$$\beta_*(\bar{y}_i) = \bar{g}_i = \beta_*(\bar{w}_i).$$

Since $l_i \sim 0$ in $S^3 - \mathring{H}$, we also have $\alpha_*(\bar{l}_i) = 0 = \alpha_*(\bar{w}_i)$. But, just as in the proof of Theorem 1, we see that $\ker(\alpha_*) \cap \ker(\beta_*) = (0)$, so the two preceding equalities give us $l_i \sim w_1$ on ∂H as required.

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