TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 180, June 1973

QUASICONFORMAL MAPPINGS AND SETS OF FINITE PERIMETER[1]

BY

JAMES C. KELLY

ABSTRACT. Let D be a domain in R^n, $n \geq 2$, f a quasiconformal mapping on D. We give a definition of *bounding surface of codimension one* lying in D, and show that, given a system Σ of such surfaces, the image of the restriction of f to "almost every" surface is again a surface. Moreover, on these surfaces, f takes H^{n-1} (Hausdorff $(n-1)$-dimensional) null sets to H^{n-1} null sets. "Almost every" surface is given a precise meaning via the concept of the module of a system of measures, a generalization of the concept of extremal length.

1. Introduction. A homeomorphism f of a domain D in R^n, $n \geq 2$, is K-quasiconformal, $K \geq 1$, if the double inequality

$$M(\Gamma)/K \leq M(\Gamma') \leq KM(\Gamma)$$

holds for every curve family Γ in D. Here Γ' is the image curve family, and $M(\Gamma)$ is the n-modulus, in the sense of Fuglede, of the system of measures obtained by restricting Hausdorff one-dimensional measure H^1 to each curve γ in Γ (see 2.3).

Suppose f is K-quasiconformal for some K, and Γ is a family of rectifiable curves in D. According to a theorem of Fuglede (applicable in a wider context, actually), there exists $\Gamma_0 \subseteq \Gamma$, $M(\Gamma_0) = 0$, such that if $\gamma \in \Gamma \setminus \Gamma_0$, then the restriction of f to γ is H^1 absolutely continuous, and γ' is rectifiable (see 2.6).

Our concern here is with the validity of an inequality of the above type for systems Σ of bounding surfaces, where, for each Σ, the corresponding system of measures is obtained by restricting Hausdorff $(n-1)$-dimensional measure H^{n-1} to each member σ. An analogous double inequality *is* valid (Theorem 6.6), and follows easily from a result in the spirit of the above theorem of Fuglede.

Received by the editors December 20, 1971 and, in revised form, August 9, 1972.

AMS (MOS) subject classifications (1970). Primary 30A60, 31B15; Secondary 26A57, 26A69.

Key words and phrases. Quasiconformal mappings, set of finite perimeter, ACL_n homeomorphism, module of system of measures.

(1) This paper is based on the author's dissertation at Indiana University, May 1971. The research was partially supported by NSF Grant GP19694.

Specifically, for σ lying outside some exceptional subsystem Σ_0, we show (i) $\sigma' = f\sigma$ is a surface, and (ii) an integral transformation formula is valid for the restriction of f to σ (Theorems 6.3, 6.4).

We take a surface σ to be the topological boundary ∂E of E, where $H^{n-1}[\partial E] < \infty$, and ∂E satisfies a certain cone condition at every point (Definition 6.1). This definition includes C^1 manifolds without boundary, and bounding polyhedral surfaces.

A consequence of the definition is that E has finite perimeter. Thus a very general version of the Gauss-Green Theorem, due to Federer, is our starting point (see 3.2). Later, we use certain integral transformation formulae involving Lipschitz mappings which are also due to Federer.

As a final note, we give an integral transformation formula for the restriction of f to almost every level surface $u^{-1}(t)$, u belonging to a certain class of real valued functions (Theorem 6.10). Here, f need only be an ACL_n homeomorphism (see 2.4). The result follows directly from some results of Ziemer.

The author wishes to take this opportunity to acknowledge his feelings of good fortune in being able to claim William P. Ziemer as both thesis adviser and friend.

2. Preliminaries.

2.1. *Notation.* R^1 will denote the real numbers, $\dot{R}^1$ its two point compactification. For $n \geq 2$, R^n is n-dimensional Euclidean space with the standard basis $e_1, e_2, \cdots, e_n$. The vector x in R^n is represented relative to this basis by the n-tuple $(x_1, \cdots, x_n)$. The inner product $\langle x, y\rangle$ of x and y is $\Sigma_{i=1}^n x_i y_i$, where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$. $|x| = \langle x, x\rangle^{1/2}$ is the norm of x.

L_n is used for n-dimensional Lebesgue measure, H^k for k-dimensional Hausdorff measure. (Only integral values of k will be of concern here. In particular, $H^n = L_n$.) Each of these measures is complete.

For $A \subseteq R^n$, ϕ one of these measures, define $\phi \,|\, A$, the restriction of ϕ to A, by $\phi \,|\, A(Y) = \phi(A \cap Y)$, $Y \subseteq R^n$. All ϕ-measurable sets are $\phi\,|A$-measurable. For $f: A \to R^n$, we will say that f is ϕ-absolutely continuous if f takes $\phi\,|\,A$ null sets to $\phi\,|\,f(A)$ null sets. ϕ-almost everywhere is abbreviated ϕ-a.e.

$\tilde{A}$ and ∂A denote, respectively, the complement and topological boundary of A. $B(x, r)$ is the closed ball of radius r centered at x, $S(x, r) = \partial B(x, r)$. $\Omega(n)$ is the L_n measure of $B(0, 1)$, ω_{n-1} is the H^{n-1} measure of $S(0, 1)$.

Whenever A is L_n-measurable, $1 \leq p < \infty$, $L^p_{\mathrm{loc}}(A)$ is the class of real valued L_n-measurable functions g such that $|g|^p$ is locally integrable over A. $L^p(A)$ is the subclass for which $\|g\|_p = (\int_A |g|^p dL_n)^{1/p} < \infty$.

The letter D will always denote a domain in R^n, and $f: D \to D'$ always means $f: D \to R^n$, $D' = f(D)$.

2.2. *The module of a system of measures.* Let $\mathfrak{M}$ be the system of Borel measures on R^n, and let E be a subsystem of $\mathfrak{M}$. For $\mu \in E$, the symbol $\rho \wedge \mu$ indicates that $\rho: R^n \to \dot{R}^1$ is a nonnegative Borel function such that $\int \rho\, d\mu \geq 1$. The symbol $\rho \wedge E$ then means that $\rho \wedge \mu$ for every $\mu \in E$. For $0 < p < \infty$, the *p-module* $M_p(E)$ *of the system* E is defined as

$$M_p(E) = \inf\left\{\int_{R^n} [\rho]^p\, dL_n;\ \rho \wedge E\right\}.$$

It is customary to omit the index when $p = n$.

A statement concerning a system E of measures is said to be true for M_p-almost every (M_p-a.e.) measure μ in E if the subsystem E_0 of E for which the statement is not true satisfies $M_p(E_0) = 0$. The system E_0 is then termed p-exceptional.

Proofs of the following elementary properties of the p-module may be found in [5].

(i) $M_p(E_1) \leq M_p(E_2)$ if $E_1 \subseteq E_2$.

(ii) $M_p(E) \leq \Sigma_{i=1}^{\infty} M_p(E_i)$ if $E = \bigcup_{i=1}^{\infty} E_i$.

(iii) If $\bar{\mu}$ is the completion of μ, and $L_n(A) = 0$, then $\bar{\mu}(A) = 0$ for M_p-a.e. $\mu \in \mathfrak{M}$.

(iv) If $\|f_i - f\|_p \to 0$ on R^n, then there is a subsequence f_{i_j} such that $\int |f_{i_j} - f|\, d\mu \to 0$ for M_p-a.e. $\mu \in \mathfrak{M}$.

2.3. *Quasiconformal mappings.* A curve γ in R^n is a nonconstant continuous mapping of the (open, half-open, or closed) unit interval $(0, 1)$ into R^n. Letting $|\gamma| = \gamma((0, 1))$, we specify a Borel measure μ_γ on R^n by setting $\mu_\gamma(B) = \int N[\gamma | \gamma^{-1}(B \cap |\gamma|), y]\, dH^1 y$ for every Borel set B. Here $\gamma|$ denotes restriction, and $N[\gamma, y]$ is the cardinality of $\gamma^{-1}\{y\}$. γ is said to lie in the domain D if $|\gamma| \subseteq D$.

If Γ is a curve family in D, and $f: D \to D'$, let $\Gamma' = \{f \circ \gamma;\ \gamma \in \Gamma\}$. Also denote by Γ and Γ' the corresponding systems of measures formed in the above manner. *The modulus of* Γ, $M(\Gamma)$, is then defined according to 2.2 (for $p = n$).

A homeomorphism $f: D \to D'$ is called *K-quasiconformal*, $K \geq 1$, if the double inequality $M(\Gamma)/K \leq M(\Gamma') \leq KM(\Gamma)$ holds for every curve family Γ in D. The mapping f is quasiconformal if it is K-quasiconformal for some K. If f is K-quasiconformal, so is f^{-1}. We refer to [14], and [7], along with [8], for basic theorems concerning quasiconformal mappings.

2.4. *ACL_p mappings.* Let $Q = \{x \in R^n;\ a_i \leq x_i \leq b_i\}$ be a closed n-interval, and let $P_i: R^n \to R^{n-1}$ be the projection mapping given by $P_i(x) = x - x_i e_i$.

A continuous mapping $f: Q \to R^m$, $m \geq 1$, is called *absolutely continuous on lines in* Q (f is ACL in Q) if, for L_{n-1}-a.e. $y \in P_i Q$, $i = 1, \cdots, n$, f is H^1-absolutely continuous on $y \times [a_i, b_i]$, and the H^1 measure of the image is finite. For U open, f is ACL in U if f is ACL in Q for each Q in U. f is ACL_p in U if, in addition, the partial derivatives of f, $\partial_j f^i$, which exist L_n-a.e., belong to $L^p_{\mathrm{loc}}(U)$.

Suppose this to be the case. A standard smoothing technique then provides a sequence $\{f_m\}_{m=1}^{\infty}$ of functions infinitely differentiable on U such that $f_m \to f$ compact-uniformly, and the partial derivatives of f_m approach the partial derivatives of f in L^p on every compact subset.

For x in U such that the $n \times m$ matrix $df(x) = (\partial_j f^i(x))$ exists, $|df(x)|$ is the maximum of $|df(x)(a)|$ over all unit vectors a, $l(df(x))$ the minimum. When $m = n$, $J(x, f) = \det(df(x))$.

2.5. *Analytic characterization of K-quasiconformal mappings.* The following theorem gives an alternate characterization of a K-quasiconformal mapping (see [14, p. 115]).

2.5.1. **Theorem.** *A homeomorphism* $f: D \to D'$ *is K-quasiconformal if and only if the following conditions hold:*

(i) *f is ACL in D,*

(ii) *f is differentiable L_n-a.e. in D,*

(iii) $|df(x)|^n/K \leq |J(x, f)| \leq Kl(df(x))^n$ *for* L_n*-a.e.* x *in* D.

2.5.2. According to [13], conditions (i) and (ii) can be replaced by the single condition that f be ACL_n in D. An arbitrary ACL_n mapping satisfying the left half of (iii) above is called a mapping of bounded distortion. If, in addition, f is a homeomorphism, then f is K_1-quasiconformal, where $K_1 = K^{n-1}$.

2.6. *Lower dimensional absolute continuity properties of quasiconformal mappings.* The notion of the modulus of a curve family can be used to extend the idea of a function being ACL (see [12, p. 14]). Thus the following special case of a theorem due to Fuglede says that a quasiconformal mapping is more than ACL (see [5, pp. 216–218]).

2.6.1. **Theorem.** *Let* $f: D \to D'$ *be quasiconformal. If* Γ *is a family of rectifiable curves in* D*, then there exists* $\Gamma_0 \subseteq \Gamma$, $M(\Gamma_0) = 0$, *such that for* $\gamma \in \Gamma \setminus \Gamma_0$, γ' *is rectifiable, and the restriction of* f *to* γ *is* H^1*-absolutely continuous.*

2.6.2. Using an extension of an argument of Gehring for the case $n = 2$, $k = 1$, Agard has shown [2] that a quasiconformal mapping f satisfies a k-dimensional analogue of the ACL property. Precisely, if $Q = Q_1 \times Q_2$ is a closed n-interval, where Q_1 and Q_2 are respectively, k-and $(n-k)$-dimensional, then the restriction of f to $Q_1 \times y$ is H^k-absolutely continuous for L_{n-k}-a.e. $y \in Q_2$, and for such y, $H^k[f(Q_1 \times y)] < \infty$.

3. **ACL_n homeomorphisms and sets of finite perimeter.** After defining a set of finite perimeter in 3.2, we then consider the image of such a set under an ACL_n homeomorphism. For $n' = n/(n-1)$ we show in Theorem 3.3.3 that the image of $M_{n'}$-a.e. set of finite perimeter is also of finite perimeter. As will be seen, the crucial property of such mappings is their L_n-absolute continuity, a result due to Reshetnyak [11].

In the same paper, Reshetnyak has also established a result analogous to 3.3.3 in case each set E_t of finite perimeter has the form $E_t = \{x;\ \Phi(x) > t\}$, subject to certain restrictions on Φ. Kopylov [9] has given a method of constructing a *quasiconformal* mapping of $B(0, 1)$ onto itself in R^3 for which a set of finite perimeter is taken to a set of infinite perimeter. Thus some type of quantifying subfamily is necessary.

3.1. Though we shall be interested primarily in the case $k = n-1$ in what follows, some general notation is now introduced.

For $1 \le k \le n$, $\bigwedge_k R^n$ is the set of k-vectors of R^n; $\bigwedge^k R^n$, the dual of the k-vectors, is the set of k-covectors of R^n. In particular, $\bigwedge_1 R^n = R^n$, $\bigwedge_n R^n = R^1$.

Let $\Lambda(n, k)$ be the set of increasing maps of $\{1, \cdots, k\}$ into $\{1, \cdots, n\}$. For $\lambda \in \Lambda(n, k)$, let $\lambda(j) = i_j$, so that $\{e_\lambda = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\}$ constitutes the standard basis of $\bigwedge_k R^n$. Let $\{e^\lambda\}$ be the dual basis. We regard $\bigwedge_k R^n$ and $\bigwedge^k R^n$ as having the standard euclidean inner product $\langle\ ,\ \rangle$ with respect to these bases, with the corresponding euclidean norm $|\ |$. If $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ is a k-vector, then the linear subspace of R^n spanned by $\alpha_1, \cdots, \alpha_k$ is the k-space of α.

Suppose $L\colon R^n \to R^m$ is linear. We suggestively replace L with the symbol $\bigwedge_1 L$, and then let $\bigwedge_k L$ denote the induced linear mapping of $\bigwedge_k R^n$ to $\bigwedge_k R^m$ given by $\bigwedge_k L(e_\lambda) = L(e_{i_1}) \wedge \cdots \wedge L(e_{i_k})$, λ in $\Lambda(n, k)$. If U is open in R^n, and $g\colon U \to R^m$ is such that $dg(x)$ is defined for some fixed x in U, we let $\bigwedge_k dg(x, \alpha)$ denote the induced mapping of k-vectors evaluated at the k-vector α, and $J_k g(x)$ the maximum absolute value over all unit k-vectors α. In particular then, whenever $m = n$, $\bigwedge_n dg(x, \alpha) = J(x, g)\alpha$ for every n-vector α, so that $J_n g(x) = |J(x, g)|$.

A differential k-form ω on U, or simply k-form, is a mapping $\omega\colon U \to \bigwedge^k R^n$. $C_0^\infty(U, \bigwedge^k R^n)$ denotes those k-forms on U which have compact support, and whose coordinate functions are C^r for every $r \ge 1$.

If $\omega = \sum \omega^\lambda e^\lambda$ is a k-form on U, then $d\omega$ is the $(k+1)$-form on U given by $d\omega = \sum d\omega^\lambda \wedge e^\lambda$. If $g\colon U \to R^n$ is such that $dg(x)$ is defined for every x in U, and ζ is a k-form on R^n, then $g^{\#}\zeta$ is the k-form on U for which $\langle g^{\#}\zeta(x), \alpha\rangle = \langle(\zeta \circ g)(x), \bigwedge_k dg(x, \alpha)\rangle$ for every k-vector α. If g is C^2 on U, then $d(g^{\#}\zeta) = g^{\#}(d\zeta)$.

For $k = n-1$, we write $e_{i'}$ instead of e_λ, where λ in $\Lambda(n, n-1)$ is such that i is not in the image of λ. Thus $e_{i'} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n$. When $k = n$, the basis element of $\bigwedge_n R^n$ is denoted by $e_I = e_1 \wedge e_2 \wedge \cdots \wedge e_n$.

If ω is an $(n-1)$-form on U, $\|\omega\|_\infty = \sup\{|\omega(x)|; x \text{ in } U\}$.

We use $*$ to denote the norm-preserving linear mapping from $\bigwedge_1 R^n$ onto $\bigwedge_{n-1} R^n$ given by $*e_i = (-1)^{i-1} e_{i'}$.

3.2. We now define a set of finite perimeter, and state the Gauss-Green Theorem.

Suppose E in R^n is an L_n-measurable set. Then $n(E, x)$ in $S(0, 1)$ is the *exterior unit normal* to E at x if, for

$$B_+(x, r) = B(x, r) \cap \{y; \langle y - x, n(E, x)\rangle \geq 0\},$$

$$B_-(x, r) = B(x, r) \cap \{y; \langle y - x, n(E, x)\rangle \leq 0\},$$

$$\lim_{r\to 0} \frac{L_n[E \cap B_-(x, r)]}{\Omega(n) r^n} = \lim_{r\to 0} \frac{L_n[\tilde{E} \cap B_+(x, r)]}{\Omega(n) r^n} = \frac{1}{2}.$$

Those points for which such an exterior unit normal exists is called the reduced boundary of E, denoted $\beta(E)$. Clearly $\beta(E) \subseteq \partial E$.

3.2.1. **Definition.** The L_n-measurable set E is said to have finite perimeter, or be of finite perimeter, if $H^{n-1}[\beta(E)] < \infty$.

Whenever E has finite perimeter, $\beta(E)$ is $(H^{n-1}, n-1)$ rectifiable, which means there exists countably many Lipschitz mappings $\psi_i\colon K_i \to R^n$, $K_i \subseteq R^{n-1}$ compact, such that $\psi_i(K_i) \cap \psi_j(K_j) = \emptyset$, $i \neq j$, $H^{n-1}[\beta(E) \backslash \bigcup_{i=1}^{\infty} \psi_i(K_i)] = 0$ (see [3, 3.2.18, 4.1.28, 4.5.6]).

If ϕ is a Borel measure on R^n, $a \in R^n$, the *k-dimensional upper and lower densities* of ϕ at a are given by

$$\theta^{*k}(\phi, a) = \limsup_{r\to 0} \Omega(k)^{-1} r^{-k} \phi(B(a, r)),$$

$$\theta^k_*(\phi, a) = \liminf_{r\to 0} \Omega(k)^{-1} r^{-k} \phi(B(a, r)).$$

The common value, $\theta^k(\phi, a)$, when it exists, is called the *k-dimensional density* of ϕ at a.

3.2.2. **Theorem (Gauss-Green).** *Let $E \subseteq R^n$ be a set of finite perimeter. Then, for $\omega \in C_0^\infty(R^n, \bigwedge^{n-1} R^n)$,*

(i) $\int_E \langle d\omega(x), e_I\rangle \, dL_n x = \int_{\beta(E)} \langle \omega(x), *n(E, x)\rangle \, dH^{n-1} x$.

(ii) $H^{n-1}[\beta(E)] = \sup\{\int_E \langle d\omega(x), e_I\rangle\, dL_n x;\ \|\omega\|_\infty \le 1\}$.

(iii) *For* H^{n-1} *– a.e.* x *in* $R^n\backslash\beta(E)$, $\theta^{n-1}(H^{n-1} \mid \beta(E), x) = 0$, *and, either* $\theta^n(L_n|E, x) = 0$ *or* $\theta^n(L_n \mid R^n\backslash E, x) = 0$.

Conclusion (ii) may be reformulated—and this reformulation is what is actually used—in the following way:

(ii)′ $H^{n-1}[\beta(E)] = \inf\{P;\ |\int_E\ d\omega(x), e_I\ dL_n x| \le P\,\|\omega\|_\infty$ for all $\omega\}$.

For the complete statement of this theorem, see [3, p. 478]. A more classical version is given in [4, p. 447].

3.3. Fix $\mathcal{P} = \{E;\ \overline{E} \subseteq D,\ H^{n-1}[\beta(E)] < \infty\}$ to be a collection of sets of finite perimeter in D. Also denote by $\mathcal{P}$ the corresponding system of measures, $\mathcal{P} = \{\mu_E;\ \mu_E = H^{n-1} \mid \beta(E)\}$. Let $n' = n/(n-1)$.

3.3.1. **Lemma.** *Let* $f: D \to D'$ *be an* ACL_n *homeomorphism. Then*

$$L_n(fA) = \int_A |J(x, f)|\, dL_n x$$

for every compact set A *in* D. *Thus* f *is* L_n*-absolutely continuous.*

Proof. See [11, §4].

3.3.2. **Lemma.** *Suppose* $g_i, g_i^m \in L^{p_i}(D)$, *such that* $\|g_i - g_i^m\|_{p_i} \to 0$ *on* D, $i = 1, \cdots, k$. *If* $p = 1/p_1 + 1/p_2 + \cdots + 1/p_k \le 1$, $q = 1/p$, *then* $\|g_1 g_2 \cdots g_k - g_1^m g_2^m \cdots g_k^m\|_q \to 0$ *on* D.

Proof. Repeatedly apply the Hölder inequality.

3.3.3. **Theorem.** *Let* $f: D \to D'$ *be an* ACL_n *homeomorphism. Then there exists* $\mathcal{P}_0 \subseteq \mathcal{P}$, $M_{n'}(\mathcal{P}_0) = 0$, *such that for* $E \in \mathcal{P}\backslash\mathcal{P}_0$,

$$H^{n-1}[\beta(f(E))] \le \int_{\beta(E)} \left|\bigwedge\nolimits_{n-1} df(x, * n(E, x))\right| dH^{n-1}x < \infty.$$

Proof. Let $f = (f^1, \cdots, f^n)$, and let $f_m = (f_m^1, \cdots, f_m^n)$, $m = 1, 2, \cdots$, be the sequence of functions as given in 2.4. After fixing an exhaustion $\{D_k\}_{k=1}^\infty$ of D by bounded subdomains, and recalling 2.2(ii), we may assume that $f_m \to f$ uniformly on D, and that the partial derivatives of f_m approach the partial derivatives of f in the L^n norm on D.

Choose x in D such that $df(x) = (\partial f^i(x)/\partial x_j)$ exists. Let $A_{ij}(x)$ be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column from $df(x)$. The induced mapping $\bigwedge_{n-1} df(x)\colon \bigwedge_{n-1} R^n \to \bigwedge_{n-1} R^n$ is then given by the $n \times n$ matrix whose elements are $A_{ij}(x)$. Similarly define $A_{ij}^m(x)$, and form $\bigwedge_{n-1} df_m(x)$.

Since $df(x)$ exists L_n-a.e., by 2.2(iii) there exists $\mathcal{P}_1 \subseteq \mathcal{P}$, $M_{n'}(\mathcal{P}_1) = 0$, such that for $\mu_E \in \mathcal{P} \backslash \mathcal{P}_1$, $\beta(E) \cap \{x: \wedge_{n-1} df(x) \text{ exists}\}$ is μ_E-full in D.

By 3.3.2, $\|A_{ij} - A_{ij}^m\|_{n'}$ approaches zero on D. Thus for

$$A_m(x) = \left[\sum_{i=1}^{n} \left[\sum_{j=1}^{n} (A_{ij}(x) - A_{ij}^m(x))\right]^2\right]^{1/2},$$

$\|A_m\|_{n'}$ approaches zero on D. By 2.2(iv), there is a subsequence, again denoted by A_m, and a subsystem $\mathcal{P}_2$ of $\mathcal{P}$, $M_{n'}(\mathcal{P}_2) = 0$, such that $\int A_m \, d\mu_E \longrightarrow 0$ for $\mu_E \in \mathcal{P} \backslash \mathcal{P}_2$.

Whenever $df(x)$ exists, the inequality

$$(1) \qquad \left|\wedge_{n-1} df(x, \alpha) - \wedge_{n-1} df_m(x, \alpha)\right| \le A_m(x)$$

is easily verified for α an $(n-1)$-vector, $|\alpha| \le 1$. For $E \in \mathcal{P}$, the $(H^{n-1}, n-1)$ rectifiability of $\beta(E)$ guarantees the H^{n-1}-measurability of the real valued function

$$x \longrightarrow \left|\wedge_{n-1} df(x, * \, n(E, x)) - \wedge_{n-1} df_m(x, * \, n(E, x))\right|$$

defined on $\beta(E)$. Thus, for $\mu_E \in \mathcal{P} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$, for every positive integer m,

$$(2) \qquad \int_D \left|\wedge_{n-1} df(x, * \, n(E, x)) - \wedge_{n-1} df_m(x, * \, n(E, x))\right| d\mu_E x$$

is well defined, and, according to (1) above, approaches zero as $m \longrightarrow \infty$. Thus $\mu_E \in \mathcal{P} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$ implies

$$\int \left|\wedge_{n-1} df(x, * \, n(E, x))\right| d\mu_E x < \infty.$$

For every $\mu_E \in \mathcal{P}$, and every positive integer m,

$$(3) \qquad \begin{aligned} &\int_E \langle (d\omega \circ f_m)(x) J(x, f_m), e_I \rangle \, dL_n x \\ &= \int_{\beta(E)} \langle (\omega \circ f_m)(x), \wedge_{n-1} df_m(x, * \, n(E, x)) \rangle \, d\mu_E x, \end{aligned}$$

whenever $\omega \in C_0^\infty(R^n, \wedge^{n-1} R^n)$, since both sides equal $\int_E \langle g^{\#} d\omega(x), e_I \rangle \, dL_n x$ by the Gauss-Green Theorem.

From 3.3.2, $\|J(\ , f) - J(\ , f_m)\|_1$ approaches zero on D. Using this, and equation (2) above, along with the uniform convergence of f_m to f on D, we observe that equation (3) holds with f_m replaced by f whenever $\mu_E \in \mathcal{P} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$.

Thus, by the L_n-absolute continuity of f,

$$\int_{f(E)} \langle d\omega(y), e_I \rangle \, dL_n y = \int_{\beta(E)} \langle (\omega \circ f)(x), \wedge_{n-1} df(x, * \, n(E, x)) \rangle \, d\mu_E x$$

for $\omega \in C_0^\infty(R^n, \wedge^{n-1} R^n)$, $\mu_E \in \mathcal{P} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$.

This equation yields the inequality

$$\left|\int_{f(E)}\langle d\omega(y), e_I\rangle\, dL_n y\right| \le \|\omega\|_\infty \int_D \left|\Lambda_{n-1} df(x, *\, n(E, x))\right| d\mu_E x,$$

which, according to (ii)$'$ of 3.2.2, means that

$$H^{n-1}[\beta(f(E))] \le \int_D \left|\Lambda_{n-1} df(x, *\, n(E, x))\right| d\mu_E x < \infty$$

for $\mu_E \in \mathcal{P}\backslash(\mathcal{P}_1 \cup \mathcal{P}_0)$.

Letting $\mathcal{P}_0 = \mathcal{P}_1 \cup \mathcal{P}_2$ then completes the proof.

4. **An integral transformation formula.** We again take $f\colon D \to D'$ to be an ACL_n homeomorphism and impose the additional requirement that f^{-1} also be ACL_n. Using the L_n-a.e. differentiability of f and f^{-1} (see 2.5.2), along with 3.3.1, we conclude that $0 < |J(x, f)| < \infty$ L_n-a.e. Since $J(x, f)$ must be positive L_n-a.e., or negative L_n-a.e. (by an argument involving topological index), we assume the former with no loss of generality. $\mathcal{P}$ remains as given in 3.3, and again, $n' = n/(n-1)$.

The additional requirement on f will enable us to show that for $M_{n'}$-a.e. $\mu_E \in \mathcal{P}$, $H^{n-1}[\beta(f(E))\backslash f(\beta(E))] = 0$. From this, we obtain a transformation formula relating the integral of the nonnegative Borel function ρ over $\beta(f(E))$ to the integral of $\rho \circ f$ over $\beta(E)$.

First, a technical lemma.

4.1. **Lemma.** *Suppose $f\colon D \to D'$ is an ACL_n homeomorphism. Let $E \in \mathcal{P}$, and let $x \in \beta(E)$ be such that f is differentiable at x, $0 < J(x, f) < \infty$. Define the vector $n_f(E, x)$ to be such that $*n_f(E, x) = \Lambda_{n-1} df(x, *n(E, x))$. Then for $s \in R^n$ such that $x + s \in D$,*

$$\langle f(x+s) - f(x), n_f(E, x)\rangle = J(x, f)\langle n(E, x), s\rangle + o(|s|).$$

Proof. By hypothesis, $df(x) = (\partial f^i(x)/\partial x_j)$ exists. Let $B_{ij}(x)$ denote the cofactor of $\partial f^i(x)/\partial x_j$. (Thus $B_{ij}(x) = (-1)^{i+j} A_{ij}(x)$, where $A_{ij}(x)$ is as in 3.3.3.) From the definition of $n_f(E, x)$, this vector has $\Sigma_{j=1}^n B_{ij}(x) n(E, x)_j$ as the ith coordinate.

For $1 \le j, k \le n$, $\Sigma_{i=1}^n B_{ij}(x)\, \partial f^i(x)/\partial x_k = \delta_{jk} J(x, f)$.

After some rearranging,

$$\langle f(x+s) - f(x), n_f(E, x)\rangle = \sum_{j=1}^n n(E, x)_j \left\{\sum_{i=1}^n B_{ij}(x)\,[f^i(x+s) - f^i(x)]\right\}. \tag{4}$$

Let $s = (s_1, \cdots, s_n)$. Since f is differentiable at x,

$$f^i(x+s) - f^i(x) = \sum_{k=1}^{n} \frac{\partial f^i}{\partial x_k}(x)s_k + o(|s|).$$

Then, again after some rearranging, the right side of (4) can be written as

$$\sum_{j=1}^{n} n(E, x)_j \left\{ s_j J(x, f) + \sum_{i=1}^{n} o(|s|)B_{ij}(x) \right\}$$

$$= J(x, f)\langle n(E, x), s\rangle + \sum_{j=1}^{n} n(E, x)_j \sum_{i=1}^{n} o(|s|)B_{ij}(x).$$

The conclusion of the lemma now follows.

4.2. We interpret this lemma geometrically in the following way. Consider $n(E, x)$ as emanating from x, and call the hyperplane passing through x which is orthogonal to $n(E, x)$ the hyperplane determined by $n(E, x)$. This hyperplane is the translate by x of the $(n-1)$-space of $*n(E, x)$. We may consider the complement of this hyperplane as consisting of a positive side and a negative side, the positive side being those vectors y such that $\langle y - x, n(E, x)\rangle > 0$.

Similarly, consider $n_f(E, x)$ as emanating from $f(x)$. The hyperplane determined by this vector is the translate by $f(x)$ of the $(n-1)$-space of $\bigwedge_{n-1} df(x, *n(E, x))$.

For $s \neq 0$, write $\langle f(x+s) - f(s), n_f(E, x)\rangle$ as

$$|s| J(x, f)\left[\left\langle n(E, x), \frac{s}{|s|}\right\rangle + \frac{1}{J(x, f)} \frac{o(|s|)}{|s|}\right],$$

which, according to the lemma, we can do.

Suppose now that $0 < \alpha < 1$ is given. Then for $|s|$ sufficiently small, $|(1/J(x, f))o(|s|)/|s|| < \alpha/2$. If, in addition, we restrict s so that $|\langle n(E, x), s/|s|\rangle| > \alpha$, then, since $J(x, f) > 0$, the sign of $\langle f(x+s) - f(x), n_f(E, x)\rangle$ must agree with the sign of $\langle n(E, x), s/|s|\rangle$. Thus, for such restricted values of s, if $y = x + s$ lies on the positive (negative) side of the hyperplane determined by $n(E, x)$, then $f(y)$ lies on the positive (negative) side of the hyperplane determined by $n_f(E, x)$.

4.3. Suppose $u: D \to \dot{R}^1$ is locally integrable. Then x in D is a *Lebesgue point* of u if $[L_n(B(x, r))]^{-1} \int_{E(x,r)} |u(x) - u(y)|\, dL_n y \to 0$ as $r \to 0$. For such a function u, L_n-a.e. x is a Lebesgue point.

4.4. Let $g = f^{-1}$. We define $N_f \subseteq D$ by specifying that $x \in N_f$ if and only if

(i) $0 < J(x, f) < \infty$,

(ii) x is a Lebesgue point of $J(\ , f)$, $f(x)$ is a Lebesgue point of $J(\ , g)$,

(iii) f is differentiable at x, g is differentiable at $f(x)$.

Define $N_g \subseteq D'$ in a similar manner, so that $f(N_f) = N_g$. Then 2.5.2, 4.3, and the L_n-absolute continuity of f and g imply $L_n(D\setminus N_f) = L_n(D'\setminus N_g) = 0$.

4.5. **Theorem.** *Let $f: D \to D'$ be an ACL_n homeomorphism such that $g = f^{-1}$ is also ACL_n, and $J(x, f) > 0$ for L_n-a.e. x. Let $E \in \mathcal{P}$, and let $x \in \beta(E) \cap N_f$. Then, with $n_f(E, x)$ as defined in 4.1, $n_f(E, x)/|n_f(E, x)|$ is the exterior unit normal to $f(E)$ at $f(x)$.*

Proof. Along with some preliminary notions we first give an overview, and then divide the proof into nine steps.

First, note that $0 < J(x, f) < \infty$ guarantees that $n_f(E, x) \neq 0$. Let

$$B_+(f(x), t) = B(f(x), t) \cap \{y;\ \langle y - f(x), n_f(E, x)\rangle \geq 0\},$$

$$B_-(f(x), t) = B(f(x), t) \cap \{y;\ \langle y - f(x), n_f(E, x)\rangle \leq 0\}.$$

To prove the theorem, we show the existence of a sequence $\{t_j\}_{j=1}^{\infty}$ of positive numbers, and two associated sequences of sets $\{\mathfrak{A}^-(j)\}_{j=1}^{\infty}$, $\{\mathfrak{A}^+(j)\}_{j=1}^{\infty}$ such that

(i) $$t_j \to 0,$$

(ii) $$f(\mathfrak{A}^+(j)) \subseteq B_+(f(x), t_j), \qquad f(\mathfrak{A}^-(j)) \subseteq B_-(f(x), t_j),$$

(iii) $$\lim_{j\to\infty} \frac{L_n[f(\mathfrak{A}^-(j)) \cap f(E)]}{\Omega(n)t_j^n} = \frac{1}{2}, \qquad \lim_{j\to\infty} \frac{L_n[f(\mathfrak{A}^+(j)) \cap f(\tilde{E})]}{\Omega(n)t_j^n} = \frac{1}{2}.$$

We also note that for any sequence $\{t_j\}_{j=1}^{\infty}$ of positive numbers such that $\lim_{j\to\infty} t_j = 0$, there is a subsequence such that the above three conditions hold. This is sufficient to prove the theorem. The proof is symmetric in that each half of (iii) can be proved in the same manner. Thus we prove only the half involving $\mathfrak{A}^-(j)$.

Suppose now that H is a closed hemisphere of radius r. For convenience, temporarily, take H to be centered at the origin, containing re_n. For $0 < \alpha < 1$, let

$$H_\alpha^+ = \{x;\ 0 \neq x \in H,\ \langle x/|x|, e_n\rangle \geq \alpha\},$$
(5)
$$H_\alpha^- = \{x;\ 0 \neq -x \in H,\ \langle x/|x|, e_n\rangle \leq -\alpha\}.$$

We may assert the existence of a sequence $\{\alpha(j)\}_{j=1}^{\infty}$ of positive numbers, $\alpha(j) \searrow 0$, such that

$$\frac{L_n[H^+_{\alpha(j)}]}{L_n[H_{\alpha(j)}]} = \frac{L_n[H^-_{\alpha(j)}]}{L_n[H_{\alpha(j)}]} \geq 1 - \frac{1}{j+1}.$$

Furthermore, this sequence can be chosen independent of the radius r of H. Fix such a sequence.

These notions still make sense when H is centered at x, has radius r, and contains $x + rn(E, x)$. Take this to be the case, and write $H^+(j)$, $H^-(j)$, instead of $H^+_{\alpha(j)}$, $H^-_{\alpha(j)}$. (Later, such hemispheres will be defined in nested pairs, H denoting the outer one, h the inner one.)

Let $R = \max|g(y) - x|$, $r = \min|g(y) - x|$, for $|y - f(x)| = t$. Since x was chosen in N_f, it follows that

(i) $\lim_{t\to 0}(R/r) = |df(x)|/l(df(x))$ (denote this quantity by $D_L(x)$),

(ii) $\lim_{t\to 0}(R/t) = |dg(f(x))|$,

(iii) $\lim_{t\to 0}(t/r) = |df(x)|$.

Using the notation of 4.2, for every j there exists $\delta_j > 0$ such that, for $0 < |s| < \delta_j$, $|(1/J(x, f))\, o(|s|)/|s|| < \alpha(j)/2$. $\{\delta_j\}_{j=1}^\infty$ is now a fixed sequence. Since $\lim_{t\to 0}(R/t)$ exists, define $\{t_j\}_{j=1}^\infty$ such that $t_j \to 0$, and for every j, $0 < R_j < \delta_j$. Thus, if $y = x + s \in B(x, R_j)$, then $0 < |s| < \delta_j$.

Also, observe that for any sequence $\{t_j\}_{j=1}^\infty$ of positive numbers such that $t_j \to 0$, there is a subsequence with the above property.

Define the following sets for $j = 1, 2, \cdots$.

$$\mathcal{H}(j) = g(B(f(x), t_j)),$$

$$\mathcal{H}^-(j) = \mathcal{H}(j) \cap \{y;\ \langle y - x, n(E, x)\rangle \le 0\},$$

$$H^-(j) = \{y;\ |y - x| \le R_j,\ \langle y - x, n(E, x)\rangle \le 0\},$$

$$h^-(j) = \{y;\ |y - x| \le r_j,\ \langle y - x, n(E, x)\rangle \le 0\},$$

$$A^-(j) = H^-(j) \cap \{y;\ \langle (y - x)/|y - x|, n(E, x)\rangle \le -\alpha(j)\},$$

$$a^-(j) = A^-(j) \cap h^-(j),$$

$$\mathcal{A}^-(j) = A^-(j) \cap \mathcal{H}^-(j).$$

For each of the sets having "$-$" as a superscript, define also its counterpart having "$+$" as a superscript. Thus, $A^+(j) = H^+(j) \cap \{y;\ \langle (y - x)/|y - x|, n(E, x)\rangle \ge \alpha(j)\}$.

Note that; by letting $y = x + s$ in the definition of $A^-(j)$, in the terminology of 4.2 we have restricted y to lie away from the hyperplane determined by $n(E, x)$. Precisely, y lies on the negative side of this hyperplane inside an n-cone which has x as vertex, $x - n(E, x)$ as axis.

We may now conclude that

(i) by definition of $\alpha(j)$,

$$L_n[a^-(j)]/L_n[h^-(j)] = L_n[A^-(j)]/L_n[H^-(j)] \ge 1 - 1/(j + 1),$$

(ii) from the definition of t_j,

$$y \in \mathfrak{A}^+(j) \Rightarrow f(y) \in B_+(f(x), t_j), \qquad y \in \mathfrak{A}^-(j) \Rightarrow f(y) \in B_-(f(x), t_j).$$

The remainder of the proof is divided into nine steps.

Step 1. $0 < \liminf L_n[\mathfrak{H}^-(j)]/L_n[\mathfrak{H}^+(j)] \le \limsup L_n[\mathfrak{H}^-(j)]/L_n[\mathfrak{H}^+(j)] < \infty$.

Proof. $0 < D_L(x) < \infty$.

Step 2. $\lim L_n[A^-(j) \cap E]/L_n[H^-(j)] = 1$.

Proof. $L_n[A^-(j) \cap E]/L_n[H^-(j)] + 1/(j+1) \ge L_n[H^-(j) \cap E]/L_n[H^-(j)]$.

Since $n(E, x)$ is the exterior unit normal to E at x, this implies

$$\liminf L_n[A^-(j) \cap E]/L_n[H^-(j)] \ge 1.$$

Step 3. $\lim L_n[\mathfrak{A}^-(j)]/L_n[\mathfrak{H}^-(j)] = 1$.

Proof. For a given j, this ratio is smallest when as much as possible of $\mathfrak{H}^-(j)$ lies in $(H^-(j)\backslash A^-(j))$. Thus, for every j,

$$\frac{L_n[\mathfrak{A}^-(j)]}{L_n[\mathfrak{H}^-(j)]} \ge \frac{L_n[\mathfrak{A}^-(j)]}{L_n[H^-(j)\backslash A^-(j)] + L_n[\mathfrak{A}^-(j)]} \ge \frac{L_n[a^-(j)]}{L_n[H^-(j)\backslash A^-(j)] + L_n[a^-(j)]}.$$

This last quantity approaches 1. Thus,

$$\liminf L_n[\mathfrak{A}^-(j)]/L_n[\mathfrak{H}^-(j)] \ge 1.$$

Step 4. $\lim L_n[\mathfrak{A}^-(j) \cap E]/L_n[\mathfrak{A}^-(j)] = 1$.

Proof.

$$1 \le \frac{L_n[\mathfrak{A}^-(j) \cap E]}{L_n[\mathfrak{A}^-(j)]} + \frac{L_n[\mathfrak{A}^-(j) \cap \tilde{E}]}{L_n[H^-(j)]} \, \frac{L_n[H^-(j)]}{L_n[b^-(j)]} \, \frac{L_n[\mathfrak{H}^-(j)]}{L_n[\mathfrak{A}^-(j)]}.$$

Since the second summand approaches zero by steps two and three,

$$\liminf L_n[\mathfrak{A}^-(j) \cap E]/L_n[\mathfrak{A}^-(j)] \ge 1.$$

Observe that, using the same arguments, each step above has a mirror conclusion in terms of "+" superscripted sets and $\tilde{E}$.

Define the sets

$$\mathfrak{G}^-(j) = g(B_-(f(x), t_j)), \qquad \mathfrak{G}^+(j) = g(B_+(f(x), t_j)).$$

4.5.1. **Lemma.** *Let* $\{C_j\}$ *denote any one of the following four sequences of sets*: $\{\mathfrak{H}(j)\}$, $\{\mathfrak{A}^-(j)\}$, $\{\mathfrak{A}^-(j) \cap E\}$, $\{\mathfrak{G}^-(j)\}$.

Then,

$$L_n[f(C_j)]/L_n[C_j] \to J(x, f) \quad \textit{as } j \to \infty.$$

Proof. $C_j \subseteq B(x, R_j)$ for every j. Suppose there exists $c > 0$ (depending on the sequence) such that $L_n[C_j]/L_n[B(x, R_j)] \geq c$ for every j. Then,

$$\left| J(x, f) - \frac{L_n[f(C_j)]}{L_n[C_j]} \right| = \left| \frac{1}{L_n(C_j)} \int_{C_j} [J(x, f) - J(y, f)] dL_n y \right|$$

$$\leq \frac{1}{cL_n[B(x, R_j)]} \int_{B(x, R_j)} |J(x, f) - J(y, f)| \, dL_n y,$$

and this last quantity approaches zero as $j \to \infty$, since x was chosen in N_f.

So to complete the proof of the lemma, we need only show that the ratios $L_n[C_j]/L_n[B(x, R_j)]$ are bounded away from zero. This is easily verified for each sequence.

Again, note that there are mirror conclusions concerning the last three sequences of sets given in the lemma.

Step 5. $\lim L_n[\mathcal{G}^-(j)]/L_n[\mathcal{G}^+(j)] = 1$.

Proof. From the preceding lemma, $\frac{1}{2}\Omega(n)t_j^n/L_n[\mathcal{G}^-(j)] \to J(x, f)$. Using the symmetry of the argument, $\frac{1}{2}\Omega(n)t_j^n/L_n[\mathcal{G}^+(j)] \to J(x, f)$.

Step 6. $\lim L_n[\mathcal{H}^-(j)]/L_n[\mathcal{G}^-(j)] = 1$.

Proof. $\mathcal{A}^-(j) \subseteq \mathcal{G}^-(j)$, so $L_n[\mathcal{H}^-(j)]/L_n[\mathcal{G}^-(j)] \leq L_n[\mathcal{H}^-(j)]/L_n[\mathcal{A}^-(j)]$.

Thus, by step three, the limit superior of the ratio in question is no greater than one. By the mirror conclusion to step three, $\lim L_n[\mathcal{H}^+(j)\backslash\mathcal{A}^+(j)]/L_n[\mathcal{H}^+(j)] = 0$, which implies, using step one, that $\lim L_n[\mathcal{H}^+(j)\backslash\mathcal{A}^+(j)]/L_n[\mathcal{H}^-(j)] = 0$. Since

$$\frac{L_n[\mathcal{H}^-(j)]}{L_n[\mathcal{G}^-(j)]} \geq \left(1 + \frac{L_n[\mathcal{H}^+(j)\backslash\mathcal{A}^+(j)]}{L_n[\mathcal{H}^-(j)]}\right)^{-1}$$

it then follows that $\liminf L_n[\mathcal{H}^-(j)]/L_n[\mathcal{G}^-(j)] \geq 1$.

Step 7. $\lim L_n[\mathcal{A}^-(j)]/L_n[\mathcal{H}(j)] = \frac{1}{2}$.

Proof. $L_n[\mathcal{H}(j)] = L_n[\mathcal{G}^+(j)] + L_n[\mathcal{G}^-(j)]$. By Steps 5 and 6, $L_n[\mathcal{H}(j)]/L_n[\mathcal{H}^-(j)] \to 2$. The conclusion now follows by Step 3.

Step 8. $\lim L_n[f(\mathcal{A}^-(j)) \cap f(E)]/L_n[f(\mathcal{A}^-(j))] = 1$.

Proof. Divide both numerator and denominator by $L_n[\mathcal{A}^-(j) \cap E]$. Then use Step 4, and 4.5.1.

Step 9. $\lim L_n[f(\mathcal{A}^-(j)) \cap f(E)]/\Omega(n)t_j^n = \frac{1}{2}$.

Proof. Divide both numerator and denominator by $L_n[f(\mathcal{A}^-(j))]$. Then use Steps 7 and 8, and 4.5.1.

4.5.2. **Corollary.** *With the hypotheses of the theorem,*

$$f(\beta(E) \cap N_f) = \beta(f(E)) \cap N_g.$$

Proof. The argument in the proof of the theorem was a local one, not using the fact that $H^{n-1}[\beta(E)] < \infty$. Thus, to prove the corollary, apply the theorem twice, using $f(N_f) = N_g$.

4.6. In the following two lemmas, let $f: D \to D'$ satisfy the conditions stipulated in 4.5.

4.6.1. **Lemma.** *There exists a sequence $\{D_j\}_{j=1}^{\infty}$ of pairwise disjoint L_n-measurable sets such that*

(i) $N_f = \bigcup_{j=1}^{\infty} D_j$,

(ii) $f|D_j$, *the restriction of f to D_j, is a Lipschitz mapping.*

Proof. See [3, p. 217].

4.6.2. **Lemma.** *Let $\rho: D' \to \dot{R}^1$ be a nonnegative Borel function. Then for every E in $\mathcal{P}$, and every H^{n-1}-measurable subset A in $\beta(E)$,*

$$\int_{A \cap N_f} (\rho \circ f)(x) |\wedge_{n-1} df(x, *n(E, x))| \, dH^{n-1}x = \int_{f(A \cap N_f)} \rho(y) \, dH^{n-1}y,$$

in the sense that both are finite and equality holds, or both are infinite.

Proof. Let A_j denote the $(H^{n-1}, n-1)$ rectifiable set $A \cap D_j$, and let $f_j = A \mid D_j$. The proof consists of applying Corollary 3.2.20 in [3] to every f_j, so we first formulate the statement of that corollary in our terminology.

The multiplicity function $N(f_j, z)$, the number of f_j-preimages of z, is identically one since f is a homeomorphism. For x in A_j, the differentiablity of f at x implies that $\operatorname{ap} d(f_j)(x)$ is the same as $df(x) \mid (n-1)$-space of $*n(E, x)$. This in turn means that $\operatorname{ap} J_{n-1} f_j(x) = |\wedge_{n-1} df(x, *n(E, x))|$. The corollary cited above then states that, for every j,

$$\int_{A_j} (\rho \circ f)(x) |\wedge_{n-1} df(x, *n(E, x))| dH^{n-1}x = \int_{f(A_j)} \rho(y) \, dH^{n-1}y.$$

The conclusion of the lemma is now immediate, since $A_i \cap A_j$ is empty for $i \neq j$.

4.7. We now enlarge the exceptional system $\mathcal{P}_0$ determined in 3.3.3, to include those sets E for which $\mu_E(D \setminus N_f) = H^{n-1}[\beta(E) \cap (D \setminus N_f)] > 0$. Since $L_n(D \setminus N_f) = 0$, by 2.2 (iii) it is still the case that $M_{n'}(\mathcal{P}_0) = 0$.

4.7.1. **Theorem.** *Let $f: D \to D'$ be an ACL_n homeomorphism such that $g = f^{-1}$ is also ACL_n, and $J(x, f) > 0$ for L_n-a.e. x in D. Let $\rho: R^n \to \dot{R}^1$ be a nonnegative Borel function. Then there exists $\mathcal{P}_0 \subseteq \mathcal{P}$, $M_{n'}(\mathcal{P}_0) = 0$, such that for $E \in \mathcal{P} \setminus \mathcal{P}_0$,*

$$\int_{\beta(E)} (\rho \circ f)(x) |\wedge_{n-1} df(x, *n(E, x))| \, dH^{n-1}x = \int_{\beta(f(E))} \rho(y) \, dH^{n-1}y.$$

Proof. Take $\mathcal{P}_0$ as specified above, and temporarily set $\rho \equiv 1$. From 4.5.2, $f(\beta(E) \cap N_f) = \beta(f(E)) \cap N_g$. By 4.6.2, with A as $\beta(E)$,

$$H^{n-1}[\beta(f(E)) \cap N_g] = \int_{\beta(E)} |\Lambda_{n-1} df(x, * \, n(E, x))| \, dH^{n-1}x,$$

since $E \in \mathcal{P} \backslash \mathcal{P}_0$.

Recalling 3.3.3, we conclude that $H^{n-1}[\beta(f(E)) \cap N_g] = H^{n-1}[\beta(f(E))]$ for $E \in \mathcal{P} \backslash \mathcal{P}_0$. The theorem now follows from 4.6.2.

4.7.2. **Remark.** From the previous theorem, $H^{n-1}[\beta(f(E)) \backslash f(\beta(E))] = 0$ for $E \in \mathcal{P} \backslash \mathcal{P}_0$. Suppose we also knew that $H^{n-1}[f(\beta(E)) \backslash \beta(f(E))] = 0$ for such E. Then, since $\underset{\sim}{f}(N_f) = N_g$, we could conclude that for A H^{n-1}-measurable in $\beta(E)$, $H^{n-1}[f(A \cap N_f)] = 0$. Since $H^{n-1}[A \backslash N_f] = 0$ for $E \in \mathcal{P} \backslash \mathcal{P}_0$, 4.6.2 would then yield the following stronger statement: For every H^{n-1}-measurable set A in $\beta(E)$, $E \in \mathcal{P} \backslash \mathcal{P}_0$,

$$\int_A (\rho \circ f)(x) |\Lambda_{n-1} df(x, * \, n(E, x))| \, dH^{n-1}x = \int_{f(A)} \rho(y) \, dH^{n-1}y.$$

In §6, the definition of surface will be such that the combination of the quasiconformality of f and conclusion (iii) of the Gauss-Green Theorem enables this to be verified for M_n, -almost all surfaces. This, in turn, will give this stronger version over such surfaces.

5. **Angle distortion under quasiconformal mappings.** We assume now that $f: D \longrightarrow D'$ is an ACL_n homeomorphism of bounded distortion, or, what is the same thing according to 2.5.2, that f is quasiconformal.

The definitions as given below of a topological cone C^*, and its inner angle measure $A(C^*)$, appear in *Angles and Quasiconformal Mappings In Space* by Agard [1].

5.1. **Definition.** By a regular cone C, having central angle β, $0 < \beta < \pi/2$, vertex the origin, is meant the graph of the equation $x_n = \sqrt{x_1^2 + \cdots + x_{n-1}^2} \cot \beta$. By a regular solid cone is meant the graph of the equation $x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2} \cot \beta$. y_0 will always denote the positive x_n-ray.

5.2. **Definition.** A topological (solid) cone C^*, having vertex at $x \in R^n$, is the image of a regular (solid) cone C under a homeomorphism h_x of a neighborhood of the origin, $h_x(C) = C^*$, $h_x(0) = x$. The measure A of the inner angle of the topological cone C^* is

$$A(C^*) = \liminf 2 \sin^{-1} (|h_x(a) - h_x(b)|/(|h_x(a) - x| + |h_x(b) - x|)),$$

where $a \in C$, $b \in y_0$, $a, b \longrightarrow 0$.

5.3. For every topolgical cone C^* there is a corresponding solid cone which C^* "encloses."

Theorems 2.2 and 2.3 in the above mentioned article by Agard guarantee that if C has central angle β, and h_x is an isometry, then $A(C^*) = \beta$. The following lemma as well is essentially contained in this article, although the conclusion, outside the range of discussion, is not.

5.4. **Lemma.** *Let C^* be a topological cone with vertex at x, $A(C^*) > 0$. Let C_s^* be the corresponding solid cone. Then $\theta^{*n}(L_n | C_s^*, x) > 0$, where θ^{*n} is the upper density defined in* 3.2.1.

Proof. First, observe that if the theorem is true, then $\theta^n(L_n \mid Y, x)$ cannot be zero for any $Y \subseteq R^n$ containing C_s^*.

Since $A(C^*) > 0$, there exists a neighborhood N_0 of the origin and a positive integer p such that, for $a, b \in N_0$, $a \in C$, $b \in \gamma_0$,

$$|h_x(b) - h_x(a)|/(|h_x(b) - x| + |h_x(a) - x|) \geq 1/2p. \tag{6}$$

Choose $r > 0$ small enough so that $h_x^{-1}(B(x, r)) \subseteq N_0$. $h_x(\gamma_0)$ is a curve issuing from x, and thus intersects $S(x, r/2)$ in a first point $h_x(c)$, $c \in \gamma_0$. We may assume that $h_x(c) = x + re_n/2$. For $0 < t < r/2$, we now estimate the cross-sectional $(n-1)$-area $g(t)$ of C_s^* that lies within the hyperplane $P_t = \{y; y_n = x_n + te_n\}$, and also within $B(x, r)$.

For each such t, there exists $b \in \gamma_0 \cap N_0$ such that $h_x(b) \in P_t \cap B(x, r/2)$. Let

$$\eta = \min\{r/2, \text{distance within } P_t \text{ from } h_x(b) \text{ to } C^*\},$$

so that for $i = 1, \cdots, n-1$, $h_x(b) + \eta e_i \in B(x, r)$. If $\eta = r/2$, then $\eta > t/p$. If $\eta \neq r/2$, choose $a \in C$ such that the above distance is given by $|h_x(b) - h_x(a)|$. Then by (6) above,

$$|h_x(b) - h_x(a)| \geq (1/2p)[|h_x(b) - x| + |h_x(a) - x|] \geq t/p.$$

Hence, $g(t) \geq \Omega(n-1)t^{n-1}/p^{n+1}$, so that

$$L_n[C_s^* \cap B(x, r)] \geq \int_0^{r/2} g(t)\,dt \geq \frac{\Omega(n-1)}{n 2^n p^{n-1}} r^n.$$

This in turn means $\theta^{*n}(L_n | C_s^*, x) > 0$.

5.5. **Lemma.** *Let $f: D \to D'$ be quasiconformal, and let C^* be a topological cone having vertex at x in D, $A(C^*) > 0$. Then $f(C^*)$ is a topological cone with vertex at $f(x)$ in D', and $A(f(C^*)) > 0$.*

Proof. See [1, §3.2].

6. **Definition of surface, main theorems.** We now give our definition of surface, and for $f: D \to D'$ quasiconformal, consider the action of f on a system of surfaces.

6.1. **Definition.** We say that $\sigma \subseteq D$ is a surface if

(i) $\sigma = \partial E$, E L_n-measurable in D,

(ii) $H^{n-1}[\partial E] < \infty$,

(iii) ∂E has the cone property at every point; that is, for every $x \in \partial E$, there exist topological cones $C^*(E, x)$ and $C^*(\tilde{E}, x)$ with vertex at x, each having positive inner angle, such that the solid cone corresponding to $C^*(E, x)$ lies within E, and the solid cone corresponding to $C^*(\tilde{E}, x)$ lies within $D \backslash E$.

Requirements (ii) and (iii) are independent. If ∂E is a surface in D, then E has finite perimeter. Moreover, the combination of 3.2.2(iii) and 5.4 implies $H^{n-1}[\partial E \backslash \beta(E)] = 0$.

6.2. Let Σ denote a system of surfaces in D, $\Sigma = \{\partial E\}$. Also denote by Σ the corresponding system of measures, $\Sigma = \{H^{n-1}|\partial E\}$. We may simultaneously consider the system $\mathcal{P} = \mathcal{P}(\Sigma)$, as in 3.3. There is an obvious one-one correspondence between $\mathcal{P}$ and Σ, so, given $\mathcal{P}_0 \subseteq \mathcal{P}$, let $\Sigma_0 \subseteq \Sigma$ denote the corresponding subsystem. As before, let $n' = n/(n-1)$.

6.3. **Theorem.** *Let $f: D \to D'$ be quasiconformal, and let Σ be a system of surfaces in D. Then there exists $\Sigma_0 \subseteq \Sigma$, $M_{n'}(\Sigma_0) = 0$, such that for $\partial E \in \Sigma \backslash \Sigma_0$, $f(\partial E) = \partial f(E)$ is a surface.*

Proof. By 5.5, requirements (i) and (iii) are easily verified for every image $f(\partial E)$. To show the existence of $\Sigma_0 \subseteq \Sigma$ such that $H^{n-1}[f\partial(E)] < \infty$ for $\partial E \in \Sigma \backslash \Sigma_0$, we use 3.3.3, and let Σ_0 correspond to the subsystem $\mathcal{P}_0$ determined there. The conclusion of the theorem then follows easily, since $M_{n'}(\mathcal{P}_0) = 0$ implies $M_{n'}(\Sigma_0) = 0$, and $H^{n-1}[f(\partial E) \backslash \beta(f(E))] = 0$ for $\partial E \in \Sigma \backslash \Sigma_0$, by 5.5, 5.4, and 3.2.2(iii).

6.4. **Theorem.** *Let $f: D \to D'$ be quasiconformal, and let Σ be a system of surfaces in D. Then there exists $\Sigma_0 \subseteq \Sigma$, $M_{n'}(\Sigma_0) = 0$, such that for $\partial E \in \Sigma \backslash \Sigma_0$, $\rho: D' \to \dot{R}^1$ a nonnegative Borel function,*

$$\int_A (\rho \circ f)(x) |\Lambda_{n-1} df(x, * \, n(E, x))| \, dH^{n-1}x = \int_{f(A)} \rho(y) \, dH^{n-1}y$$

whenever $A \subseteq \partial E$ is H^{n-1}-measurable.

Proof. We again simultaneously consider the system $\mathcal{P} = \mathcal{P}(\Sigma)$, take $\mathcal{P}_0 \subseteq \mathcal{P}$ as given in 4.7, and let Σ_0 be the corresponding subsystem.

Since $H^{n-1}[f(\beta(E))\setminus\beta(f(E))] = 0$ for $\partial E \in \Sigma\setminus\Sigma_0$, the improved version of 4.6 given in 4.7.2 is now valid. The conclusion of the present theorem is then immediate, because $H^{n-1}[\partial E\setminus\beta(E)]$ and $H^{n-1}[\partial f(E)\setminus f(\beta(E))]$ are both zero for $\partial E \in \Sigma\setminus\Sigma_0$.

6.5. Suppose now that $f: D \to D'$ is K-quasiconformal, and x is chosen in D so that $|df(x)|^n/K \le J(x, f)$. (According to 2.5.1 this is the case for L_n-a.e. x in D.) Since $J(x, f) \le |df(x)|^n$ always holds, the K-quasiconformality of f effectively yields the double inequality $J(x, f) \le |df(x)|^n \le KJ(x, f)$. Using the notation of 3.1, this may be written as $J_nf(x) \le (J_1f(x))^n \le KJ_nf(x)$.

A similar relation holds for $J_{n-1}f(x)$. It is always the case that $J_nf(x) \le (J_{n-1}f(x))^{n'}$. K-quasiconformality of f then yields the double inequality $J_nf(x) \le (J_{n-1}f(x))^{n'} \le K^{1/(n-1)}J_n(f(x))$.

6.6. **Theorem.** *Let $f: D \to D'$ be K-quasiconformal, and let Σ be a system of surfaces in D. Let Σ' denote the image system. Then*

$$M_{n'}(\Sigma)/K^{1/(n-1)} \le M_{n'}(\Sigma') \le K^{1/(n-1)}M_{n'}(\Sigma).$$

Proof. Suppose $\rho: R^n \to \dot{R}^1$ is such that $\rho \wedge (H^{n-1}|f(\partial E))$, $\partial E \in \Sigma$. By 6.4, the nonnegative Borel function $\tilde{\rho}: D \to \dot{R}^1$ given by $\tilde{\rho}(x) = (\rho \circ f)(x)J_{n-1}f(x)$ is such that $\tilde{\rho} \wedge (H^{n-1}|\partial E)$, $\partial E \in \Sigma\setminus\Sigma_0$. Since Σ_0 is exceptional,

$$M_{n'}(\Sigma) \le \int_D [\tilde{\rho}(x)]^{n'}dL_nx \le K^{1/(n-1)}\int_D [(\rho \circ f)(x)]^{n'}J(x, f)dL_nx$$
$$\le K^{1/(n-1)}\int_{R^n}[\rho(y)]^{n'}dL_ny.$$

Thus $M_{n'}(\Sigma) \le K^{1/(n-1)}M_{n'}(\Sigma')$, since ρ was arbitrary.

The inequality $M_{n'}(\Sigma') \le K^{1/(n-1)}M_{n'}(\Sigma)$ is proved in a similar manner following the observation that $M_{n'}\{f(\partial E);\ H^{n-1}[f(\partial E)] = \infty\} = 0$ (see [12, p. 6]).

6.7. As a final note, we give a result similar to 6.4, which would perhaps be useful in application. In this case, the surfaces involved are the level sets of a certain real valued function. We first state two pertinent results due to Ziemer. For the first, see [15], along with [16]; for the second, see [17] for the general result.

6.7.1. **Theorem.** *Suppose $u: D \to R^1$ is ACL_1. Then for L_1-a.e. s, $u^{-1}(s)$ is of finite perimeter, and, if $A \subseteq D$ is L_n-measurable,*

$$\int_A |\nabla u(x)|\, dL_n(x) = \int_{-\infty}^{\infty} H^{n-1}[u^{s-1}(s) \cap A]dL_1s. \tag{7}$$

6.7.2. **Theorem.** *Suppose $f: D \to D'$ is an ACL_n homeomorphism, and*

$u: D \to R^1$ is ACL_n. Then $v = u \circ f^{-1}$ is ACL_1.

For such a function u, let $E(s) = \{x;\ u(x) > s\}$.

6.8. **Theorem.** *Suppose $f: D \to D'$ is an ACL_n-homeomorphism, and u: $D \to R^1$ is ACL_n. Then for L_1-a.e. s,*

$$\int_{u^{-1}(s)\cap A} (\rho \circ f)(x) |\Lambda_{n-1} df(x, * n(E(s), x))| \, dH^{n-1}x$$
$$= \int_{f(u^{-1}(s)\cap A)} \rho(y)\, dH^{n-1}y,$$

whenever $\rho: D' \to \dot{R}^1$ is a nonnegative Borel function, and $A \subseteq u^{-1}(s)$ is H^{n-1}-measurable. (Equality is taken here in the general sense.)

Proof. Recalling 2.5.2, we proceed as in 4.6.1, 4.6.2, and write $D = D_0 \cup \bigcup_{j=1}^{\infty} D_j$, where $f | D_j$ is a Lipschitz mapping, f is differentiable at each $x \in \bigcup_{j=1}^{\infty} D_j$, $D_i \cap D_j = \varnothing$, $i \neq j$, and $L_n(D_0) = 0$. We then observe that it is sufficient to show that for s outside some L_1-null set S, $H^{n-1}[u^{-1}(s) \cap D_0] = 0$ implies $H^{n-1}[f(u^{-1}(s) \cap D_0)] = 0$.

This is easily shown. Since $L_n(D_0) = 0$, (7) above implies $H^{n-1}[u^{-1}(s) \cap D_0] = 0$ for s outside S_1, $L_1(S_1) = 0$. Using 6.7.2 and the L_n-absolute continuity of f, an application of 6.7.1 with u replaced by v yields $H^{n-1}[f(u^{-1}(s) \cap D_0)] = H^{n-1}[v^{-1}(s) \cap f(D_0)] = 0$ for s outside S_2, $L_1(S_2) = 0$. Letting $S = S_1 \cup S_2$ then completes the proof.

BIBLIOGRAPHY

1. S. Agard, *Angles and quasiconformal mappings in space*, J. Analyse Math. 22 (1969), 177–200. MR 40 #5854.

2. ———, *Quasiconformal mappings and the moduli of p-dimensional surface families*, Proc. Romanian-Finnish Sem. Teichmüller Spaces Quasiconform. Mappings, Brasov 1969, 1971, pp. 9–48.

3. H. Federer, *Geometric measure theory*, Die Grudlehren der math. Wissenschaften, Band 153, Springer-Verlag, New York, 1969. MR 41 #1976.

4. ———, *A note on the Gauss-Green theorem*, Proc. Amer. Math. Soc. 9 (1958), 447–451. MR 20 #1751.

5. B. Fuglede, *Extremal length and functional completion*, Acta Math. 98 (1957), 171–219. MR 20 #4187.

6. F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. 101 (1961), 499–519. MR 24 #A2677.

7. ———, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), 353–393. MR 25 #3166.

8. ———, *Extremal length definitions for the conformal capacity of rings in space*, Michigan Math. J. 9 (1962), 137–150. MR 25 #4098.

9. A. P. Kopylov, *The behavior of a quasiconformal mapping of three-dimensional space on plane sections of its domain of definition*, Dokl. Akad. Nauk SSSR 167 (1966), 743–746 = Soviet Math. Dokl. 7 (1966), 463–466. MR 33 #7529.

10. H. M. Riemann, *Über das Verhalten von Flächen unter quasikonformen Abbildungen im Raum*, Ann. Acad. Sci. Fenn. Ser. AI No. 470 (1970), 26 pp.

11. Ju. G. Rešetnjak, *Some geometrical properties of functions and mappings with generalized derivatives*, Sibirsk. Mat. Ž. 7 (1966), 886–919 = Siberian Math. J. 7 (1966), 704–732. MR 34 #2872.

12. J. Väisälä, *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. AI No. 298 (1961), 36 pp. MR 25 #4100a.

13. ———, *Two new characterizations for quasiconformality*, Ann. Acad. Sci. Fenn. Ser. AI No. 362 (1965), 12 pp. MR 30 #4975.

14. ———, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math., vol. 229, Springer-Verlag, Berlin and New York, 1971.

15. W. P. Ziemer, *Some lower bounds for Lebesgue area*, Pacific J. Math. 19 (1966), 381–390. MR 34 #2830.

16. ———, *Extremal length and conformal capacity*, Trans. Amer. Math. Soc. 126 (1967), 460–473. MR 35 #1776.

17. ———, *Change of variables for absolutely continuous functions*, Duke Math. J. 36 (1969), 171–178. MR 38 #6006.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65201