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BOUNDARY VALUES IN THE FOUR COLOR PROBLEM[1]

BY

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ABSTRACT. Let G be a planar graph drawn in the plane so that its outer boundary is a k-cycle. A four coloring of the outer boundary γ is admissible if there is a four coloring of G which coincides with γ on the boundary. If ψ is the number of admissible boundary colorings, we show that the 4CC implies $\psi \geq 3 \cdot 2^k$ for $k = 3, \cdots, 6$. We conjecture this to be true for all k and show ψ is $\geq c((1 + 5^{1/2})/2)^k$.

A graph is totally reducible (t.r.) if every boundary coloring is admissible. There are triangulations of the interior of a k-cycle which are t.r. for any k. We investigate a class of graphs called annuli, characterize t.r. annuli and show that annuli satisfy the above conjecture.

1. **Introduction.** Let G be a planar graph, drawn in the plane so that its outer boundary is a k-cycle. Suppose the Four-Color-Conjecture (4CC) is true. As we run through all possible 4-colorings of the vertices of G the colorings of the points of the boundary of G (∂G) run through a subset of the 4-colorings of the bounding k-cycle. This subset of the set of colorings of ∂G which can be extended to the full graph G we shall refer to as the set of "admissible" colorings of ∂G.

Let ψ denote the number of admissible boundary colorings of G (note that ψ depends upon the face F which we choose to draw as the infinite face as well as on the abstract graph G; we suppose the choice of F is fixed). Evidently ψ cannot exceed the number of 4-colorings of a k-cycle, and so if the 4CC is true we have the two-sided inequality

$$0 < \psi \leq 3^k + (-1)^k \cdot 3. \tag{1}$$

We may now concisely state the goals of this paper: to improve the left-hand inequality in (1) and to investigate the possibility of the "=" sign on the right side.

Since graphs with nontriangular regions are normally not considered in chromatic graph theory a word of motivation may be in order. If one wished to synthesize large graphs which were in a sense "hard to color," a not unreasonable approach might be to begin with a supply of graphs with the properties (a)

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their boundaries are large cycles and (b) relative to the size of the bounding cycle the graph has few admissible boundary colorings. Then, by gluing together two of these along their boundaries one might arrive at a graph with relatively even fewer admissible boundary colorings because the short admissible lists of the original pair of graphs would contain few pairs which agree on the common stretch of boundary after gluing. Our purpose, then, was to explore the limitations of this approach, though the results obtained are not without their own intrinsic interest.

Suppose first that G is a k-cycle whose interior is triangulated by diagonals, so that G consists of exactly k vertices, $k-3$ interior edges, and $k-2$ interior (triangular) faces (see Figure 1). We can fix three consecutive vertices on C_k which form a triangle, and assign colors A, B, C to them. There are two choices for the color of the unique vertex whose coloring will complete the coloring of a second triangle. For each of these two choices there are two for the next such vertex, etc. Hence such a triangulation of a k-cycle without interior vertices, as shown in Figure 1 $(k=8)$, has precisely $4 \cdot 2^{k-3} = 3 \cdot 2^k$ distinct admissible colorings. (This remains true if any of the interior triangles is barycentrically subdivided.) (2)

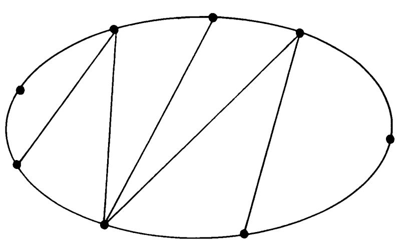

Figure 1

We believe that no graph G can have fewer admissible boundary colorings, and so we formulate

Conjecture A. *A planar graph G whose outer boundary is a k-cycle admits at least $3 \cdot 2^k$ distinct boundary 4-colorings.*

We remark that just the case $k=3$ of the conjecture is identical with the 4CC, so what we will try to prove is that the 4CC implies the truth of the conjecture for all $k \geq 4$, i.e. that Conjecture A is equivalent to the 4CC.

In this paper we shall prove that this conjecture is true if the 4CC is, when $k = 4, 5, 6$, as well as the truth of a weaker proposition, for all k.

In the latter sections of the paper we investigate graphs all of whose boundary colorings are admissible, so that the right-hand inequality in (1) is actually

(2) We are indebted to the referee for this observation.

an equality. We call these graphs *totally reducible*. We further consider a special set of graphs called *annuli* whose total reducibility we can characterize, and conclude by showing that annuli satisfy Conjecture A.

2. **Notation and preliminary results.** We shall use the letters A, B, C, D for the four colors. If the boundary of our graph consists of, say, 5 points, we shall speak of the boundary coloring $ABACD$, for instance, to mean the coloring of the 5 boundary points in which the first vertex has color $A, \cdots$, the 5th vertex has color D.

By an *extension* of a boundary coloring we mean any coloring of the full graph G in which the given boundary coloring actually appears on ∂G.

We shall use Kempe chain arguments: If, for example, $ABACD$ is an admissible boundary coloring of G, then we may contemplate, in some extension of $ABACD$, the subgraph induced by those vertices of G which are colored either color B or color D, say. If vertices 2 and 5 of ∂G both lie in the same connected component of this subgraph ("Kempe chain") then we can exchange colors A, C in vertices 3 and 4 of ∂G to find that $ABCAD$ is another admissible coloring. If 2 and 5 are not connected by the chain, we can, for instance, exchange B and D in the component containing vertex 5, to find that $ABACB$ is also admissible. This *entire paragraph* will be abbreviated, in this paper, by the statement

$$(ABACD;\ BD;\ 25) \Rightarrow ABCAD \quad \text{or} \quad ABACB.$$

Lemma 1. *Let* $\partial G = \{v_1, \cdots, v_k\}$, *suppose* v_1, v_3 *are not joined by an edge of* G, *and suppose the* 4CC. *If Conjecture* A *is true for* $3, 4, \cdots, k-1$ *then there are at least* $3 \cdot 2^{k-1}$ *admissible colorings of* ∂G *in which* v_1, v_3 *have different colors and at least* $3 \cdot 2^{k-2}$ *admissible colorings of* ∂G *in which* v_1 *and* v_3 *have the same color.*

Proof. Join v_1, v_3 by an edge in the infinite face and apply the inductive hypothesis to the new graph, which has $k-1$ boundary points. If instead we identify v_1, v_3 the graph has $k-2$ boundary points and the second claim follows. □

Lemma 2. *Let* ψ_k *denote the minimum number of admissible boundary colorings of any planar graph whose boundary is a* k*-cycle. Then*

$$\psi_k \geq \psi_{k-1} + \psi_{k-2} \qquad (k \geq 5). \tag{2}$$

Proof. Choose two boundary points v_i, v_{i+2} which are not connected by an edge and apply the proof of the preceding lemma. □

Lemma 3. *If Conjecture* A *is true for* $3, 4, \cdots, k-1$, *and if two nonconsecutive points of* ∂G $(\#\partial G = k)$ *are joined by an edge of* G *then the conjecture is true for* G *also.*

Proof. The diagonal splits G into G_r ($\#G_r = r$) and G_s ($\#G_s = s$) where $r + s = k + 2$. There are $3 \cdot 2^r$ admissible colorings of ∂G_r and each of these matches $(1/12) \cdot 3 \cdot 2^s = 2^{s-2}$ of the admissible colorings of ∂G_s at their two junction points, which yield at least

$$(3 \cdot 2^r) \cdot (2^{s-2}) = 3 \cdot 2^k$$

admissible colorings of ∂G. □

3. **Low values of** k.

Theorem 1. *The 4CC implies the truth of Conjecture* A *for* $k = 4$.

Proof. If vertices 1 and 2 of ∂G are fixed at colors A and B, there are seven possible colorings of ∂G:

$$0:\ ABAB; \qquad 2:\ ABCB,\ ABDB;$$
$$1:\ ABAC,\ ABAD; \qquad 3:\ ABCD,\ ABDC.$$

(We have grouped them into pairs because exchange of C and D does not influence admissibility.) The set S of admissible colorings which begin with AB has at least two colorings $ABXY$ where $X \neq A$, at least two where $Y \neq B$, at least one where $X = A$ and at least one where $Y = B$ (Lemma 1). If $\#S < 4$, the only possibility is $S = \{ABCD, ABDC, ABAB\}$. But

$$(ABCD;\ BD;\ 24) \Longrightarrow ABAD \quad \text{or} \quad ABCB,$$

a contradiction, so $\#S \geq 4$. □

Theorem 2. *The 4CC implies the truth of Conjecture* A *for* $k = 5$.

Proof. If $ABXYZ$ is a typical admissible boundary coloring, there are twenty possibilities for XYZ which can be grouped into ten pairs because of invariance under exchange of colors C and D:

(1) ABC, ABD; (2) ACB, ADB; (3) ACD, ADC; (4) CAB, DAB;

(5) CAC, DAD; (6) CAD, DAC; (7) CBC, DBD; (8) CBD, DBC;

(9) CDB, DCB; (10) CDC, DCD.

If S is the set of admissible boundary colorings then in S we have $X = A$ (≥ 2 times), $X = Z$ (≥ 2), $Y = B$ (≥ 2), $Z = B$ (≥ 2), $Y = A$ (≥ 2), $X \neq A$ (≥ 4), $X \neq Z$ (≥ 4), $Y \neq B$ (≥ 4), $Z \neq B$ (≥ 4), $Y \neq A$ (≥ 4). Thus $\#S = 6$ or else $\#S \geq 8$ and we are finished. If $\#S = 6$ then S consists of exactly three of the ten pairs listed above. If we take into account the requirements for S listed above, then there are exactly five possibilities for the three color-pairs which constitute S:

S = (1), (4), (10) or (1), (5), (9) or (2), (5), (8) or (2), (6), (7) or (3), (4), (7). However by

$(ABABC;\ BD;\ 24)$: $(1) \Rightarrow (3)$ or (7),
$(ABADB;\ BD;\ 24)$: $(2) \Rightarrow (1)$ or (9),
$(ABACD;\ BC;\ 24)$: $(3) \Rightarrow (1)$ or (10)
which excludes all five possibilities for S. □

Theorem 3. *The 4CC implies the truth of Conjecture* A *for* $k = 6$.

Proof. There are 61 possible colorings of a 6-cycle which are of the form $ABXYZW$. We group these into 30 pairs ($C - D$ symmetric) and a singleton $ABABAB$. The list of 30 pairs is Appendix 1 of this paper.

Let S be the set of admissible colorings of ∂G. Let S_1 denote the set of all color-pairs $ABXYZW$ for which $X = A$, and similarly let S_2 denote those for which $Y = B$, S_3 $(Z = X)$, S_4 $(W = Y)$, S_5 $(Z = A)$, S_6 $(W = B)$. Each of these sets contains ten pairs which are listed in Appendix 2.

Now S contains at least two color-pairs of each of these 6 types, i.e.

$$\#S \cap S_i \geq 2 \qquad (i = 1, \cdots, 6)$$

and furthermore

$$\#S \cap S_i^c \geq 4 \qquad (i = 1, \cdots, 6)$$

by Lemma 1.

Appendix 3 consists of the results of Kempe chain arguments. Typically such an argument will show that if a certain pair is in S, then so are several other pairs, the object being to reach 8. Thus, for example, $ABACAD$ belongs to color-pair 6, and so we show that if pair 6 is in S so are pairs 1 or 25, 5 or 8, and 4 or 14, because
$(ABACAD;\ BC;\ 24) \Rightarrow 1$ or 25,
$(ABACAD;\ CD;\ 46) \Rightarrow 5$ or 8,
$(ABACAD;\ BD;\ 26) \Rightarrow 4$ or 14.
This would be noted in Appendix 3 in standard Boolean notation as

$$6 = 6(1 + 25)\ (5 + 8)\ (4 + 14).$$

The proof proceeds as follows. We know there are at least two elements from S_1 in S. Hence for every pair of elements in S_1 we show that S must contain at least 8 color pairs. There are 45 possible pairs from S_1 and we present the arguments for the first case.

Suppose 1 and 2 are in S. Then either (1, 2, 6, 8), (1, 2, 6, 23), (1, 2, 18, 8) or (1, 2, 18, 23) are in S. If (1, 2, 6, 8) are in S, since S must contain four elements from S_1^c also, S must contain 8 elements. If (1, 2, 6, 23) are in S, we concentrate on color-pair 6. Since 6 is in S then at least one of 8 or 5 must be in S. Thus there are at least four elements from S_1 in S, hence at least 8 elements in S. If (1, 2, 18, 8) are in S, since 8 is in S, one of 6 and 7 is in S

and we are done. If (1, 2, 18, 23) are in S then since (1, 2, 18, 23) are in S_2 there must be at least four elements from S_2^c hence at least 8 elements in S.

The remaining cases are straightforward though some are more complex than the above.

The complete proof including the derivation of Appendix 3 has been deposited in the mathematics library of the University of Pennsylvania.

We conclude this section with the simple

Theorem 4. *The 4CC implies that if G is planar and bounded by a k-cycle then ∂G admits at least*

$$4!F_{k-1} \geq C((1+\sqrt{5})/2)^k$$

colorings, where $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $\cdots$ are the Fibonacci numbers.

Proof. Follows from Lemma 2 and Theorem 1. □

Corollary. *If $P(\lambda)$ is the chromatic polynomial of a planar graph G, and if the 4CC is true, then $P(4) \geq 24F_{M-1}$ where M is the maximum number of vertices of any face in G.*

4. **Totally reducible graphs.** We shift now to the right-hand inequality in (1), and we make the following definition.

Definition. *A planar graph G, together with a fixed face F drawn as the infinite face, is totally reducible (t.r.) if every 4-coloring of ∂F is admissible, i.e. extends to a coloring of all of G.*

Evidently a single cycle is t.r. We prove, however, that there are t.r. graphs with triangulated interiors and having any preassigned number k of boundary points.

Lemma 4. *The graph G shown in Figure 2 below is t.r.*

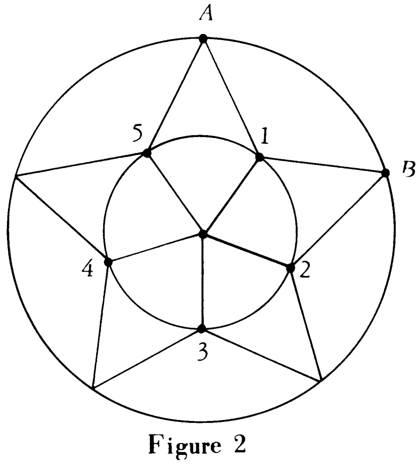

Figure 2

Proof (brute force). There are twenty colorings of a five-cycle which begin with AB. For each of these we must exhibit a compatible 3-coloring of the inner five-cycle. It suffices to choose one representative from each of the ten $C - D$-interchange classes which are listed in the proof of Theorem 2. These colorings of the outer and inner five-cycle are ($ABABC$, $CDCDB$); ($ABACB$, $DCBDC$); ($ABACD$, $DCDBC$); ($ABCAB$, $CDBCD$); ($ABCAC$, $CDBDB$); ($ABCAD$, $CDBCB$); ($ABCBC$, $DADAB$); ($ABCDB$, $CDACD$); ($ABCBD$, $CADAC$); ($ABCDC$, $DABAB$). □

Lemma 5. *Let G_r, G_s be two t.r. graphs with $\#\partial G_r = r$, $\#\partial G_s = s$, and let $q \geq 3$ be fixed. Let H be a graph which is obtained by gluing together G_r and G_s along q consecutive boundary points of each. Then H is t.r. also.*

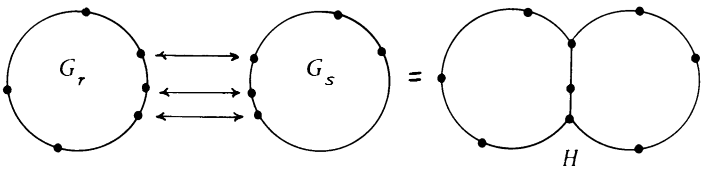

Figure 3

Proof. Choose any coloring of ∂H. Extend it along the interior path of H which forms the interface between the former G_r, G_s. We now have colorings of ∂G_r, ∂G_s. Extend them. □

Theorem 5. *Fix $k \geq 3$. There is a graph G_k which* (a) *is t.r.,* (b) *has precisely k points in ∂G_k,* (c) *has triangulated interior.*

Proof. Let K be the set of k for which the theorem is true. By Lemma 4, $5 \in K$. By Lemma 5, $r, s \in K$ and $q \geq 3 \Rightarrow r + s - q - 1 \in K$. Clearly $3 \in K$. Take $q = 3$, $s = 5$ to find that $r + 1 \in K$ if $r \in K$. □

5. Annuli.

Definition. An *annulus* G_{kl} consists of two disjoint cycles of k and l vertices ($k, l \geq 3$) with the k-cycle drawn exterior to the l-cycle and with a maximum number of nonintersecting edges connecting the two cycles.

In clockwise order we denote the vertices of the l-cycle by $u_1, \cdots, u_l$ and the vertices of the k-cycle by $v_1, \cdots, v_k$. We remark that an annulus contains $2k + 2l$ edges. Figure 4 shows a G_{64}.

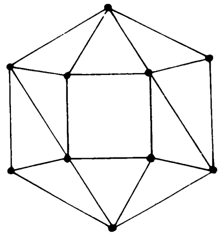

Figure 4

As promised we shall investigate the structure of t.r. annuli.

Lemma 6. *If G_{kl} is an annulus and* $\deg(u_1) \geq 6$, *then G_{kl} is not t.r.*

Proof. Suppose $v_1, \cdots, v_4$ are adjacent to u_1. Then $(A, B, C, D, \cdots)$ is not admissible. □

Lemma 7. *Suppose G_{kl} is an annulus,* $\deg(u_1) = \deg(u_j) = 5$, $\deg(u_i) = 4$ *for all $1 < i < j$, and $j + 3 \leq k$; then G_{kl} is not t.r.*

Proof. Suppose $v_1, \cdots, v_{j+3}$ are adjacent to $u_1, \cdots, u_j$. If j is odd

$$(A, B, C, B, C, \cdots, B, C, \underset{\substack{\uparrow \\ j+3}}{D}, \cdots)$$

is not admissible.

If j is even

$$(A, B, C, B, C, \cdots, B, C, A, \underset{\substack{\uparrow \\ j+3}}{B}, \cdots)$$

is not admissible. □

Lemma 8. *Suppose G_{kl} is an annulus,* $\deg(u_1) = \deg(u_j) = 5$, $\deg(u_i) = 4$ *for all $1 < i < j$, $j + 3 = k + 1$, and j is even; then G_{kl} is not t.r.*

Proof. $(A, B, C, B, C, \cdots)$ is not admissible. □

Lemma 9. *Suppose G_{kl} is an annulus,* $\deg(u_1) = 5$, $\deg(u_i) = 4$ *for all $1 < i \leq l$ and l is odd; then G_{kl} is not t.r.*

Proof. $(A, B, A, B, \cdots)$ is not admissible. □

We now attack the converse problem of finding conditions that force an annulus to be t.r. Typically we will be demonstrating that an arbitrary coloring of the k-cycle is admissible. For a fixed coloring of the k-cycle and for each vertex u_i of the l-cycle we let S_i be the set of colors *not* used in coloring the vertices of the k-cycle adjacent to u_i. Evidently $\#S_i \geq 6 - \deg(u_i)$. We also note that all our subscripts will be considered modulo l.

Lemma 10. *Suppose G_{kl} is an annulus and* $\deg(u_i) \leq 4$ *for all $i = 1, \cdots, l$; then G_{kl} is t.r.*

Proof. Fix a coloring for the k-cycle. Suppose there is a u_j with $\deg(u_j) = 3$. Since $\#S_{j+1} \geq 2$ we can assign a color to u_{j+1}. Since $\#S_{j+2} \geq 2$ we can assign u_{j+2} a color from S_{j+2} different than the color already assigned to u_{j+1}. We can continue in this manner until u_{j-1} is colored. Now G_{kl} is colored except for u_j; however u_j is adjacent to only three vertices in G_{kl}, hence it can be colored.

Now we assume $\deg(u_i) = 4$ for $i = 1, \cdots, l$. For any coloring of the k-cycle $\#S_i = 2$ for all i. If $\#(S_i \cup S_{i+1}) = 2$ for all i then the k-cycle is two colored. Since in this case $k = l$ the l-cycle can also be two colored and the coloring is admissible. Hence we may assume there is a u_j with $\#(S_j \cup S_{j+1}) \geq 3$. Assign a color from $S_{j+1} - S_j$ to u_{j+1}. Assign colors as above to $u_{j+2}, \cdots,$ $u_l, u_1, \cdots, u_{j-1}$. It is possible that u_{j-1} is assigned a color from S_j; however since $\#S_j = 2$ and u_{j+1} is not assigned a color from S_j, u_j can be colored. □

Lemma 11. *Suppose G_{kl} is an annulus with $\deg(u_i) < 6$ for $i = 1, \cdots, l$. If $\deg(u_i) = \deg(u_j) = 5$ implies that there is at least one vertex u_l with $i < l < j$ such that $\deg(u_l) = 3$; then G_{kl} is t.r.*

Proof. If there are no vertices of degree three we are done by Lemma 10. If there are vertices of degree three consider the segments of the l-cycle that arise from removing these vertices. Each segment contains at most one vertex of degree 5. Hence we can extend the coloring of the k-cycle to each segment as follows. If there is a vertex u_p in the segment with $\#S_p = 1$, assign u_p the color it is forced to have. Since $\#S_{p-1}, \#S_{p+1} \geq 2$, u_{p-1} and u_{p+1} can be colored. Continue coloring the vertices of each segment in both directions until reaching vertices of degree 3. In this manner all vertices of the l-cycle except those of degree 3 can be colored, hence all the interior vertices can be colored. □

Lemma 12. *Suppose G_{kl} is an annulus*, $\deg(u_1) = 5$, $\deg(u_i) = 4$ *for all $1 < i \leq l$ and l is even; then G_{kl} is t.r.*

Proof. If $\#S_l = 2$ there must be some u_j with $\#(S_j \cup S_{j+1}) \geq 3$. If not, the k-cycle would have to be two colored but k is odd. If such a vertex exists we proceed as in the last part of the proof of Lemma 10. Hence we may assume $\#S_1 = 1$. Assign u_1 the color it must receive. If u_1 has not been assigned a color from S_2 we proceed as follows. Assign u_l a color from S_l different from the color assigned u_1. Continue in this manner properly assigning colors to $u_{l-1}, u_{l-2}, \cdots, u_3$. Clearly u_2 can also be colored as only one element of S_2 has been used to color u_1 and u_3. If u_1 has been assigned a color from S_2 assign u_2 the color it must receive. Continue assigning colors until the color assigned to u_j is not a member of S_{j+1}. If this occurs color $u_l, u_{l-1}, \cdots, u_{j+2}$. It is clear that u_{j+1} can be properly colored as its neighbors u_j and u_{j+2} use at most one element from S_{j+1}. Suppose it is the case that u_i is always assigned a color from S_{i+1}. The coloring can be completed unless u_1 and u_{l-1} are assigned different colors from S_l. Suppose u_{l-1} is forced to have the color C. Then either u_{j-3} or v_{j-2} is also forced to be assigned C. Similarly either u_{l-5}

or v_{j-4} must be assigned C. Whence we know that either u_3 or v_4 is assigned color C.

Similarly we assume v_j is colored B. Then either v_{j-2} is B or u_{l-3} is forced to be B. Continuing we have either v_4 is B or u_3 is forced to be B. Thus u_3 and v_4 are assigned colors B and C in some order. This means that u_2 and v_4 must be assigned colors A and D in some order. But u_1 must be assigned one of A or D. If u_1 is assigned D, v_4 must also be D and then u_2 cannot be forced to be color A. □

Lemma 13. *Suppose* G_{kl} *is an annulus*, $\deg(u_1) = \deg(u_j) = 5$, $\deg(u_i) = 4$ *for all* $1 < i < j$, $j < l$, $j + 3 = k + 1$, *and* j *is odd; then* G_{kl} *is t.r.*

Proof. If $\#S_j = 2$, assign u_1 a color from S_1, u_2 a color from S_2 different than the color assigned $u_1, \cdots$. Since u_1 and u_j are not adjacent the coloring of the boundary is admissible. Hence we assume that $\#S_i = \#S_j = 1$. We may also assume that if we assign u_1 the color it must have that color is also in S_2. Similarly the color that u_i is forced to be must be in S_{i+1}.

Now suppose u_j is adjacent to v_{j-1}, v_j, and v_1. Let $S_j = \{D\}$. If the boundary coloring is not admissible u_{j-1} must be forced to be colored D. Then as before either v_{k-3} or u_{j-3} must be colored D. Continuing, either u_2 or v_4 must be colored D. Similarly if v_{j+1} is colored C then either u_2 or v_4 must also be colored C. As in the proof of Lemma 12, both of these cases lead to contradictions. □

We now summarize our results as follows:

Theorem 6. *An annulus* G_{kl} *is totally reducible if and only if it satisfies the hypotheses of one of Lemmas* 10–13.

Theorem 7. *If* G_{kl} *is an annulus then* G_{kl} *satisfies Conjecture* A.

Proof. If all of the vertices of the l-cycle have degree less than five then G_{kl} is t.r. and we are done. Hence we may assume $\deg(u_1) = d \geq 5$. Suppose $v_1, \cdots, v_{d-2}$ are adjacent to u_1, v_1 is adjacent to u_l and v_{d-2} is adjacent to u_2. See Figure 5.

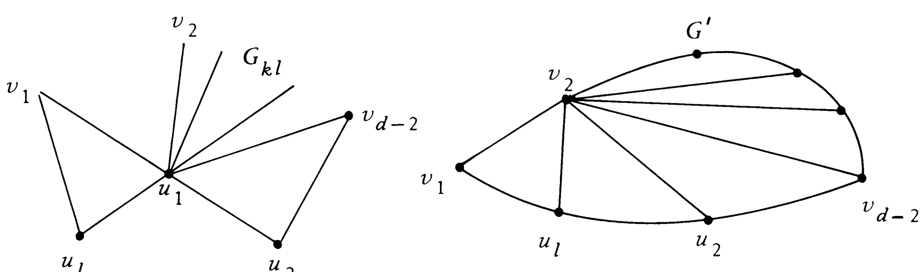

Figure 5

Form G' by contracting along the edge connecting u_1 with v_2 and adding an edge connecting u_l with u_2. G' contains as a subgraph a $G_{k+5-d,l-1}$. If the $G_{k+5-d,l-1}$ satisfies Conjecture A there are at least $3 \cdot 2^{k+5-d}$ admissible boundary colorings. For each admissible boundary coloring there are exactly two ways of coloring $v_{d-1}, \cdots$, hence there are $3 \cdot 2^k$ admissible boundary colorings of G'. Each admissible boundary coloring of G' will yield at least one admissible boundary coloring of G_{kl}. Hence the theorem will be true if any G_{k3} has enough admissible boundary colorings. If in a G_{k3}, $\deg(u_1) = \deg(u_2) = 3$, then the number of admissible boundary colorings is at least four times the number of ways of three coloring a k-cycle. Suppose u_1 is adjacent to $v_1, \cdots, v_p$; u_2 is adjacent to $v_p, \cdots, v_q$; and u_3 is adjacent to $v_q, \cdots, v_k, v_1$. There are $3 \cdot 2^{p-1}$ ways of assigning 3 colors to $v_1, \cdots, v_p$, there are 2^{q-p} ways of assigning 3 colors to $v_{q+1}, \cdots, v_{k-1}$. Since there are four possible choices of colors not to use in coloring u_1, u_2, and u_3 there are at least $3 \cdot 2^k$ admissible boundary colorings. □

Appendix 1. This is a list of all 30 color-pairs admitted by a 6-cycle. We list *XYZW*; the full coloring is *ABXYZW*.

1. *ABAC, ABAD*	11. *CABC, DABD*	21. *CBCD, DBDC*
2. *ABCD, ABDC*	12. *CABD, DABC*	22. *CBDB, DBCB*
3. *ABCB, ABDB*	13. *CACB, DADB*	23. *CBDC, DBCD*
4. *ACAB, ADAB*	14. *CACD, DADC*	24. *CDAB, DCAB*
5. *ACAC, ADAD*	15. *CADB, DACB*	25. *CDAC, DCAD*
6. *ACAD, ADAC*	16. *CADC, DACD*	26. *CDAD, DCAC*
7. *ACBC, ADBD*	17. *CBAB, DBAB*	27. *CDBC, DCBD*
8. *ACBD, ADBC*	18. *CBAC, DBAD*	28. *CDBD, DCBC*
9. *ACDB, ADCB*	19. *CBAD, DBAC*	29. *CDCB, DCDB*
10. *ACDC, ADCD*	20. *CBCB, DBDB*	30. *CDCD, DCDC*

Appendix 2.

$S_1 = \{1 - 10\}$

$S_2 = \{1 - 3, 17 - 23\}$

$S_3 = \{1, 4 - 6, 13 - 14, 20 - 21, 29 - 30\}$

$S_4 = \{3, 5, 7, 10, 17, 20, 22, 26, 28, 30\}$

$S_5 = \{1, 4 - 6, 17 - 19, 24 - 26\}$

$S_6 = \{3, 4, 9, 13, 15, 17, 20, 22, 24, 29\}$

Appendix 3.

1 = 1(6 + 18)	16 = 16(14 + 23)(10 + 11)(15 + 25)
2 = 2(8 + 23)	17 = 17(18 + 22)
3 = 3(7 + 22)	18 = 18(11 + 19)(1 + 25)(17 + 23)
4 = 4(6 + 13)	19 = 19(12 + 18)
5 = 5(6 + 7)	20 = 20(13 + 22)
6 = 6(1 + 25)(5 + 8)(4 + 14)	21 = 21(14 + 23)
7 = 7(10 + 11)(5 + 8)(3 + 28)	22 = 22(15 + 20)(3 + 28)(17 + 23)
8 = 8(6 + 7)(9 + 12)(2 + 27)	23 = 23(16 + 21)(2 + 27)(18 + 22)
9 = 9(8 + 15)	24 = 24(15 + 25)
10 = 10(7 + 16)	25 = 25(26 + 27)(6 + 18)(16 + 24)
11 = 11(13 + 27)(12 + 18)(7 + 16)	26 = 26(25 + 28)
12 = 12(11 + 19)(8 + 15)(14 + 28)	27 = 27(25 + 28)(8 + 23)(11 + 29)
13 = 13(15 + 20)(4 + 14)(11 + 29)	28 = 28(26 + 27)(7 + 22)(12 + 30)
14 = 14(16 + 21)(6 + 13)(12 + 30)	29 = 29(13 + 27)
15 = 15(13 + 22)(9 + 12)(16 + 24)	30 = 30(14 + 28)

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