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GLOBAL REGULARITY FOR $\bar\partial$ ON WEAKLY PSEUDO-CONVEX MANIFOLDS[1]

BY

J. J. KOHN

ABSTRACT. Let M' be a complex manifold and let $M \subset\subset M'$ be an open pseudo-convex submanifold with a smooth boundary which can be exhausted by strongly pseudo-convex submanifolds. The main result of this paper is the following: Given a $\bar\partial$-closed (p, q)-form α, which is C^∞ on $\bar M$ and which is cohomologous to zero on M then for every m there exists a $(p, q-1)$-form $u_{(m)}$ which is C^m on $\bar M$ such that $\bar\partial u_{(m)} = \alpha$.

1. **Introduction.** In this paper we study global regularity properties of the Cauchy-Riemann equations in serveral complex variables. This work deals with forms on manifolds but in this introduction we will describe our results for the special case of functions on domains in $\mathbf{C}^n$.

Given a domain $M \subset\subset \mathbf{C}^n$ and functions α_j, $j = 1, \cdots, n$, on M we consider the inhomogeneous Cauchy-Riemann equations:

$$u_{\bar z_j} = \alpha_j, \qquad j = 1, \cdots, n, \tag{1.1}$$

where $z_1, \cdots, z_n$ are coordinate functions on $\mathbf{C}^n$, $x_j = \operatorname{Re}(z_j)$, $y_j = \operatorname{Im}(z_j)$, $u_{z_j} = \frac{1}{2}(\partial u/\partial x_j - \sqrt{-1}\partial u/\partial y_j)$ and $u_{\bar z_j} = \frac{1}{2}(\partial u/\partial x_j + \sqrt{-1}\partial u/\partial y_j)$. We are concerned with the dependence of u on α; in particular, if the α_j are "smooth" can we find a "smooth" u satisfying (1.1). Since the system (1.1) is elliptic, the interior regularity is well understood—for example if the $\alpha_j \in C^\infty(M)$ then any solution u of (1.1) is in $C^\infty(M)$ since u satisfies the equation $\Delta u = 4\Sigma\, \alpha_{jz_j}$. So that our question really is about smoothness at the boundary of M, here we cannot expect that every solution u is smooth, since $u + h$ is also a solution whenever h is holomorphic on M. We will assume that bM, the boundary of M, is smooth; more precisely, we assume that in a neighborhood U of bM there exists a real-valued function $r \in C^\infty(U)$ such that $r > 0$ outside of $\bar M$, $r < 0$ inside M and $dr \neq 0$ everywhere, so that bM is given by the equation $r = 0$. We further assume that M is pseudo-convex, which means that if $P \in bM$ and $(a^1, \cdots, a^n) \in \mathbf{C}^n$ such that $\Sigma\, r_{z_j}(P)a^j = 0$ then

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$$\sum r_{z_j \bar{z}_k}(P) a^j \bar{a}^k \geq 0. \tag{1.2}$$

This hermitian form is called the Levi-form. Pseudo-convexity is a natural condition which, among other things, implies that the system (1.1) always has a solution, provided the α_j satisfy the necessary compatibility conditions:

$$\alpha_{j\bar{z}_k} = \alpha_{k\bar{z}_j}. \tag{1.3}$$

Now we want to distinguish between two types of regularity theorems: global and local. By global regularity we mean that, given smooth α_j on $\bar{M}$, there exists a smooth u on $\bar{M}$ satisfying (1.1). By local regularity in a neighborhood U of $P \in bM$, we mean that, given α_j, there exists u satisfying (1.1) such that the set of points in U at which the α_j are smooth contains the set in U at which u is smooth.

Local regularity holds whenever the Levi-form (1.2) is positive definite at P, in fact if the Levi-form is positive definite at every point of bM then the unique solution u of (1.1) which is orthogonal to the holomorphic functions possesses this property (as proved in [7]). Recently there has been great interest in boundary behavior of solutions of (1.1) on strongly pseudo-convex domains (i.e. (1.2) positive definite) and quite precise results have been obtained in terms of continuity and Hölder norms (see [6], [3] and [4]).

When the Levi-form is semidefinite but not strictly positive definite then further invariants control the question of local regularity (see [8]). If the Levi-form is identically zero in a neighborhood then local regularity does not hold (see [8]). On the other hand, here we obtain the following result, which is a special case of our main theorem (3.19).

(1.4) **Theorem.** *If* $M \subset\subset \mathbf{C}^n$ *is pseudo-convex with smooth boundary,* $\alpha_j \in C^\infty(\bar{M})$, $j = 1, \cdots, n$, *and if the* α_j *satisfy the compatibility conditions* (1.3) *then for each integer* m *there exists a function* $u_{(m)} \in C^m(\bar{M})$ *satisfying* (1.1).

The question immediately arises whether, under the assumptions of the above theorem, there exists a solution $u \in C^\infty(\bar{M})$. This remains an open problem; in particular, it would be interesting to know whether the solution u which is orthogonal to holomorphic functions is smooth.

It is convenient to use the standard notation:

$$\bar{\partial} u = \sum u_{\bar{z}_j} d\bar{z}_j, \qquad \alpha = \sum \alpha_j d\bar{z}_j$$

(1.5)

$$\bar{\partial}\alpha = \sum_{j \leq k} (\alpha_{j\bar{z}_k} - \alpha_{k\bar{z}_j}) d\bar{z}_j \wedge d\bar{z}_k.$$

The system (1.1) then becomes $\bar\partial u = \alpha$ and the compatibility condition (1.3) becomes $\bar\partial\alpha = 0$. The questions posed here are studied via the $\bar\partial$-Neumann problem with respect to the type of weights introduced by Hörmander in [5]. The essential tools are the techniques developed in [7] and [9]. In $\mathbf{C}^n$ this means considering the "weighted" hermitian form:

$$R_t(\phi, \psi) = (\bar\partial\phi, f_t\bar\partial\psi) + (\vartheta\phi, \vartheta(f_t\psi)) \tag{1.6}$$

where

$$f_t(z) = \exp(-\tfrac{1}{2}t|z|^2), \qquad \phi = \sum \phi_j d\bar{z}_j, \qquad \vartheta\phi = -\sum \phi_{jz_j},$$

and (,) is the sum of the L_2-inner products of the components. Given α we then show there exists a solution ϕ_t to the variational problem

$$R_t(\phi_t, \psi) = (\alpha, f_t\psi), \quad \text{for all } \psi, \tag{1.7}$$

under the boundary conditions

$$\sum r_{z_i}\phi_i = \sum r_{z_i}\psi_i = 0 \quad \text{on } bM. \tag{1.8}$$

Then a solution of (1.1) is given by

$$u_t = f_{-t}\vartheta(f_t\phi_t) \tag{1.9}$$

and we show that given m then $u_t \in C^m(\bar{M})$ if t is sufficiently large.

In case M is a manifold the above outline has to be somewhat modified. First, to define f_t, we must assume that there exists a function with properties analogous to $|z|^2$ (i.e. which is strongly plurisubharmonic in a neighborhood of bM). Second, in general, the $\bar\partial$-cohomology of M is different from zero and this is related to the fact that then R_t is not positive definite. We then consider the positive definite form $Q_t(\phi, \psi) = R_t(\phi, \psi) + (\phi, f_t\psi)$ and solve the variational problem (1.7) for this form. Using this solution we then obtain the desired solution of (1.1) under the additional necessary condition that α is orthogonal to the "harmonic space". Throughout this paper we follow the type of program set up in [8]; for more detailed explanations we refer to [1].

2. **The $\bar\partial$-Neumann problem with weights.** Let M' be a complex hermitian manifold of dimension n, and let $M \subset\subset M'$ be an open submanifold of M' whose closure, $\bar{M}$, is compact. We denote by bM the boundary of M and we assume that there exists a neighborhood U of bM and a real-valued function $r \in C^\infty(U)$ such that $dr \neq 0$ and $r(P) = 0$ if and only if $P \in bM$. We normalize r so that

$$|dr| = 1 \quad \text{on } bM, \tag{2.1}$$

where $| \ |$ denotes the length defined by the hermitian metric. The sign of r is chosen so that $r < 0$ in M and $r > 0$ outside of $\overline{M}$.

By $\mathfrak{A}^{p,q}$ we denote the space of forms of type (p, q) on $\overline{M}$ which are C^∞ up to and including the boundary, i.e. they are restrictions to $\overline{M}$ of C^∞ forms on M'. We set

$$\mathfrak{A} = \sum \mathfrak{A}^{p,q}. \tag{2.2}$$

In terms of local holomorphic coordinates $z_1, \cdots, z_n$ on a coordinate neighborhood V we can express $\phi \in \mathfrak{A}$ by

$$\phi = \sum \phi_{IJ} dz^I \wedge d\overline{z}^J, \tag{2.3}$$

where $\phi_{IJ} \in C^\infty(V \cap \overline{M})$; $I = (i_1, \cdots, i_p)$ with $1 \le i_1 < \cdots < i_p \le n$, $1 \le p \le n$; $J = (j_1, \cdots, j_q)$ with $1 \le j_1 < \cdots < j_q \le n$, $1 \le q \le n$; $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\overline{z}^J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$.

The operator $\overline{\partial}: \mathfrak{A} \to \mathfrak{A}$ is expressed in terms of local coordinates by

$$\overline{\partial}\phi = \sum \frac{\partial \phi_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz^I \wedge d\overline{z}^J = \sum (-1)^{|I|} \frac{\partial \phi_{IJ}}{\partial \overline{z}_k} \epsilon^{kJ}_{\langle kJ \rangle} dz^I \wedge d\overline{z}^{\langle kJ \rangle}, \tag{2.4}$$

where $|I| = p$, $\langle kJ \rangle$ is the ordered $(q + 1)$-tuple consisting of k, and the elements of J and

$$\epsilon^A_B = \begin{cases} 0 & \text{if } A \ne B, \\ 0 & \text{if } A \text{ has repeated elements}, \\ \text{sign of the permutation } A \to B & \text{otherwise}. \end{cases} \tag{2.5}$$

Then

$$\overline{\partial}(\mathfrak{A}^{p,q}) \subset \mathfrak{A}^{p,q+1} \quad \text{and} \quad \overline{\partial}^2 = 0. \tag{2.6}$$

The hermitian structure induces an inner product on $\mathfrak{A}$ for each $P \in M$. Thus if $\phi, \psi \in \mathfrak{A}$ we denote by $\langle \phi, \psi \rangle_P$ their inner product at P and by $\langle \phi, \psi \rangle$ the function whose value at P is $\langle \phi, \psi \rangle_P$. In terms of coordinates we have

$$\langle \phi, \psi \rangle = \sum \phi_{IJ} \overline{\psi}_{KL} g^{IJ,KL}, \tag{2.7}$$

where

$$g^{IJ,KL} = \langle dz^I \wedge d\overline{z}^J, dz^K \wedge d\overline{z}^L \rangle$$

$$= \begin{cases} g^{i_1 k_1} \cdots g^{i_{|I|} k_{|K|}} \overline{g}^{j_1 l_1} \cdots \overline{g}^{j_{|J|} l_{|L|}} & \text{if } |I| = |K| \text{ and } |J| = |L|; \\ 0 \quad \text{if either } |I| \ne |K| \text{ or } |J| \ne |L|, \end{cases} \tag{2.8}$$

and $g^{ik} = \langle dz_i, dz_k \rangle$, $\overline{g}^{jl} = \langle d\overline{z}_j, d\overline{z}_l \rangle$.

Further if $\phi, \psi \in \mathfrak{A}$ the L_2-inner product and norm are defined by

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle dV \quad \text{and} \quad \|\phi\|^2 = (\phi, \phi), \tag{2.9}$$

where dV is the volume element induced by the hermitian metric. We denote by $\tilde{\mathfrak{A}}$ the inner hilbert space obtained by completing $\mathfrak{A}$ under the norm $\| \ \|$.

The formal adjoint of $\bar\partial$ is denoted by ϑ: $\mathfrak{A} \to \mathfrak{A}$ and defined by the requirement that

$$(\vartheta\phi, \psi) = (\phi, \bar\partial\psi) \tag{2.10}$$

for all ψ with compact support in M. Then

$$\vartheta(\mathfrak{A}^{p,q}) \subset \mathfrak{A}^{p,q-1} \quad \text{and} \quad \vartheta^2 = 0. \tag{2.11}$$

If $P \in \bar{M}$ we denote by $\mathfrak{A}_P$ the space of forms at P, by T_P (and T_P^*) the real tangent (and cotangent) space to M' at P. Then for each cotangent $\eta \in T_P^*$ with $\eta \neq 0$ the symbol of $\bar\partial$ is a linear map $\sigma_P(\bar\partial, \eta)$: $\mathfrak{A}_P \to \mathfrak{A}_P$ given by

$$\sigma_P(\bar\partial, \eta)\theta = (\Pi_P\eta) \wedge \theta, \tag{2.12}$$

where Π_P: $\mathfrak{A}_P^{1,0} \oplus \mathfrak{A}_P^{0,1} \to \mathfrak{A}_P^{0,1}$ is the projection. The symbol of ϑ, denoted by $\sigma_P(\vartheta, \eta)$: $\mathfrak{A}_P \to \mathfrak{A}_P$, is then the adjoint of $\sigma_P(\bar\partial, \eta)$ with respect to the inner product $\langle \ , \ \rangle_P$. Integration by parts gives us

$$(\vartheta\phi, \gamma) = (\phi, \bar\partial\psi) + \int_{bM} \langle \sigma(\vartheta, dr)\phi, \psi \rangle dS, \tag{2.13}$$

for all $\phi, \psi \in \mathfrak{A}$, where dS is the volume element on bM. We define the space $\mathfrak{D} \subset \mathfrak{A}$ by

$$\mathfrak{D} = \{\phi \in \mathfrak{A} \mid \sigma_P(\vartheta, (dr)_P)\phi_P = 0 \text{ for all } P \in bM\}. \tag{2.14}$$

Thus we have

$$(\vartheta\phi, \psi) = (\phi, \bar\partial\psi) \tag{2.15}$$

whenever $\phi \in \mathfrak{D}$ and $\psi \in \mathfrak{A}$. On $\mathfrak{D}$ we define the hermitian form Q: $\mathfrak{D} \times \mathfrak{D} \to \mathbf{C}$ by

$$Q(\phi, \psi) = (\bar\partial\phi, \bar\partial\psi) + (\vartheta\phi, \vartheta\psi) + (\phi, \psi). \tag{2.16}$$

Let $\tilde{\mathfrak{D}}$ be the hilbert space obtained by completing $\mathfrak{D}$ under the norm $Q(\phi, \phi)^{1/2}$. Since we have

$$Q(\phi, \phi) \geq \|\phi\|^2, \tag{2.17}$$

there is a natural imbedding of $\tilde{\mathfrak{D}}$ in $\tilde{\mathfrak{A}}$. Then by a well-known theorem in hilbert space theory there exists a unique operator F with $\text{Dom}(F) \subset \tilde{\mathfrak{D}}$ such that

(2.18) $$Q(\phi, \psi) = (F\phi, \psi)$$

for all $\phi \in \mathrm{Dom}(F)$ and $\psi \in \mathfrak{D}$, and $\mathfrak{R}(F) = \tilde{\mathfrak{A}}$, where $\mathfrak{R}(F)$ denotes the range of F. F is called the Friedrichs representative of Q; it is selfadjoint and its inverse is bounded. For smooth forms in $\mathrm{Dom}(F)$ we have

(2.19) $$F\phi = \Box\phi + \phi,$$

where $\Box$ is the complex laplacian defined by

(2.20) $$\Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}.$$

In [7] (see also in [9] and [1]) the smooth elements of $\mathrm{Dom}(F)$ are characterized by the following boundary conditions:

(2.21) $$\mathrm{Dom}(F) \cap \mathfrak{A} = \{\phi \in \mathfrak{D} |\ \bar{\partial}\phi \in \mathfrak{D}\}.$$

The space of harmonic forms $\mathfrak{H}$ is defined by

(2.22) $$\mathfrak{H} = \{\phi \in \tilde{\mathfrak{D}} \mid Q(\phi, \phi) = \|\phi\|^2\}.$$

It then follows that $\mathfrak{H}$ can also be characterized by the following:

(2.23) $$\mathfrak{H} = \{\phi \in \tilde{\mathfrak{D}} |\ \bar{\partial}\phi = 0 \text{ and } \bar{\partial}^*\phi = 0\} = \{\phi \in \mathrm{Dom}(F) |\ F\phi = \phi\};$$

here $\bar{\partial}$ denotes the L_2-closure of $\bar{\partial}$ and $\bar{\partial}^*$ its L_2-adjoint (noting that $\mathrm{Dom}(\bar{\partial}^*) \subset \tilde{\mathfrak{D}}$). Now we formulate the *L_2-$\bar{\partial}$-Neumann problem* as follows:

(2.24) **Problem.** Find a bounded operator $N: \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ such that

(I) $\mathfrak{R}(N) \subset \mathrm{Dom}(F)$,

(II) $\mathfrak{N}(N) = \mathfrak{H}$, where $\mathfrak{N}(N)$ denotes the null space of N,

(III) $\mathfrak{R}(N) \perp \mathfrak{H}$,

(IV) if $\alpha \in \tilde{\mathfrak{A}}$ and $\alpha \perp \mathfrak{H}$, then

(2.25) $$Q(N\alpha, \psi) = (\alpha, \psi) + (N\alpha, \psi) \quad \text{for all } \psi \in \mathfrak{D}.$$

It is clear that if such an operator N exists, satisfying (I) through (IV), then it is unique. Furthermore, N also satisfies the following:

(A) If $H: \tilde{\mathfrak{A}} \to \mathfrak{H}$ denotes the orthogonal projection then whenever $\alpha \in \tilde{\mathfrak{A}}$, we have the orthogonal decomposition

(2.26) $$\alpha = \bar{\partial}\vartheta N\alpha + \vartheta\bar{\partial}N\alpha + H\alpha.$$

(B) $\bar{\partial}N = N\bar{\partial}$, $\bar{\partial}^*N = N\bar{\partial}^*$ and $FN = NF$.

(C) If $\alpha \in \tilde{\mathfrak{A}}$ and $\bar{\partial}\alpha = 0$ (i.e. α is in the domain of the L_2 closure of $\bar{\partial}$) and if $\alpha \perp \mathfrak{H}$ then

(2.27) $$\alpha = \bar{\partial}\vartheta N\alpha,$$

and $\phi = \vartheta N\alpha$ is the unique solution of the equation $\alpha = \bar{\partial}\phi$ which is orthogonal to $\mathfrak{H}$.

Let λ be a C^∞ nonnegative function on M'. With λ fixed and for each $t \geq 0$ we will define the $\bar\partial$-Neumann problem of weight t; the above problem will then correspond to $t = 0$. For $\phi \in \mathfrak{A}$

$$(2.28)\qquad (\phi, \psi)_{(t)} = (\phi, e^{-t\lambda}\psi), \qquad \|\phi\|^2_{(t)} = (\phi, \phi)_{(t)}.$$

Observe that the norms $\|\ \|_{(t)}$ are equivalent to the norm $\|\ \|_0 = \|\ \|$, hence a function is in the completion under any of these norms if and only if it is square integrable. So that for each t the inner product (2.28) can be extended to $\tilde{\mathfrak{A}}$. We define $\vartheta_t\colon \mathfrak{A} \to \mathfrak{A}$ by

$$(2.29)\qquad \vartheta_t\phi = e^{\lambda t}\vartheta(e^{-\lambda t}\phi) = \vartheta\phi - t\sigma(\vartheta, d\lambda)\phi$$

so we see that

$$(2.30)\qquad \sigma(\vartheta_t, \eta) = \sigma(\vartheta, \eta).$$

Further, we have as in (2.13),

$$(2.31)\qquad (\vartheta_t\phi, \psi)_{(t)} = (\phi, \bar\partial\psi)_{(t)} + \int_{bM} \langle\sigma(\vartheta, dr)\phi, e^{-\lambda t}\psi\rangle\, dS,$$

so that, when $\phi \in \mathfrak{D}$, the boundary term vanishes.

We define $Q_t\colon \mathfrak{D} \times \mathfrak{D} \to \mathbf{C}$ by

$$(2.32)\qquad Q_t(\phi, \psi) = (\bar\partial\phi, \bar\partial\psi)_{(t)} + (\vartheta_t\phi, \vartheta_t\psi)_{(t)} + (\phi, \psi)_{(t)}.$$

Again, the norm $Q_t(\phi, \phi)^{1/2}$ is equivalent to $Q(\phi, \phi)^{1/2}$ for each t and hence, for each t, the completion of $\mathfrak{D}$ under Q_t may be identified with $\tilde{\mathfrak{D}}$ and, for each t, the inner product (2.32) extends to $\tilde{\mathfrak{D}}$.

As in (2.18), we have for each t and each $\alpha \in \tilde{\mathfrak{A}}$ a unique $\phi_t \in \tilde{\mathfrak{D}}$ such that

$$(2.33)\qquad Q_t(\phi_t, \psi) = (\alpha, \psi)_{(t)} \quad \text{for all } \psi \in \mathfrak{D}.$$

Then F_t, the Friedrich's representative of Q_t, is defined by $F_t\phi_t = \alpha$ and $\mathrm{Dom}(F_t) = \mathrm{Dom}(F)$ is independent of t. We have

$$(2.34)\qquad F_t\phi = \square_t\phi + \phi,$$

where

$$(2.35)\qquad \square_t = \bar\partial\vartheta_t + \vartheta_t\bar\partial.$$

Analogously to (2.22) we define the space of weighted harmonic forms $\mathcal{H}_t$ by

$$(2.36)\qquad \mathcal{H}_t = \{\phi \in \tilde{\mathfrak{D}} \mid Q_t(\phi, \phi) = \|\phi\|^2_{(t)}\}.$$

We can now formulate the *$\bar\partial$-Neumann problem of weight t* with respect to the function λ as follows: Find a bounded operator N_t: $\tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ which satisfies properties I, II, III and IV of 2.24 and hence also (A), (B) and (C) with N, F, $\mathcal{H}$, Q, H and ϑ replaced by N_t, F_t, $\mathcal{H}_t$, Q_t, H_t and ϑ_t respectively, and $\perp$ and adjoint are understood with respect to the inner product $(\ ,\ )_t$.

Our interest centers principally on property (C) which we can reformulate as follows:

(2.37) **Proposition.** *If $\alpha \in \tilde{\mathfrak{A}}$ and if α is in the domain of the L_2-closure of $\bar\partial$ with $\bar\partial\alpha = 0$ and if*

$$(\alpha, \psi) = 0, \tag{2.38}$$

wherever $\vartheta\psi = 0$ and $\psi \in \mathfrak{D}$ and if for some t the $\bar\partial$-Neumann problem of weight t is solvable in the above sense, then there exists u such that $\bar\partial u = \alpha$ and the unique u satisfying this equation which is orthogonal to $\mathcal{H}_t$ (in the $(\ ,\ )_{(t)}$ inner product) is given by $u = \vartheta_t N_t \alpha$.

Proof. We simply observe that if $\vartheta_t\theta = 0$, i.e. $\vartheta(e^{-\lambda t}\theta) = 0$, then (2.38) implies

$$(\alpha, \theta)_{(t)} = 0, \text{ hence } \alpha \perp \mathcal{H}_t, \text{ under } (\ ,\ )_{(t)},$$

and since $\bar\partial\alpha = 0$ we have the desired result by property (C).

3. **The main theorem.** The concepts of plurisubharmonic and pseudo-convex are well-known in the theory of several complex variables, we recall them below.

(3.1) **Definition.** If $P \in M'$ we denote by $T_P^{1,0}$ the vectors of type $(1, 0)$ at P; if $L \in T_P^{1,0}$ then in local coordinates we have

$$L = \sum_{j=1}^{n} a^j \frac{\partial}{\partial z_j}. \tag{3.2}$$

If $\lambda \in C^2(M')$ we define the *complex Hessian* of λ at P to be the hermitian form on $T_P^{1,0}$ given by

$$L \mapsto \langle(\partial\bar\partial\lambda)_P, L \wedge \bar{L}\rangle. \tag{3.3}$$

The function λ is call *plurisubharmonic* at P if the complex Hessian is positive semidefinite and it is called *strongly plurisubharmonic* at P if the complex Hessian is positive definite.

In terms of local coordinates the complex Hessian at P is represented by the matrix $((\partial^2\lambda/\partial z_j\partial\bar z_k)_P)$.

(3.4) **Definition.** M is called *pseudo-convex* if for each $P \in bM$ the restriction of the complex Hessian of r to $T_P^{1,0} \cap \mathbf{C}T_P(bM)$ is semidefinite and M is called *strongly pseudo-convex* if this restriction is positive definite. Here $CT_P(bM)$ denotes the complexified tangent space of bM at P.

In terms of local coordinates, we are looking at the hermitian form,

$$(a^1, \cdots, a^n) \mapsto \sum_{j,k} \left(\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}\right)_P a^j \bar{a}^k, \tag{3.5}$$

where

$$\sum_j a^j \left(\frac{\partial r}{\partial z_j}\right)_P = 0. \tag{3.6}$$

(3.7) **Theorem.** *If M is pseudo-convex and if there exists a function λ on M' which is strongly plurisubharmonic in some neighborhood of bM, then there exist a nonnegative function $\mu \in C^\infty(M)$ and a compact set $K \subset M$ such that μ is strongly plurisubharmonic outside of K, and, if for each $c \geq 0$, M_c is defined by*

$$M_c = \{P \in M |\ \mu(P) < c\}, \tag{3.8}$$

then $\bar{M}_c \subset\subset M$ is compact.

Proof. We will show that the function $\mu \in C^\infty(M)$ which outside of a sufficiently large compact set K is given by

$$\mu = -\log |r| + C\lambda, \tag{3.9}$$

where C is a large constant, satisfies the properties required above.

Let $z_1, \cdots, z_n$ be a holomorphic coordinate system with origin on bM and domain U. Suppose $P \in U$ and $L \in T_P^{1,0}$ is given by (3.2). We write

$$L = \tau_P(L) + \nu_P(L), \tag{3.10}$$

where

$$\nu_P(L) = \frac{\sum a^m r_{z_m}(P)}{\sum |r_{z_m}(P)|^2} \sum r_{\bar{z}_j}(P) \frac{\partial}{\partial z_j}. \tag{3.11}$$

Then the vector $\tau_P(L)$ is in the tangent space at P of the surface $r = r(P)$. The pseudo-convexity of M implies that there exist $\epsilon > 0$ and $A > 0$ such that

$$\sum r_{z_j \bar{z}_k}(P)(\tau_P(L))_j \overline{(\tau_P(L))_k} \geq - A|L|^2 |r(P)| \tag{3.12}$$

whenever $|r(P)| < \epsilon$ and $P \in U$, where

$$|L|^2 = \sum_{m=1}^n |a^m|^2. \tag{3.13}$$

To see this choose ϵ so small that the points P with $|r(P)| < \epsilon$ are in a smooth product neighborhood V of bM. Denote by $\pi\colon V \longrightarrow bM$ the projection in this product. Let $k(P)$ denote the left side of (3.12); since k is smooth we have

$$k(P) - k(\pi(P)) = O(|L|^2|r(P)|)$$

when $|r(P)| < \epsilon$ and $P \in V$. This combined with the pseudo-convexity of M yields (3.12).

From (3.10) we obtain

$$\begin{aligned} \sum \mu_{z_j \bar z_k} a^j \bar a^k &= \frac{1}{|r|} \sum r_{z_j \bar z_k} a^j \bar a^k + \frac{1}{r^2} \left| \sum r_{z_i} a^i \right|^2 + C \sum \lambda_{z_j \bar z_k} a^j \bar a^k \\ &= \frac{1}{|r|} \sum r_{z_j \bar z_k} \tau(L)_j \overline{\tau(L)_k} + \frac{1}{r^2} |L(r)|^2 + C \sum \lambda_{z_j \bar z_k} a^j \bar a^k \\ &\quad + O\left(\frac{1}{|r|} \{ |\tau(L)| |\nu(L)| + |\nu(L)|^2 \} \right). \end{aligned} \tag{3.14}$$

The strong plurisubharmonicity of μ for small r is then a consequence of (3.12), (3.14), the fact that $|L(r)| = |\nu(L)|$, the estimate

$$(1/|r|)|\tau(L)||\nu(L)| \leq \text{large const}\, |\tau(L)|^2 + \text{small const}\, (1/r^2)|\nu(L)|^2,$$

and the fact that λ is strongly plurisubharmonic near bM. The remaining properties that are needed are immediate consequences of the definition of μ.

Observe that whenever bM has a neighborhood U which is Stein then there exists a function λ which is strongly plurisubharmonic on a neighborhood $V \subset\subset U$ of bM. Namely, let $f_1, \cdots, f_N$ be a set of holomorphic functions which separate points of $\bar V$ and such that the $\{df_j\}$ span the whole n-dimensional space of (1, 0)-forms at each point of $\bar V$. Then the function

$$\lambda = \sum_{m=1}^{N} |f_m|^2 \tag{3.15}$$

is strongly plurisubharmonic on V since

$$\sum \lambda_{z_j \bar z_k} a^j \bar a^k = \sum_m \left| \sum_j f_{m z_j} a^j \right|^2. \tag{3.16}$$

In [2] Grauert gives examples of pseudo-convex manifolds on which there does not exist a function μ as described in the above theorem.

To formulate the main theorem we use Sobolev norms on the space $\mathfrak{A}$. For each nonnegative integer s we define the norm $\| \ \|_s$ on $\mathfrak{A}$ by

$$\|\phi\|_s^2 = \sum_\nu \sum_{|\alpha| \leq s} \sum_{I,J} \int_{U_\nu \cap M} |D^\alpha \phi_{IJ}^{(\nu)}|^2 \, dV, \tag{3.17}$$

where U_ν is a finite covering of $\bar{M}$ by holomorphic coordinate systems, $\phi_{IJ}^{(\nu)}$ are the components of ϕ (as in (2.3)) with respect to the coordinates in U_ν; as usual $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_{2n}^{\alpha_{2n}}$, with $x_j = \mathrm{Re}\,(z_j)$, $x_{j+n} = \mathrm{Im}\,(z_j)$, $j = 1, \cdots, u$, and $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$. The topology defined by this norm is independent of the choice of coordinate cover; we denote by $\tilde{\mathfrak{A}}_s$ the completion of $\mathfrak{A}$ under $\| \ \|_s$. Here we recall the well-known relation between differentiability and the Sobolev spaces. For each integer r we denote by $\mathfrak{A}_r$ the space of forms whose components in each coordinate neighborhood U are in $C^r(U \cap \bar{M})$, and set $\mathfrak{A}_\infty = \mathfrak{A}$.

(3.18) **Lemma.** *$\tilde{\mathfrak{A}}_s \subset \mathfrak{A}_{s-n}$ whenever $s \geq n$. In particular, if $\phi \in \tilde{\mathfrak{A}}_s$ for all s then $\phi \in \mathfrak{A}$.*

The principal result of this paper is given in the following.

(3.19) **Main theorem.** *If $M \subset\subset M'$ is a pseudo-convex manifold with a smooth boundary and if there exists a nonnegative function λ on M' which is strongly plurisubharmonic in a neighborhood of bM, then there exists a number T_0 such that the $\bar{\partial}$-Neumann problem of weight t is solvable (in the sense of §2) for each $t \geq T_0$. Furthermore, for each s there exists a number T_s such that*

$$N_t(\tilde{\mathfrak{A}}_s) \subset \tilde{\mathfrak{A}}_s, \quad H_t(\tilde{\mathfrak{A}}_s) \subset \tilde{\mathfrak{A}}_s, \quad \mathcal{H}_t \subset \tilde{\mathfrak{A}}_s \tag{3.20}$$

whenever $t \geq T_s$. Finally, whenever $t \geq T_0$ then $\mathcal{H}_t^{p,q}$, with $q \geq 1$, is finite dimensional, its dimension is independent of t and it represents the $\bar{\partial}$-cohomology of M.

As in Proposition (2.37), we obtain the following consequence of the main theorem by applying Lemma (3.18).

(3.21) **Corollary.** *Under the same hypotheses as above, the operators N_t and H_t map $\tilde{\mathfrak{A}}_s$ into $\mathfrak{A}_{s-n}$ provided $t \geq T_s$; in particular they map $\mathfrak{A}$ into $\mathfrak{A}_{s-n}$. Furthermore if $\alpha \in \tilde{\mathfrak{A}}_s$, $\bar{\partial}\alpha = 0$ and* (2.38) *holds then there exists $u \in \tilde{\mathfrak{A}}_s$ (and hence $u \in \mathfrak{A}_{s-n}$) such that $\bar{\partial}u = \alpha$. In particular if $\alpha \in \mathfrak{A}$ with $\bar{\partial}\alpha = 0$ and* (2.38) *holding then, for each m, there exists a $u \in \mathfrak{A}_m$ such that $\bar{\partial}u = \alpha$.*

4. **A priori estimates.** Our starting point is the following estimate which is proved by Hörmander in [5].

(4.1) **Proposition.** *If $M \subset M'$ is pseudo-convex and if there exists a nonnegative function λ on M' which is strongly plurisubharmonic in a neighborhood of bM then there exists a function $f \in C_0^\infty(M)$ and constants T, C and for each t a constant C_t such that*

$$t\|\phi\|_{(t)}^2 \leq CQ_t(\phi, \phi) + C_t\|f\phi\|_1^2, \tag{4.2}$$

whenever $\phi \in \mathfrak{D}^{p,q}$ *with* $q \geq 1$ *and* $t \geq T$, *where* $\| \; \|_{(t)}$ *and* $\| \; \|_1$ *are the norms defined by* (2.28) *and* (3.17) *respectively.*

First we have the following lemma which is an immediate consequence of the ellipticity of Q.

(4.3) **Lemma.** *If* $g \in C_0^\infty(M)$ *and* $g' \in C_0^\infty(M)$ *so that* $g' = 1$ *on the support of* g *and if* $\alpha \in \mathfrak{A}$, $\phi \in \mathfrak{D}$ *with*

$$Q_t(\phi, \psi) = (\alpha, \psi)_{(t)} \tag{4.4}$$

for all $\psi \in \mathfrak{D}$ *and some fixed* t, *then, for each* $s \geq 0$, *there exists a constant* $C > 0$ *so that*

$$\|g\phi\|_{s+2} \leq C\|g'\alpha\|_s. \tag{4.5}$$

(4.6) **Theorem.** *Under the same hypotheses as Proposition* (4.1), *for each nonnegative integer* s *there exist constants* T_s *and* C_s *such that if* $\alpha \in \mathfrak{A}^{p,q}$ *with* $q \geq 1$ *and if* (4.3) *holds with* $\phi \in \mathfrak{D}^{p,q}$ *for all* $\psi \in \mathfrak{D}^{p,q}$ *and* $t \geq T_s$ *then we have*

$$\|\phi\|_s \leq C_s\|\alpha\|_s. \tag{4.7}$$

Proof. The argument is along the lines of the proof of Theorem 2 in [9]. We choose U to be a suitably small neighborhood of $P \in bM$ on which there is defined a boundary coordinate system as well as a boundary complex frame, which we describe below. A *boundary coordinate system* is a system of real coordinates $t_1, \cdots, t_{2n-1}, r$, where r is the function defining the boundary described in §2. A *boundary complex frame* is a set of orthonormal vector fields, $L_1, \cdots, L_n$ on U of type $(1, 0)$, such that $L_j(r) = 0$ if $j = 1, \cdots, 2n-1$ and $L_n(r) = 1$. We let $\omega^1, \cdots, \omega^n$ be the bases of $(1, 0)$-forms on U which is dual to $L_1, \cdots, L_n$. Any form ϕ on $U \cap \overline{M}$ can be expressed in terms of the ω as follows:

$$\phi = \sum \phi_{IJ}\omega^I \wedge \bar{\omega}^J, \tag{4.8}$$

where $\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$ and $\bar{\omega}^J = \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q}$. The restrictions of forms in $\mathfrak{D}$ to $U \cap \overline{M}$ are then characterized by

$$\phi_{IJ}|_{bM} = 0 \quad \text{whenever } n \in J. \tag{4.9}$$

If α is a $(2n)$-tuple of integers we denote by D^α the operator given by

$$D^\alpha = (-i)^{|\alpha|}\partial^{|\alpha|}/\partial t_1^{\alpha_1} \cdots \partial t_{2n-1}^{\alpha_{2n-1}}\partial r^{\alpha_{2n}} \tag{4.10}$$

and if α is a $(2n-1)$-tuple we define D_b^α by

$$D_b^\alpha = (-i)^{|\alpha|}\partial^{|\alpha|}/\partial t_1^{\alpha_1}\cdots\partial t_{2n-1}^{\alpha_{2n-1}}. \tag{4.11}$$

These operators will be interpreted as acting on forms on $U \cap \bar M$ by applying them to each component in (4.8). Thus if ϕ is the restriction to $U \cap \bar M$ of a form in $\mathfrak{D}$ then so is $D_b^\alpha\phi$. Now if $\zeta, \zeta' \in C_0^\infty(U \cap \bar M)$ and $\zeta' = 1$ on the support of ζ, we wish to prove the estimate

$$t \sum_{|\alpha|\le s} \|\zeta D^\alpha\phi\|_{(t)}^2 \le C_s \sum_{|\alpha|\le s} \|\zeta' D^\alpha\phi\|_{(t)}^2 + C_{s,t}\|\alpha\|_s^2 \tag{4.12}$$

for $\alpha, \phi \in \mathfrak{D}^{p,q}$ satisfying (4.3), where C_s is independent of t.

Before proving (4.12) we will show how the desired estimate (4.7) follows from (4.12) and Lemma (4.2). Let $U_1, \cdots, U_N$ be a covering of bM by neighborhoods such as described above, and let $\zeta_0, \zeta_1, \cdots, \zeta_N$ be functions in $C^\infty(\bar M)$ such that $\operatorname{supp}(\zeta_0) \subset\subset M$ and $\operatorname{supp}(\zeta_j) \subset\subset U_j \cap \bar M$ for $j = 1, \cdots, N$ and such that

$$\sum_{j=0}^N \zeta_j = 1. \tag{4.13}$$

It follows from (4.12) that

$$\begin{aligned} &t \sum_{j=1}^N \sum_{|\alpha|\le s} \|\zeta_j D^\alpha\phi\|_{(t)}^2 \\ &\le C_s \sum_{j=1}^N \sum_{|\alpha|\le s} \{\|\zeta_j'(\zeta_0'-1)D^\alpha\phi\|_{(t)}^2 + \|\zeta_j'\zeta_0' D^\alpha\phi\|_{(t)}^2\} + C_{s,t}\|\alpha\|_s^2, \end{aligned} \tag{4.14}$$

where $\operatorname{supp}(\zeta_0') \subset\subset M$, $\operatorname{supp}(\zeta_j') \subset\subset U_j \cap \bar M$ for $j = 1, \cdots, N$ and $\zeta_j' = 1$ on the support of ζ_j for $j = 0, \cdots, N$.

Now since $\Sigma_{j=1}^N \zeta_j'|\zeta_0' - 1| \le \text{const}\ \Sigma_{j=1}^N \zeta_j$ then for t sufficiently large the first term on the right is smaller than the left-hand side of (4.14) and hence can be "absorbed". The second term on the right is less that $\|g\phi\|_s^2$ for a suitable $g \in C_0^\infty(M)$ and hence can be estimated by (4.5) which yields the desired (4.7).

Now we will prove (4.12) by induction on s. We observe that (4.3) implies

$$Q_t(\phi, \phi) = (\alpha, \phi)_t \le \|\phi\|_{(t)}^2 + \|\alpha\|_{(t)}^2; \tag{4.15}$$

this combined with (4.2) and (4.5) yields the desired result for $s = 0$ if t is sufficiently large. Suppose that (4.7) has been proven for $s - 1$. Since $\phi \in \mathfrak{D}^{p,q}$ then $\zeta D_b^\gamma\phi \in \mathfrak{D}^{p,q}$ and substituting this in (4.2) we have

$$t\|\zeta D_b^\gamma\phi\|_{(t)}^2 \le CQ_t(\zeta D_b^\gamma\phi, \zeta D_b^\gamma\phi) + C_t\|f\zeta D_b^\gamma\phi\|_1^2 \tag{4.16}$$

if $|\gamma| = s$, the last term can be estimated by $\|g\phi\|_{s+1}^2$ with a suitable $g \in C_0^\infty(M)$ and hence by const $\|\alpha\|_s^2$ using (4.5). Integration by parts gives

$$
\begin{aligned}
Q_t(\zeta D_b^\gamma\phi, \zeta D_b^\gamma\phi) \\
&= Q_t(\phi, D_b^\gamma \zeta^2 D_b^\gamma \phi) \\
&+ O\Big\{(\|[\bar\partial, \zeta D_b^\gamma]\phi\|_{(t)} + \|[\boldsymbol{\vartheta}_t, \zeta D_b^\gamma]\phi\|_{(t)} + \|\zeta D_b^\gamma\phi\|_{(t)}) \\
(4.17)\qquad &+ C_t \sum_{|\beta|<s} \sqrt{Q_t(\zeta' D_b^\beta\phi, \zeta' D_b^\beta\phi)}\,\sqrt{Q_t(\zeta D_b^\gamma\phi, \zeta D_b^\gamma\phi)} \\
&+ (\bar\partial\phi[[\zeta D_b^\gamma, \bar\partial], \zeta D_b^\gamma]\phi)_{(t)} + (\boldsymbol{\vartheta}_t\phi, [[\zeta D_b^\gamma, \boldsymbol{\vartheta}_t], \zeta D_b^\gamma]\phi)_{(t)} \\
&+ \|[\zeta D_b^\gamma, \bar\partial]\phi\|_{(t)}^2 + \|[\zeta D_b^\gamma, \boldsymbol{\vartheta}_t]\phi\|_{(t)}^2\Big\}.
\end{aligned}
$$

By (2.29) we have

$$
(4.18)\qquad [\boldsymbol{\vartheta}_t, \zeta D_b^\gamma] = [\boldsymbol{\vartheta}, \zeta D_b^\gamma] - t[\sigma(\boldsymbol{\vartheta}, d\lambda), \zeta D_b^\gamma].
$$

Note that the first term on the right is an operator of order s and is independent of t, and that the second term on the right is of order $s-1$. Thus the error term in (4.17) can be estimated by

$$
\begin{aligned}
&\text{small const } Q_t(\zeta D_b^\gamma\phi, \zeta D_b^\gamma\phi) + \text{large const} \sum_{|\beta|=s} \|\zeta' D^\beta\phi\|_{(t)}^2 \\
(4.19)\qquad &+ C(t)\|\zeta'\phi\|_{s-1}^2,
\end{aligned}
$$

where the first two constants are independent of t and ζ' has compact support in $U \cap \bar M$ with $\zeta' = 1$ on the support of ζ. By (4.3) and integration by parts, we have

$$
(4.20)\qquad Q_t(\phi, D_b^\gamma\zeta^2 D_b^\gamma\phi) = (\zeta D_b^\gamma\alpha, \zeta D_b^\gamma\phi).
$$

Choosing a covering and a partition of unity as above, summing over γ with $|\gamma| \le s$ and over the partition we obtain, using the inductive hypothesis,

$$
(4.21)\qquad t\sum_{j,|\gamma|\le s} \|\zeta_j D_b^\gamma\phi\|_{(t)}^2 \le \text{const} \sum_{j,|\beta|\le s} \|\zeta_j' D^\beta\phi\|_{(t)}^2 + C(t)\|\alpha\|_s^2,
$$

where the first constant on the right is independent of t. To conclude the proof of the desired estimate (4.7) we must control the normal derivatives. Since Q is elliptic the boundary bM is noncharacteristic and so we have

$$
(4.22)\qquad \left\|\frac{\partial}{\partial r}(\zeta\phi)\right\|_{(t)}^2 \le \text{const}\Big\{Q_t(\zeta\phi, \zeta\phi) + \sum_{|\theta|\le 1} \|\zeta' D_b^\theta\phi\|_{(t)}^2\Big\}
$$

for all $\phi \in \mathfrak{D}$, the constant being independent of t. Replacing ϕ with $D_b^\sigma\phi$, where $|\sigma| = s-1$, applying (4.17), (4.19) summing over the partitions of unity we obtain

$$(4.23)\quad \sum_{j,|\sigma|<s-1} \left\|\zeta_j \frac{\partial}{\partial r} D_b^\sigma \phi\right\|_{(t)}^2 \le \text{const} \sum_{j,|\gamma|\le s} \|\zeta_j' D_b^\gamma \phi\|_{(t)}^2 + C(t)\|\alpha\|_{s-1}^2,$$

again the first constant is independent of t. The condition (4.3) implies that

$$(4.24)\qquad \alpha = \Box_t + \phi;$$

this is a determined elliptic system in which the second order terms are independent of t, hence in $U \cap \overline{M}$, we can solve for $\partial^2\phi/\partial r^2$ obtaining

$$(4.25)\quad \frac{\partial^2\phi}{\partial r^2} = A\alpha + \sum_{|\theta|=2} B_\theta D_b^\theta \phi + \sum_{|\theta|=1} C_\theta \frac{\partial}{\partial r} D_b^\theta \phi + \text{first order terms},$$

where A, B_θ and C_θ are matrices independent of the parameter t. Applying D_b^σ with $|\sigma| = s-2$ to (4.25), taking the norm $\| \ \|_{(t)}$ and combining with (4.23) we obtain

$$(4.26)\qquad \sum_{j,|\sigma|\le s-2} \left\|\zeta_j \frac{\partial^2}{\partial r^2} D_b^\sigma \phi\right\|_{(t)}^2 \le \cdots,$$

where the dots stand for the right-hand side of (4.23). By successive differentiations of (4.25) and proceeding similarly we obtain

$$(4.27)\qquad \sum_{j,|\sigma|\le s-k} \left\|\zeta_j \left(\frac{\partial}{\partial r}\right)^k D_b^\sigma \phi\right\|_{(t)}^2 \le \cdots,$$

for $k = 1, \cdots, s$, where again the dots represent the right side of (4.26). The desired inequality (4.17) is finally obtained by using the inequalities (4.27) to replace the first term on the right of (4.21) by the first term on the right of (4.23); then t is chosen large enough to absorb this term in the left-hand side. This gives the desired estimate for the tangential derivatives of order s and hence, by (4.27), also of all other derivatives of order s.

5. **Proof of the main theorem.** We begin by "regularizing" the a priori estimates of the previous section. That is, we want to show that the derivatives for which we obtained a priori bounds actually exist.

Throughout this section we will assume that $M \subset\subset M'$ is pseudo-convex and that there exists $\lambda \in C^\infty(M')$ which is strongly plurisubharmonic in a neighborhood of bM.

(5.1) **Proposition.** *For each nonnegative integer s there exists a number T_s such that for any fixed $t \ge T_s$ and $\alpha \in \tilde{\mathfrak{A}}^{p,q}$ with $q \ge 1$ if $\phi_t \in \tilde{\mathfrak{D}}^{p,q}$ such that*

$$(5.2)\qquad Q_t(\phi_t, \psi) = (\alpha, \psi)_{(t)}$$

for all $\psi \in \mathfrak{D}$ *then* $\phi \in \tilde{\mathfrak{A}}_s$. *Furthermore* ϕ *satisfies the estimate* (4.17).

Proof. The proof follows exactly the same lines as §4 in [9]. We outline it here. For $\delta \geq 0$ let Q_t^δ be the quadratic form defined by

$$(5.3)\qquad Q_t^\delta(\phi, \psi) = Q_t(\phi, \psi) + \delta(\phi, \psi)_1$$

for $\phi, \psi \in \mathfrak{D}$, where $(\ ,\ )_1$ denotes the inner product associated with the Sobolev norm $\|\ \|_1$. Let $\tilde{\mathfrak{D}}_\delta$ be the completion of $\mathfrak{D}$ under Q_t^δ; this is independent of t and for $\delta \geq 0$ it is also independent of δ and $\tilde{\mathfrak{D}}_\delta \subset \tilde{\mathfrak{D}}$. If $\alpha \in \tilde{\mathfrak{A}}^{p,q}$ then, by the Friedrichs' representation theorem, there exists a unique $\phi_t^\delta \in \tilde{\mathfrak{D}}_\delta^{p,q}$ such that

$$(5.4)\qquad Q_t^\delta(\phi_t^\delta, \psi) = (\alpha, \psi)_{(t)}$$

for all $\psi \in \mathfrak{D}^{p,q}$. Furthermore, if $\delta > 0$ then ϕ_t^δ is a solution of a coercive boundary value problem and hence $\phi_t^\delta \in \tilde{\mathfrak{A}}_2^{p,q}$ if $\alpha \in \tilde{\mathfrak{A}}_s^{p,q}$ then $\phi_t^\delta \in \tilde{\mathfrak{A}}_{s+2}^{p,q}$. By the same method as in Theorem (4.6) it can be shown that if $t \geq T_s$ and $\delta > 0$ then

$$(5.5)\qquad \|\phi_t^\delta\|_s \leq \text{const}\|\alpha\|_s.$$

Then we can choose a sequence $\{\delta_\nu\}$ with $\lim \delta_\nu = 0$ such that the sequence $\{\nu^{-1}(\phi_t^{\delta_1} + \cdots + \phi_t^{\delta_\nu})\}$ converges in $\|\ \|_s$; it is easy to see that the limit is the unique ϕ_t satisfying (5.2) and hence $\phi_t \in \tilde{\mathfrak{A}}_s^{p,q} \cap \tilde{\mathfrak{D}}$.

(5.6) **Lemma.** *For any integer* s, $\tilde{\mathfrak{A}}_s^{p,q} \cap \text{Dom}(F_t)$ *is dense in* $\text{Dom}(F_t) \cap \tilde{\mathfrak{D}}^{p,q}$ *in the norm* Q_t *if* $q \geq 1$ *and* $t \geq T_s$.

Proof. If $\phi \in \text{Dom}(F_t) \cap \tilde{\mathfrak{D}}^{p,q}$ let $\alpha = F_t\phi$, let $\alpha_\nu \in \mathfrak{A}^{p,q}$ such that $\alpha = \lim \alpha_\nu$ in $\tilde{\mathfrak{A}}^{p,q}$, let ϕ_ν be such that $Q_t(\phi_\nu, \psi) = (\alpha_\nu, \psi)_{(t)}$ for all $\psi \in \mathfrak{D}^{p,q}$. By the above $\phi_\nu \in \tilde{\mathfrak{A}}_s^{p,q}$ and we have

$$Q_t(\phi_\nu - \phi, \phi_\nu - \phi) \leq \|\alpha_\nu\|_{(t)}\|\phi_\nu - \phi\|_{(t)} \leq \|\alpha_\nu\|_{(t)}\sqrt{Q_t(\phi_\nu - \phi, \phi_\nu - \phi)}$$

which proves that $\phi = \lim \phi_\nu$ in the norm Q_t.

(5.7) **Lemma.** *If* $q \geq 1$ *and* t *sufficiently large then* $\mathcal{H}_t^{p,q}$ *is finite dimensional and there exists* $C > 0$ *such that for all* $\phi \in \tilde{\mathfrak{D}}^{p,q}$ *with* $\phi \perp \mathcal{H}_t^{p,q}$ *we have*

$$(5.8)\qquad \|\phi\|_{(t)}^2 \leq C(\|\bar{\partial}\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2).$$

Proof. From (4.1) and Lemma (4.2) we have

$$(5.9)\qquad \|\phi\|_{(t)}^2 \leq C(\|\bar{\partial}\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2 + \|gF_t(\phi)\|_{-1}^2),$$

where $g \in C_0^\infty(M)$. If $\phi \in \mathcal{H}_t^{p,q}$ then the above reduces to $\|\phi\|_{(t)}^2 \leq C\|g\phi\|_{-1}^2$. This implies that unit sphere in $\mathcal{H}_t^{p,q}$ is compact and hence $\mathcal{H}_t^{p,q}$ is finite dimensional. Further, we have

$$(5.10) \qquad \|gF_t(\phi)\|_{-1}^2 \leq \text{const}\,(\|\bar\partial\phi\|^2 + \|\vartheta\phi\|^2 + \|g\phi\|_{-1}^2).$$

So (5.8) is deduced by the usual contradiction argument, i.e. take a sequence $\phi_\nu \perp \mathcal{H}_t^{p,q}$ with $\|\phi_\nu\|_{(t)} = 1$ and

$$(5.11) \qquad \|\phi_\nu\|_{(t)}^2 > \nu(\|\bar\partial\phi_\nu\|_{(t)}^2 + \|\vartheta_t\phi_\nu\|_{(t)}^2),$$

then combining this with (5.9) and (5.10) we have $\|\phi_\nu\|_{(t)} \leq \text{const}\,\|g\phi_\nu\|_{-1}$ for ν large. This shows that a subsequence of the $\{\phi_\nu\}$ converges in L_2 and because of (5.11) also in Q_t to an element ϕ but $\phi \perp \mathcal{H}_t^{p,q}$, $\|\phi\|_{(t)} = 1$ and, by (5.11), $\phi \in \mathcal{H}_t^{p,q}$ which is a contradiction proving that (5.8) must hold.

(5.2) **Proposition.** *For each s there exists a number T_s such that if $q \geq 1$ and $t \geq T_s$ then $\mathcal{H}_t^{p,q} \subset \tilde{\mathfrak{A}}_s$ and if $\alpha \perp \mathcal{H}_t^{p,q}$ then there exists a unique $\phi \in \tilde{\mathfrak{D}}^{p,q} \cap \tilde{\mathfrak{A}}_s$ such that $\phi \perp \mathcal{H}_t^{p,q}$ and*

$$(5.13) \qquad Q_t(\phi, \psi) - (\phi, \psi)_t = (\alpha, \psi)_t$$

for all $\psi \in \mathfrak{D}$.

Proof. The proof is by the same method as is (5.4) of [9]. Let $\theta_1, \cdots, \theta_N$ be a basis of $\mathcal{H}_t^{p,q}$, $q > 1$. For each positive ν and $\phi, \psi \in \tilde{\mathfrak{D}}^{p,q}$ we define the quadratic form P_t^ν by

$$(5.14) \qquad P_t^\nu(\phi, \psi) = Q_t(\phi, \psi) - (\phi, \psi)_{(t)} + \frac{1}{\nu}\sum_{j=1}^N (\phi, \theta_j)_{(t)}(\theta_j, \psi)_{(t)}.$$

We write

$$(5.15) \qquad \phi = \phi' + \phi'',$$

where $\phi' \in \mathcal{H}_t^{p,q}$ and $\phi'' \perp \mathcal{H}_t^{p,q}$; then

$$(5.16) \qquad P_t^\nu(\phi, \phi) = Q_t(\phi'', \phi'') - (\phi'', \phi'')_{(t)} + \nu^{-1}\|\phi'\|_{(t)}^2,$$

hence by Lemma (5.7) we obtain

$$(5.17) \qquad \|\phi\|_{(t)}^2 \leq \text{const}\; P_t^\nu(\phi, \phi);$$

note that this constant depends on ν. Proceeding as in Proposition (5.1), we conclude that for sufficiently large t, if $\alpha \in \tilde{\mathfrak{A}}_s^{p,q}$, $q \geq 1$, then there exists a unique $\phi_\nu \in \tilde{\mathfrak{D}}^{p,q}$ such that

(5.18) $$P_t^{\nu}(\phi_\nu, \psi) = (\alpha, \psi)_{(t)}$$

for all $\psi \in \tilde{\mathfrak{D}}^{p,q}$ and $\phi_\nu \in \tilde{\mathfrak{A}}_s^{p,q}$. Furthermore

(5.19) $$\|\phi_\nu\|_s \leq \text{const}\, \|\alpha\|_s,$$

where the constant again depends on ν. It is easy to see from the derivation of (5.19) that there exists $C > 0$ such that

(5.20) $$\|\phi_\nu\|_s \leq C(\|\alpha\|_s + \|\phi_\nu\|),$$

for all ν.

Now we shall prove that $\mathcal{H}_t^{p,q} \subset \tilde{\mathfrak{A}}_s$. If $\dim \mathcal{H}_t^{p,q} = 0$ there is nothing to prove, otherwise set $\theta_0 = 0$ and assume $\theta_j \in \tilde{\mathfrak{A}}_s$ for $j = 0, \cdots, k$ with $k < N$; we shall construct $\theta \in \mathcal{H}_t^{p,q} \cap \tilde{\mathfrak{A}}_s$ with $\|\theta\| = 1$ and $(\theta, \theta_j)_{(t)} = 0$ for $j \leq k$. In this way we will construct a basis of $\mathcal{H}_t^{p,q}$ which is contained in $\tilde{\mathfrak{A}}^s$. Let $\alpha \in \tilde{\mathfrak{A}}_s$ such that α is orthogonal to θ_j for $j \leq k$ but so that α is not orthogonal to θ_{k+1}. Let ϕ_ν satisfy (5.18), then the sequence $\{\|\phi_\nu\|\}$ is unbounded, for if it were bounded we could by (5.20) find a subsequence converging to a ϕ in $\|\ \|_{s-1}$ and then ϕ would satisfy

(5.21) $$Q_t(\phi, \psi) - (\phi, \psi)_{(t)} = (\alpha, \psi)_{(t)}$$

for all $\psi \in \tilde{\mathfrak{D}}^{p,q}$. Setting $\psi = \theta_j$, the left-hand side is zero for all j and the right-hand side is different from zero for $j = k + 1$, which is a contradiction. Thus the sequence $\{\|\phi_\nu\|\}$ is unbounded and hence we can find a subsequence $\{\phi_{\nu_i}\}$ such that $\lim_i \|\phi_{\nu_i}\| = \infty$. Setting $\beta_i = \phi_{\nu_i}/\|\phi_{\nu_i}\|$, it follows that

(5.22) $$P^{\nu_i}(\beta_i, \psi) = (\alpha, \psi)/\|\phi_{\nu_i}\|.$$

From (5.20) we see that $\|\beta_i\|_s$ is bounded; now choose a subsequence of the β_i such that its arithmetic means converge to θ in $\|\ \|_s$ and such that the subsequence converges to θ in $\|\ \|_{s-1}$. Then $\theta \in \tilde{\mathfrak{A}}_s$, $\|\theta\| = 1$ and $Q_t(\theta, \psi) = (\theta, \psi)$ so that $\theta \in \mathcal{H}_t^{p,q}$. Further setting $\psi = \theta_m$ in (5.18) we have,

(5.23) $$\nu^{-1}|(\phi_\nu, \theta_m)|^2 = (\alpha, \theta_m)$$

and hence ϕ_ν is orthogonal to θ_j for $j = 1, \cdots, k$ so that θ is also orthogonal to θ_j for $j = 1, \cdots, k$. Therefore $\mathcal{H}_t^{p,q} \subset \tilde{\mathfrak{A}}_s$.

If $\alpha \perp \mathcal{H}_t^{p,q}$, $\alpha \in \tilde{\mathfrak{A}}_s$, we proceed as above to find ϕ_ν satisfying (5.18). Now, however, the sequence $\{\|\phi_\nu\|\}$ is bounded, for if it were unbounded then we could construct β as above but now $\beta \perp \mathcal{H}_t^{p,q}$, $\|\beta\| = 1$ and $\beta \in \mathcal{H}_t^{p,q}$ which is a contradiction. So that by (5.20) we see that there is a subsequence of the ϕ_ν which converges in $\|\ \|_{s-1}$ to ϕ and whose arithmetic means converge in $\|\ \|_s$ to ϕ so that $\phi \in \tilde{\mathfrak{D}}^{p,q} \cap \tilde{\mathfrak{A}}_s$, $\phi \perp \mathcal{H}_t^{p,q}$ and ϕ satisfies (5.13) as required.

(5.24) **Lemma.** *Let* $L_t = F_t - 1$, *then the range of* L_t *is closed if* t *is sufficiently large.*

Proof. It suffices to prove that there exists a constant $C > 0$ such that for every $\phi \in \mathrm{Dom}(F_t)$ with $\phi \perp \mathcal{H}_t$ we have

$$\|\phi\|_{(t)} \leq C\|L_t\phi\|_{(t)}. \tag{5.25}$$

We write $\mathrm{Dom}(F_t) = \bigoplus \mathrm{Dom}(F_t) \cap \tilde{\mathfrak{D}}^{p,q}$ and denote by $L_t^{p,q}$ the restriction of L_t to $\mathrm{Dom}(F_t) \cap \tilde{\mathfrak{D}}^{p,q}$. Then L_t is the direct norm of the $L_t^{p,q}$ and (5.25) follows by proving it for each $L_t^{p,q}$ separately. The case $q \geq 1$ follows from the construction given above, or we can deduce (5.25) directly from (5.8) by noting that

$$\|\bar\partial\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2 = (L_t^{p,q}\phi, \phi)_{(t)}. \tag{5.26}$$

To show that the range of $L_t^{p,0}$ is closed we proceed as in [7]. Namely, we first observe that if $\theta \in \mathrm{Dom}(L_t^{p,0})$ then $\bar\partial\theta \in \tilde{\mathfrak{D}}^{p,0}$, thus we can apply (5.8) to $\bar\partial\theta$ and obtain

$$\|\bar\partial\theta\|_{(t)}^2 \leq C\|L_t^{p,0}\theta\|_{(t)}^2. \tag{5.27}$$

Furthermore, since $L_t^{p,0}$ is selfadjoint its range is dense in the orthogonal complement to $\mathcal{H}_t^{p,0}$. Hence, if $\phi \perp \mathcal{H}_t^{p,0}$ and $\epsilon > 0$ there exists $\theta \in \mathrm{Dom}(L_t^{p,0})$ so that $\|\phi - L_t^{p,0}\theta\| < \epsilon$, so if $\phi \in \mathrm{Dom}(L_t^{p,0})$ we have

$$\begin{aligned}\|\phi\|_{(t)}^2 &\leq |(\phi, L_t^{p,0}\theta)_{(t)}| + \epsilon\|\phi\|_{(t)} \leq \|\bar\partial\phi\|_{(t)}\|\bar\partial\theta\|_{(t)} + \epsilon\|\phi\|_{(t)} \\ &\leq \|L_t^{p,0}\phi\|_{(t)}\|L_t^{p,0}\theta\|_{(t)} + \epsilon\|\phi\|_{(t)} \leq \|L_t^{p,0}\phi\|_{(t)} + \epsilon(\|\phi\|_{(t)} + \|L_t^{p,0}\phi\|)\end{aligned}$$

which concludes the proof.

We can now define the operator N_t required in the main theorem (3.19). Namely, we set $N_t(\mathcal{H}_t) = 0$ and $N_t\alpha = \phi$ if $\alpha = L_t\phi$ with $\phi \perp \mathcal{H}_t$. Proposition (5.12) and Lemma (5.24) show that N_t is well defined and satisfies the required regularity as well as the properties (I) through (IV) of (2.24). The further properties (A), (B) and (C) of (2.24) then follow easily from the properties (I) through (IV) and the regularity (cf. [7] of [1]).

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540