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THE LATTICE TRIPLE PACKING OF SPHERES IN EUCLIDEAN SPACE

BY

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ABSTRACT. We say that a lattice Λ in n-dimensional Euclidean space E_n provides a k-fold packing for spheres of radius 1 if, when open spheres of radius 1 are centered at the points of Λ, no point of space lies in more than k spheres. The multiple packing constant $\Delta_k^{(n)}$ is the smallest determinant of any lattice with this property. In the plane, the first three multiple packing constants $\Delta_2^{(2)}$, $\Delta_3^{(2)}$, and $\Delta_4^{(2)}$ are known, due to the work of Blundon, Few, and Heppes. In E_3, $\Delta_2^{(3)}$ is known, because of work by Few and Kanagasabapathy, but no other multiple packing constants are known. We show that $\Delta_3^{(3)} \leq 8\sqrt{38}/27$ and give evidence that $\Delta_3^{(3)} = 8\sqrt{38}/27$. We show, in fact, that a lattice with determinant $8\sqrt{38}/27$ gives a local minimum of the determinant among lattices providing a 3-fold packing for the unit sphere in E_3.

1. **Introduction.** Let Λ be an n-dimensional lattice in n-dimensional Euclidean space E_n, such that, if open spheres of radius 1 are centered at the points of Λ, then no point of space is covered more than k times. That is, for any point X in E_n there do not exist distinct points $L_1, L_2, \cdots, L_{k+1}$ of Λ such that $|X - L_1|, \cdots, |X - L_{k+1}| < 1$. Then we say that Λ provides a k-fold packing for spheres of radius 1. The terms *single*, *double* and *triple* are synonymous with *k-fold* for $k = 1, 2$ and 3.

Let $d(\Lambda)$ denote the determinant of Λ, and let $\Delta_k^{(n)}$ denote the lower bound of $d(\Lambda)$, taken over all lattices Λ that provide a k-fold packing for spheres of radius 1. (Thus $\Delta_1^{(n)}$ is the critical determinant of a sphere of radius 2.) It is well known and easy to see (e.g., divide one generator of the lattice by k) that $\Delta_k^{(n)} \leq \Delta_1^{(n)}/k$.

It has been shown by Few [1] that $\Delta_2^{(2)} = (1/2)\Delta_1^{(2)}$, and Heppes [5] showed that $\Delta_k^{(2)} = \Delta_1^{(2)}/k$ if and only if $k \leq 4$.

In [4] Few and Kanagasabapathy determined the exact value of $\Delta_2^{(3)}$, namely $3\sqrt{3}/2$, which is less than $\Delta_1^{(3)}/2 = 2\sqrt{2}$. By constructing particular lattices they also showed that $\Delta_2^{(n)} < \Delta_1^{(n)}/2$ for every $n \geq 3$.

Few remarks in [2] that $\Delta_2^{(3)}$ is the only multiple packing constant known exactly in three dimensions or more, and in this note I shall prove that $\Delta_3^{(3)} \leq$

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$8\sqrt{38}/27 < \Delta_1^{(3)}/3 = 4\sqrt{2}/3$ and give evidence suggesting that $\Delta_3^{(3)} = 8\sqrt{38}/27$.
In fact, I prove

Theorem 1. *A certain lattice* Λ_0 *of determinant* $d_0 = 8\sqrt{38}/27$ *provides a triple packing for the unit sphere* S. *Also* Λ_0 *has generators* P, Q, R *with* $|P| = 2/3$.

Theorem 2. *Any lattice* Λ *having generators* P', Q', R' *with* $|P'| \le 0.95$ *providing a triple packing for* S *must have determinant* $d(\Lambda) \ge d_0$ *with equality only when* $\Lambda = \Lambda_0$. *Hence* Λ_0 *gives a local minimum of* $d(\Lambda)$ *for triple packing of unit spheres.*

Remark. There is extensive numerical evidence that $d(\Lambda)$ does not fall below d_0 for any triple packing with S.

2. An economical lattice Λ_0.

Theorem 1. *The best lattice triple packing for spheres in* E^3 *has determinant* $d(\Lambda) \le 8\sqrt{38}/27 = \sqrt{2432/729} = 1.82649\cdots$, *since indeed the lattice* Λ_0 *generated by* P, Q *and* R *where* $P = (a, 0, 0) = (2/3, 0, 0)$, $Q = (b, b, 0) = (1/3, \sqrt{3}, 0)$ *and* $R = (g, f, c)$, *where* $g = 1/3$, $f = (11\sqrt{3})/27$ *and* $c^2 = 3 - f^2$, *provides a triple packing for the unit sphere* S.

Proof. Convention: The letters λ, μ and ν will denote integers. $S(A, r)$ will be the open sphere of radius r centered at A; $S(A)$ will denote $S(A, 1)$; thus $S = S(\text{origin})$. Suppose that the point $X = (x, y, z)$ is covered four times. Translating X by a lattice point, we may suppose that $X \in S$, and replacing X by $-X$ if necessary we may suppose $z \ge 0$. The three other spheres covering X can be written $S(\lambda P + \mu Q + \nu R) = S + \lambda P + \mu Q + \nu R$ where $(\lambda, \mu, \nu) \ne (0, 0, 0)$. We must have $|\lambda P + \mu Q + \nu R| < 2$ since they must intersect S. Therefore

$$(\lambda a + \mu b + \nu g) + (\mu b + \nu f)^2 + \nu^2 c^2 < 4 \tag{$*$}$$

and $|\nu| < 2/c$. Since $c > 1$, we have $\nu \in \{-1, 0, 1\}$. Now ν cannot be -1, since otherwise $|X - (\lambda P + \mu Q - R)| \ge |c + z| > 1$. Hence $\nu \in \{0, 1\}$. From $(*)$ we also get $|\mu b + \nu f| < 2$. Since $0 \le \nu \le 1$ and $0 \le f \le b/2$ and $b = \sqrt{3} > 4/3$ this gives $-2 < \mu < 2$, $\mu \in \{-1, 0, 1\}$. We divide the proof into two parts.

Part 1. $y \ge 0$. Then $\mu \in \{0, 1\}$; in fact if $\mu = -1$, then for $X \in S(\lambda P + \mu Q + \nu R)$ we would have $|X - (\lambda P - Q + \nu R)|^2 \ge (b + y - \nu f)^2 > (16b/27)^2 = 256/243$, since $\nu \in \{0, 1\}$. Also $(\mu, \nu) \ne (1, 1)$, since $|\lambda P + Q + R|^2 \ge b^2 + c^2 > 4$. Hence $(\mu, \nu) \in \{(0, 0), (1, 0), (0, 1)\}$.

Type 1 *spheres*. Suppose $(\mu, \nu) = (0, 0)$. Then $\lambda P + \mu Q + \nu R = \lambda P$, and $S(\lambda P) \cap S = \varnothing$ if $|\lambda| > 2$, since $|3P| = 3a = 2$. The $S(\lambda P)$ such that $0 < |\lambda| \le 2$ are called Type 1 spheres.

Type 2 *spheres*. Suppose $(\mu, \nu) = (1, 0)$. Then $\lambda P + \mu Q + \nu R = \lambda P + Q$, and $S(\lambda P + Q) \cap S = \emptyset$ if $\lambda \notin \{0, -1\}$, since then $|\lambda P + Q|^2 = b^2 + (\lambda a + b)^2 = 3 + |2\lambda/3 + 1/3|^2 \geq 4$. The $S(Q - P)$ and $S(Q)$ are called Type 2 spheres.

Type 3 *spheres*. Suppose $(\mu, \nu) = (0, 1)$. Then $\lambda P + \mu Q + \nu R = R + \lambda P$, $S(R + \lambda P) \cap S = \emptyset$ if $\lambda \notin \{0, -1\}$. To see this, observe that if $S(R + \lambda P) \cap S \neq \emptyset$ then $(\lambda a + g)^2 + f^2 + c^2 < 4$; since $f^2 + c^2 = 3$, $|2\lambda/3 + 1/3| < 1$ and $\lambda \in \{0, -1\}$. We call $S(R)$ and $S(R - P)$ Type 3 spheres.

It follows from the discussion of Type 2 and Type 3 spheres that $S \cap S(\lambda P + E) = \emptyset$ if $\lambda \notin \{-1, 0\}$ where $E \notin \{R, Q\}$. In particular,

$$\emptyset = S \cap S(E + P) = S(-P) \cap S(E) = S(-2P) \cap S(E - P),$$

$$\emptyset = S \cap S(E + 2P) = S(-2P) \cap S(E),$$

$$\emptyset = S \cap S(E - 2P) = S(P) \cap S(E - P) = S(2P) \cap S(E),$$

$$\emptyset = S \cap S(E - 3P) = S(2P) \cap S(E - P).$$

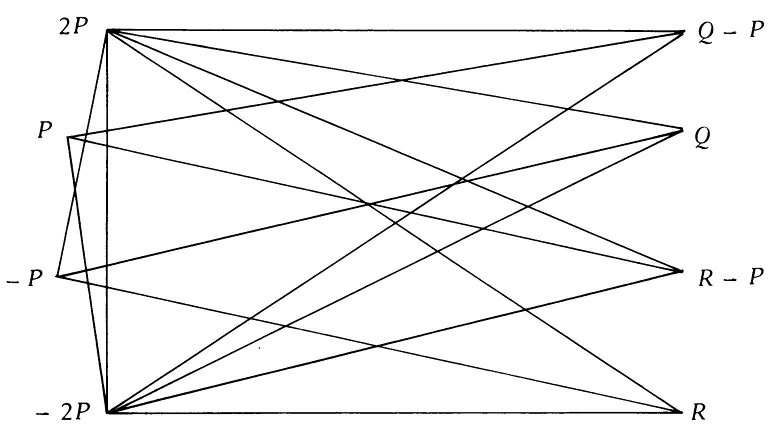
From the discussion of Type 1 spheres $S \cap S(\lambda P) = \emptyset$ for $|\lambda| > 2$ so that

$$\emptyset = S \cap S(3P) = S(-P) \cap S(2P) = S(-2P) \cap S(P),$$

$$\emptyset = S \cap S(4P) = S(-2P) \cap S(2P).$$

We now draw a graph G where edges A and B are joined only if we know that $S(A) \cap S(B) = \emptyset$.

G:



We next observe that $\emptyset = S(Q + \lambda P) \cap S \cap S(R + \lambda' P + \nu Q)$. For $|Q + \lambda P| \geq |Q| = \sqrt{b^2 + b^2} > \sqrt{3}$. Therefore the height (maximal value of the z coordinate of the closure) of $S(Q + \lambda P) \cap S$ is less than $\sqrt{1 - 3/4} = \frac{1}{2} < c - 1$, since $c = 1.5 \cdots > 3/2$. Since $R + \lambda' P$ has z component c, the above intersection is void. Hence we cannot have X simultaneously inside a sphere of Type 2 and a sphere of Type 3, so

$$X \in S(\lambda_1 P) \cap S(\lambda_2 P) \cap S(E + \lambda_3 P) \quad \text{with } 0 < |\lambda_1|, \quad |\lambda_2| \le 2, \ -1 \le \lambda_3 \le 0,$$

or

$$X \in S(\lambda_1 P) \cap S(E + \lambda_2 P) \cap S(E + \lambda_3 P) \quad \text{with } 0 < |\lambda_1| \le 2, \ -1 \le \lambda_2, \lambda_3 \le 0,$$

where $E \in \{R, Q\}$. Both of these contradict the graph G, and Part 1 follows.

Part 2. We now suppose that $y < 0$. Recall that if $S \cap S(\lambda P + \mu Q + \nu R) \neq \varnothing$, then $-1 \le \mu \le 1$ and $0 \le \nu \le 1$. For those $S(\lambda P + \mu Q + \nu R)$ containing X we must have $-1 \le \mu \le 0$. For suppose that $\mu = 1$; since $X = (x, y, z)$, $y < 0$, we would have $|\lambda P + \mu Q + \nu R - X| \ge |b + \nu f - y| > b > 1$. The spheres $S(\lambda P + \mu Q + \nu R)$ containing X other than S may therefore be divided into four types, as follows:

Type 1 *spheres, when* $(\mu, \nu) = (0, 0)$. As before the only spheres $S(\lambda P)$ intersecting S satisfy $0 < |\lambda| \le 2$, i.e., the Type 1 spheres are $S(2P)$, $S(P)$, $S(-P)$ and $S(-2P)$.

Type 2 *spheres, when* $(\mu, \nu) = (-1, 0)$. If $S(\lambda P + \mu Q + \nu R)$ is to intersect S we must have $4 > |\lambda P + \mu Q + \nu R|^2 = (\lambda a - b)^2 + b^2 = (2\lambda/3 - 1/3)^2 + 3$. Hence $0 \le \lambda \le 1$, i.e., the Type 2 spheres are $S(-Q)$ and $S(P - Q)$.

Type 3 *spheres, when* $(\mu, \nu) = (0, 1)$. As in Part 1, $-1 \le \lambda \le 0$ if $S(\lambda P + \mu Q + \nu R)$ intersects S, i.e., the Type 3 spheres are $S(R)$ and $S(R - P)$.

Type 4 *spheres, when* $(\mu, \nu) = (-1, 1)$. If $S(\lambda P + \mu Q + \nu R)$ intersects S, then $(\lambda a + g - b)^2 + (f - b)^2 + c^2 < 4$, $4\lambda^2/9 < 4 - (b - f)^2 - c^2 = 4/9$, $\lambda^2 < 1$, $\lambda = 0$, and $S(R - Q)$ is the only sphere of Type 4.

As we did in Part 1, we deduce several new disjoint pairs of spheres. From the discussion of Type 4 spheres, $S \cap S(-Q + R + \lambda P) = \varnothing$ if $\lambda \neq 0$, so we have $S(\lambda P) \cap S(-Q + R) = \varnothing$ if $\lambda \neq 0$. From the Type 2 spheres we have

$$\varnothing = S \cap S(-Q + 2P) = S(-P) \cap S(-Q + P) = S(-2P) \cap S(-Q),$$

$$\varnothing = S \cap S(-Q + 3P) = S(-2P) \cap S(-Q + P),$$

$$\varnothing = S \cap S(-Q - P) = S(P) \cap S(-Q) = S(2P) \cap S(-Q + P),$$

$$\varnothing = S \cap S(-Q - 2P) = S(2P) \cap S(-Q).$$

If we combine these with some of the disjoint pairs that we already know from Part 1 and draw a graph, G', in which A is joined to B only if we know $S(A) \cap S(B) = \varnothing$, we obtain the following graph.

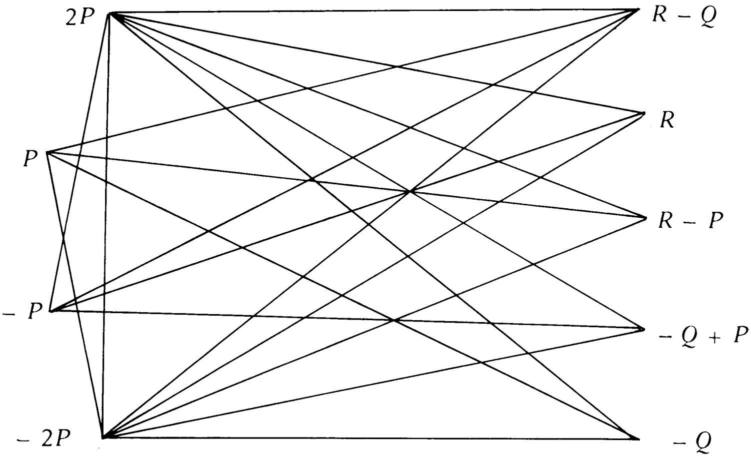
In addition to these disjoint spheres, we observe that, for any λ and λ', $S(-Q + \lambda P) \cap S(R + \lambda' P) \cap S = \varnothing$, since $|-Q + \lambda P| \ge |Q|$, so that the height of $S(-Q + P) \cap S$ is not greater than the height of $S \cap S(Q)$, which is less than $\frac{1}{2} < c - 1$, and c is the height of $R + \lambda' P$.

Also, for any λ, $\varnothing = S \cap S(Q + \lambda P) \cap S(R) = S \cap S(-Q - \lambda P) \cap S(-R) = S(R) \cap S(R - Q - \lambda P) \cap S$.

In particular,

$$(1) \qquad S(R) \cap S(R - Q) \cap S = \varnothing.$$

G':



The following enumeration of possibilities shows that X cannot be contained in the necessary spheres, and Theorem 1 follows:

Clearly Types 2 and 3 cannot both occur by the fourth paragraph above, and two spheres of Type 1 cannot occur with anything else by the graph G'. Again by the graph G', if two spheres of Type 2 (or two spheres of Type 3) occur, the remaining sphere cannot have Type 1. Hence the only remaining possibilities are

(2) $$X \in S(\lambda_1 P) \cap S(\lambda_2 P - Q) \cap S(R - Q), \quad 0 < |\lambda_1| \le 2, \ 0 \le \lambda_2 \le 1,$$

(3) $$X \in S(\lambda_1 P) \cap S(\lambda_2 P + R) \cap S(R - Q), \quad 0 < |\lambda_1| \le 2, \ -1 \le \lambda_2 \le 0,$$

(4) $$X \in S(P - Q) \cap S(-Q) \cap S(R - Q), \quad \text{or}$$

(5) $$X \in S(R) \cap S(-P + R) \cap S(R - Q).$$

Now (2) and (3) contradict G', and (1) excludes (5). To eliminate (4) we observe that, since $S(P) \cap S \cap S(R) \cap S(Q) = \emptyset$, we must have $S(P - Q) \cap S(-Q) \cap S(R - Q) \cap S = \emptyset$.

3. The lattice Λ_0 is locally optimal.

Remark 1. An arbitrary lattice Λ in E_3 has a basis P, Q, R where $|P| \le |Q| \le |R|$ are the successive minima of the unit sphere, $P = (a, 0, 0)$, $Q = (h, b, 0)$, $R = (g, f, c)$, $a, b, c > 0$, $0 \le h \le a/2$, $0 \le f \le b/2$, and $-a/2 < g \le a/2$. Such a basis is said to be *reduced in the sense of Gauss* or simply *reduced*. For a proof, see [6, p. 163 et seq., "Seebers inequality"].

Remark 2. If Λ has a reduced basis P, Q, R with $P = (a, 0, 0)$, $Q = (h, b, 0)$, $R = (g, f, c)$ and if $d(\Lambda) \le d_0$, then $b^2 \le b_m^2$, where $b_m^2 = a^2/6 + (2/3)\sqrt{a^4/16 + 3d_0^2/a^2}$.

Proof. Using $|R^2| = g^2 + f^2 + c^2 \ge |Q|^2 = b^2 + h^2$, and the other inequalities of reduction, we have $d_0^2 \ge d^2(\Lambda) = a^2 b^2 c^2 \ge a^2 b^2 (b^2 + h^2 - g^2 - f^2) \ge$

$a^2b^2(3b^2/4 - a^2/4)$. Putting $t = b^2$, we get $3a^2t^2 - a^4t - 4d_0^2 \leq 0$. Hence b^2 must lie between the roots $a^2/6 - (2/3)\sqrt{a^4/16 + 3d_0^2/a^2}$ and $a^2/6 + (2/3)\sqrt{a^4/16 + 3d_0^2/a^2}$ of the quadratic.

Let ρ^+ denote $\max\{0, \rho\}$.

Theorem 2. *If* (Λ, S) *is a triple packing and* $P = (a, 0, 0)$, $Q = (h, b, 0)$ *and* $R = (g, f, c)$ *gives a basis for* Λ *reduced in the sense of Gauss, and* $a \leq 1$, *then*

$$d(\Lambda) = abc \geq ab\sqrt{\{4 - (a+h)^2 - ((b^2 - 2ah - h^2)/(2b))^2\}^+} = f_1(a, b, h) \tag{6}$$

when $0 \leq g \leq h$, *and*

$$d(\Lambda) = abc \geq ab\sqrt{\{4 - 9a^2/4 - (b^2 + h^2 - 3ah)^2/(2b)^2\}^+} = f_2(a, b, h) \tag{7}$$

when $-a/2 \leq g \leq 0$ *and when* $h \leq g \leq a/2$. *Furthermore* $f_1(a, b, h) \geq f_2(a, b, h)$, *so that in fact*

$$d(\Lambda) \geq f_2(a, b, h) \tag{8}$$

in all cases. Also, if $d(\Lambda) \leq d_0$ *and* $2/3 \leq a \leq 0.9508$, *we have*

$$f_2^2(a, b, h) \geq \min\{p(a), d_0^2 + 1/100\}, \tag{9}$$

where $p(a) = 2a^6 - 11a^4 + 12a^2$, $p(2/3) = d_0^2$, *and*

$$p(a) > d_0^2 \quad \text{for } 2/3 < a \leq 0.9508. \tag{10}$$

Hence $d(\Lambda) \geq d_0$ *for* $2/3 \leq a \leq 0.9508$, *and with equality only if* $a = 2/3$.

Proof. Suppose that (Λ, S) gives a triple packing and that P, Q, R form a reduced basis of Λ. From reduction, we have

$$|P| \leq |Q| \leq |R|,\ 0 \leq h \leq a/2,\ 0 \leq f \leq b/2, \text{ and } |g| \leq a/2. \tag{11}$$

We also have $a \geq 2/3$, since otherwise the point $(\frac{1}{2})P$ would be covered by $S(-P)$, S, $S(P)$ and $S(2P)$. Observe that the center of the parallelogram with vertices P, Q, $Q+P$ and the origin will be covered by the four spheres S, $S(P)$, $S(Q)$, and $S(Q+P)$ unless one of the diagonals $|Q+P|$, $|Q-P|$ is at least 2. Since $|Q+P| \geq |Q-P|$ by (11), it follows that $4 \leq |Q+P|^2 = b^2 + (a+h)^2$; hence

$$b^2 \geq 4 - (a+h)^2 \quad \text{and} \quad a \geq 2/3. \tag{12}$$

Case 1. In this case we assume

(13) $0 \leq g \leq h$.

A consequence of (13) is that $|R - Q - P|$ is not less than $|R - Q + P|$. They cannot both be less than 2, since then the center of the parallelogram having vertices R, Q, $Q + P$, $R + P$ would be covered four times. Hence we have

(14) $|R - Q - P| \geq 2.$

Another consequence of (13) is that $|R + P|$ is not less than $|R - P|$. Considering the parallelogram with vertices P, R, $R + P$ and the origin shows that

(15) $|R + P| \geq 2.$

With a view to proving (6) we imagine a, h, b to be fixed and find the point $R = (g, f, c)$ having least nonnegative c such that (13), (14) and (15) hold and also

(16) $0 \leq f \leq b/2.$

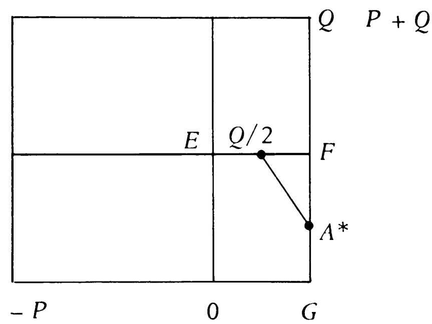
We are in fact looking for the lowest point $X = (x, y, z)$ inside the rectangular prism given by

(17) $0 \leq x \leq h$, $0 \leq y \leq b/2$, $z \geq 0$,

subject to the additional constraint

(18) $|X - P - Q| \geq 2$, $|X + P| \geq 2$.

The problem is somewhat simplified by the fact that the centers $P + Q$, $-P$ of the spheres lies on the plane $z = 0$, outside the prism.



Case 1

If the right-hand side of (6) is zero, there is nothing to prove. Let us suppose, therefore, that it is positive. We shall show that the point X^* that lies on the intersection of the boundary of $S(-P, 2)$ and $S(P + Q, 2)$ and the plane $x = h$ is the lowest point satisfying (17) and (18). We start by finding X^*. Let $X^* = (x^*, y^*, z^*)$. Let π be the radical plane of $S(-P, 2)$ and $S(P + Q, 2)$ (the plane obtained by subtracting the equations of the two spheres). Then π passes through $(\frac{1}{2})Q = (h/2, b/2, 0)$, which is halfway between the center of the two spheres, and has the equation $y - b/2 = (-(2a + h)/b)(x - h/2)$. Putting $x^* = h$, we obtain $y^* = b/2 - (2ab + h^2)/(2b)$. We must show that $0 \leq y^*$ so that (17) is satisfied. By (12) we have $b^2 \geq 4 - (a + h)^2 \geq 2ah + h^2 + \frac{1}{2}$, since $h \leq a/2$; hence $2by^* = b^2 - (2ah + h^2) > 0$ and (17) follows. We see that $z^* = \sqrt{4 - (a + h)^2 - ((b^2 - 2ah - h^2)/(2b))^2} > 0$ by a previous assumption.

The first step in showing that X^* is optimal is to show that the bottom of the prism is covered by $S(-P, 2)$ and $S(P+Q, 2)$. This means that there is no X satisfying (17) and (18) with $z = 0$. Let $A^* = (x^*, y^*, 0)$, $E = (0, b/2, 0)$, $F = (b, b/2, 0)$ and $G = (b, 0, 0)$. Then $2 = |P+Q-X^*| > |P+Q-A^*| \geq |P+Q-F|$, and also $|P+Q-(\frac{1}{2})Q| < 2$, since the spheres $S(-P, 2)$ and $S(P+Q, 2)$ intersect and $(\frac{1}{2})Q$ is halfway between their centers. Hence the triangle with vertices $(\frac{1}{2})Q$, A^*, F lies in the interior of $S(P+Q, 2)$.

Similarly, $2 = |-P-X^*| > |-P-A^*| \geq |-P-G| \geq |-P|$ and $2 > |-P-(\frac{1}{2})Q| \geq |-P-E|$ so that the convex pentagon with vertices G, A^*, $(\frac{1}{2})Q$, E and the origin lies in the interior of $S(-P, 2)$. Hence the bottom of the prism is covered.

We now let $X_1 = (x_1, y_1, z_1)$ be a lowest point satisfying (17) and (18). We know that X_1 exists, because the set of solutions is nonempty and closed. The point X_1 must be on the boundary of $S(-P, 2)$ or $S(P+Q, 2)$ since otherwise it could be lowered and still satisfy (18).

Let us suppose first that X_1 is on the boundary of $S(-P, 2)$. We shall deduce that X_1 is on the boundary of $S(P+Q, 2)$. Suppose not. Then (x_1, y_1) must be the point satisfying (17) that is farthest from $-P$, namely $(b, b/2)$. But then the point $X_1 = (b, b/2, \sqrt{4-(b+a)^2-b^2/4})$ is easily seen to be inside $S(P+Q, 2)$, contrary to (18).

Suppose that X_1 is not on the boundary of $S(-P, 2)$. Then $|X_1 - P - Q| = 2$, and $X_1 = (0, 0, \sqrt{4-(a+b)^2-b^2})$ lies inside $S(-P, 2)$ contrary to (18).

Hence X_1 lies on the arc of the intersection of $S(-P, 2)$ and $S(P+Q, 2)$ with $z \geq 0$. The highest point of the arc is the point directly above $(\frac{1}{2})Q$, which is on the boundary of the prism, and the lowest point in the prism is X^*, where the arc cuts the $x = b$ plane. Hence $X_1 = X^*$ and (6) follows.

Case 2. Assume

(19) $-a/2 \leq g \leq 0$.

Since the center of the parallelogram with vertices R, $Q-P$, $R+P$, Q must not be covered four times, we know that one of its two diagonals $|R-Q|$, $|R-Q+2P|$ must be at least 2. Assuming Gauss reduction, we always have $|R-Q+2P| \geq |R-Q|$, since the vectors $R-Q+2P$ and $R-Q$ differ only in the first component, and $|g-b+2a| \geq 2a - |g| - b \geq a \geq |g-b|$. Hence $|R-Q+2P| \geq 2$.

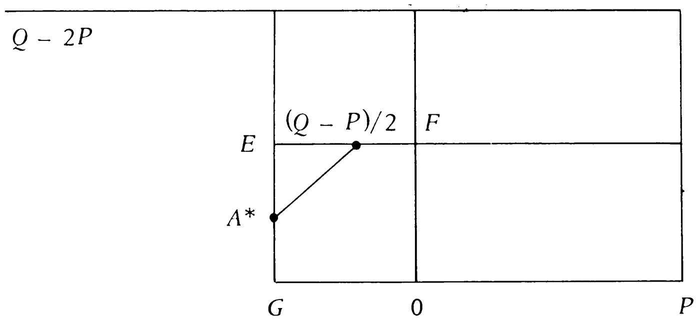
As in Case 1, one of $|R+P|$, $|R-P|$ must be at least 2, and from (19) we know that $|R-P| \geq |R+P|$ so that $|R-P| \geq 2$.

In a manner similar to Case 1, we are looking for the lowest point $X = (x, y, z)$ inside the rectangular prism given by

(20) $-a/2 \leq x \leq 0$, $0 \leq y \leq b/2$, $z \geq 0$,

such that

(21) $|X-P| \geq 2$, and $|X-Q+2P| \geq 2$.



Case 2

The procedure is the same as in Case 1. We may suppose that the right-hand side of (7) is positive. Let $X^* = (x^*, y^*, z^*)$ be the point on the two spheres and the plane $x = -a/2$, with $z^* \geq 0$.

The radical plane π of the two spheres passes through $(\frac{1}{2})(Q - P)$. The equation of π is $2(h - 3a)x + 2by = b^2 + (h - 3a)(h - a)$. Putting $x = x^* = -a/2$ yields $y^* = (b^2 + h^2 - 3ah)/(2b)$. We must show that $0 \leq y \leq b/2$. Now $b^2 \geq 4 - (a + h)^2$ for a triple packing; hence $2by^* = b^2 + h^2 - 3ah \geq 4 - a^2 - 5ah \geq \frac{1}{2}$. On the other hand $h^2 \leq ah/2 \leq 3ah$, $2by^* = b^2 + h^2 - 3ah \leq b^2$, $y^* \leq b/2$. We see that $z^* = \sqrt{4 - (3a/2)^2 - ((b^2 + h^2 - 3ah)/(2b))^2} > 0$ by assumption.

We now show that the bottom of the prism is covered by the two spheres $S(P, 2)$ and $S(Q - 2P, 2)$. Let $A^* = (x^*, y^*, 0)$, $E = (-a/2, b/2, 0)$, $F = (0, b/2, 0)$ and $G = (-a/2, 0, 0)$. Then $2 = |Q - 2P - A| > |Q - 2P - A^*| \geq |Q - 2P - E|$, and also $|(Q - 2P) - (\frac{1}{2})(Q - P)| < 2$ since the spheres $S(P, 2)$ and $S(Q - P, 2)$ intersect and $(\frac{1}{2})(Q - P)$ is halfway between their centers, Hence the triangle with vertices $(\frac{1}{2})(Q - P)$, E, A^* lies in the interior of $S(Q - 2P, 2)$. Similarly $2 > |P - (\frac{1}{2})(Q - P)| \geq |P - F| \geq |P|$ and $2 > |P - A^*| \geq |P - G|$, so that the convex pentagon with vertices G, A^*, $(\frac{1}{2})(Q - P)$, F, and the origin lies in the interior of $S(P, 2)$. Hence the bottom of the prism is covered.

We now let $X_1 = (x_1, y_1, z_1)$ be a lowest point satisfying (20) and (21); X_1 exists because the set of points satisfying (20) and (21) is closed and nonempty. Since $z_1 > 0$, X_1 must be on the boundary of one of the spheres. We suppose first that $|X_1 - P| = 2$, and $|X_1 - Q + 2P| > 2$. Then (x_1, y_1) must be as far from $(a, 0)$ as possible still satisfying (20). Hence $X_1 = (-a/2, b/2, \sqrt{4 - b^2/4 - (3a/2)^2})$, and calculation shows that $|X_1 - Q + 2P| < 2$, which is a contradiction.

Suppose now that $|X_1 - P| > 2$, so that $|X_1 - Q + 2P| = 2$. Then (x_1, y_1) must be as far as possible from $(h - 2a, b)$ and still satisfy (20). Now $-2a \leq h - 2a \leq -(3/2)a < -a/2$. Hence $X_1 = (0, 0, \sqrt{4 - (h - 2a)^2 - b^2})$. Hence $|X_1 - P|^2 = a^2 + 4 - (h - 2a)^2 - b^2 < a^2 + 4 - 9a^2/4 < 4$, which is a contradiction.

Hence X_1 must be on the boundary of both spheres. In a manner similar to Case 1, X_1 lies on a circular arc whose highest point $(\frac{1}{2})(Q - P)$ is on the boundary of the prism and whose lowest point inside the prism is X^*. Hence $X_1 = X^*$ and (7) is proved for this case.

Case 3. Assume

(22) $h \leq g \leq a/2$.

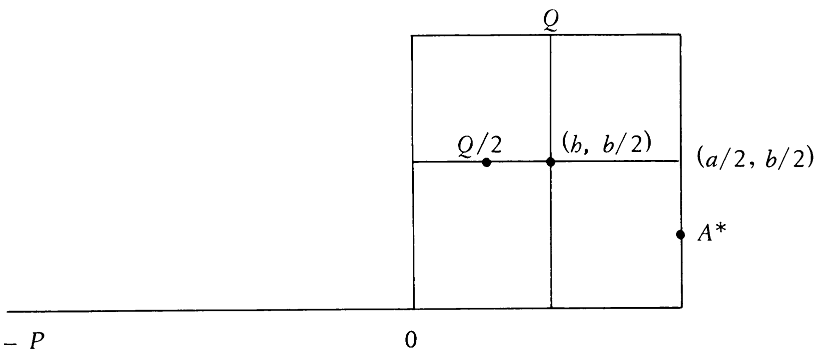
The vectors $R - Q + P$, $R - Q - P$ differ only in the first component and, by (22), $|g - h + a| \geq |g - h - a|$. Since the center of the parallelogram with vertices P, $R - Q$, $R - Q + P$ and the origin must not be covered four times, we conclude that $|R - Q + P| \geq 2$. On the other hand, as in Case 1, $|R + P| \geq 2$.

We are seeking the lowest point X_1 of the prism

(23) $h \leq x \leq a/2,\ 0 \leq y \leq b/2,\ z \geq 0,$

such that

(24) $|X + P| \geq 2,\ |X - Q + P| \geq 2.$



Case 3

If the right-hand side of (7) is zero, there is nothing to prove. Let us suppose that it is positive. Let X be the point on the two spheres and the plane $x = a/2$. To find $X = (x, y, z)$ we solve $9a^2/4 + y^2 + z^2 = 4$, $(3a/2 - h)^2 + (y - b)^2 + z^2 = 4$, and obtain $x^* = a/2$, $y^* = (b^2 + h^2 - 3ah)/(2b)$, and $z^* = \sqrt{4 - 9a^2/4 - (y^*)^2}$. This is the same y^* that appears in Case 2, so $0 \leq y^* \leq b/2$ and X^* satisfies (23) and (24).

Now let X_1 be a lowest point satisfying (23) and (24), and we shall show that $X_1 = X^*$.

We must show that the bottom of the prism is covered. Since $x = a/2$ maximizes the horizontal distance from both $-P$ and $Q - P$, it is sufficient to show that the line segment $\{(a/2, t, 0): 0 \leq t \leq b/2\}$ is covered. The spheres $S(-P, 2)$ and $S(Q - P, 2)$ intersect the plane $x = a/2$ in circles which intersect at $(a/2, y^*, z^*)$. Since $z^* > 0$, the segment is covered.

For a lowest point X_1 we must have $x_1 = a/2$. Since the spheres intersect the $x = a/2$ plane in circles whose centers have y components 0 and b respectively, it is clear that $X_1 = X^*$ and inequality (7) holds. This finishes Case 3.

Hence $d(\Lambda) \geq \min\{f_1, f_2\}$. We observe immediately that $f_1 \geq f_2$. This would follow if $\psi(h) \geq 0$ where $\psi(h) = 9a^2/4 + (b/2 - (3a-h)h/(2b))^2 - (h+a)^2 - \{(b/2 - (h/2)(h+2a)/b)\}^2$. Now $\psi(0) = 9a^2/4 - a^2 = (5/4)a^2 > 0$ and $\psi(a/2) = 0$. Differentiating, we have $\psi'(h) = -\{(5a)/(2b^2)\}(b^2 + 3h^2 - ah) < 0$ since $b^2 \geq a^2 - h^2 \geq (3/4)a^2$.

We now prove

$$(9) \qquad f_2^2 \geq \min\{d_0^2 + 1/100,\ 2a^6 - 11a^4 + 12a^2\}$$

under the hypothesis that $d(\Lambda) \leq d_0$ and $2/3 \leq a \leq 0.9508$. Write $F(a, h, b) = f_2^2 = a^2b^2\{4 - 9a^2/4 - (b/2 + h(h-3a)/(2b))^2\}$ and put $t = b^2$. Clearly $\partial^2F/\partial t^2 = -a^2/2 < 0$. Hence $F(a, h, b) \geq \min\{\phi(a, h), \psi(a, h)\}$, where $\phi(a, h) = F(a, h, \sqrt{4-(a+h)^2})$, and $\psi(a, h) = F(a, h, b_m)$, where b_m was defined in Remark 2. Now

$$\phi(a, h) = a^2\{4 - (a+h)^2\}\{4 - 9a^2/4 - (1/4)(4 - (a+h)^2) - (1/2)h(h-3a)\} - (a^2/4)h^2(h-3a)^2$$

and calculation shows that $\partial^2\phi/\partial h^2 = -8a^2 - 8a^4 < 0$. Since $\phi(a, 0) = \phi(a, a/2) = p(a)$ where $p(x) = 2x^6 - 11x^4 + 12x^2$, we have $\phi \geq p(a)$.

To complete the proof of (9), we will show that $\psi(a, h) \geq d_0 + 1/100$. Now

$$\psi(a, h) = F(a, h, b_m) = a^2b_m^2\{4 - 9a^2/4 - (b_m/2 + h(h-3a)/(2b_m))^2\}$$

and calculation shows that $\partial^2\psi/\partial h^2 = -a^2b_m^2 + 3a^2h^2 - 9a^3h + 9a^4/4$. We shall show that $\partial^2\psi/\partial h^2 < 0$, so that $\psi(a, h)$ is a concave function of h.

We digress for a moment to show that $h \geq h_m(a)$, where $h_m = \sqrt{4 - b_m^2} - a$. To see this, recall that for triple packing we must have $b^2 \geq 4 - (a+h)^2$, whereas $b^2 \leq b_m^2$, since $d(\Lambda) \leq d_0$. The juxtaposition $b_m^2 \geq 4 - (a+h)^2$ yields $h \geq h_m$. Putting $h = a/2$ we see that $h_m \leq a/2$. It is conceivable that h_m is negative, even though h never is. For what follows, it is useful to know that $h_m \geq -a/2$. To see this, note that

$$b_m^2 = a^2/6 + (2/3)\sqrt{a^4/16 + 3d_0^2/a^2} < 1/6 + (2/3)\sqrt{1/16 + 27} < 3.64,$$

since $2/3 \leq a \leq 1$ and $3 < d_0^2 < 4$, so that $4 - b_m^2 > 1/4 \geq a^2/4$, $h_m + a = \sqrt{4 - b_m^2} > a/2$, and $h_m > -a/2$.

We now return to showing that $\psi(a, h)$ is a concave function of h for $h_m \leq h \leq a/2$. Since $\partial^3\psi/\partial h^3 = 6a^2h - 9a^3 \leq 3a^3 - 9a^3 = -6a^3 < 0$, it is enough to show that $f(a) = (1/a^2)\partial^2\psi/\partial h^2|_{h=h_m} < 0$.

Since $f(a)$ is a rather complicated function of the single variable a, we shall simply find an upper bound for its derivative as a function of a and use a computer to evaluate it on a fine grid.

Let $F(x, y, z) = -z + 3y^2 - 9xy + 9x^2/4$ so that $F(a, b_m, b_m^2) = f(a)$. Then

$$|f'(a)| \le \left|\frac{\partial F}{\partial x}\right|_0 + \left|\frac{\partial F}{\partial y}\right|_0 \left|\frac{db_m}{da}\right| + \left|\frac{\partial F}{\partial z}\right|_0 \left|\frac{d}{da}b_m^2\right|,$$

where the subscript 0 indicates that the partial derivatives are evaluated at $(x, y, z) = (a, b_m, b_m^2)$.

Then

$$\left|\frac{\partial F}{\partial x}\right|_0 = |-9b_m + 9a/2| \le 9a \le 9, \quad \left|\frac{\partial F}{\partial y}\right|_0 \le |6b_m| + 9a \le 12, \quad \text{and} \quad \left|\frac{\partial F}{\partial z}\right|_0 = 1.$$

Let $u = a^2$, $g(u) = b_m^2 = u/6 + (2/3)\sqrt{u^2/16 + 3d_0^2/u}$. Then

$$g'(u) = 1/6 + (1/3)(u/8 - 3d_0^2/u^2)/\sqrt{u^2/16 + 3d_0^2/u},$$

and

$$|g'(u)| \le 1/6 + (1/3)(1/8 + 3 \times 4/(4/9)^2)/\sqrt{(4/9)^2/16 + 9}$$
$$= 1/6 + (1/3)(1/8 + 243/4)/\sqrt{9} < 7.$$

Therefore $|dg(a^2)/da| = |2ag'(a^2)| < 14$. That is, $|db_m^2/da| < 14$.

Finally, $b_m = \sqrt{4 - b_m^2} - a$, $db_m/da = -1 - (½)(db_m^2/da)/\sqrt{4 - b_m^2}$, and $|db_m/da| \le 1 + 7\{4 - b_m^2\}^{-½} < 1 + 7\{0.36\}^{-½} < 13$, since $b_m^2 < 3.64$. Hence

$$|f'(a)| \le 9 + 12 \times 13 + 1 \times 14 = 179.$$

Using a computer we verified that (allowing for roundoff error) $f(a_i) < -0.2$ at the points $2/3 = a_0 < a_1 < \cdots < a_n = 1$, where $n = 500$, and $|a_{i-1} - a_i| < 1/1200$ for $1 \le i \le n$.

Let a be an arbitrary number in the interval $[2/3, 1]$. Then $a \in [a_{i-1}, a_i]$ for some i, and therefore

$$f(a) = f(a_{i-1}) + \int_{a_{i-1}}^{a} f'(t)\,dt \le -0.2 + \frac{179}{1200} < -0.05.$$

Hence $\psi(a, b)$ is a concave function of b as claimed, so $\psi(a, b) \ge \min\{\psi(a, b_m), \psi(a, a/2)\}$.

The functions $\psi(a, b_m)$ and $\psi(a, a/2)$ are also rather complicated functions of the single variable a, and they are both above $d_0^2 + 1/100$. We shall simply find an upper bound for their derivatives as functions of a^2 or a and use a computer to evaluate them on a fine mesh.

Let $u = a^2$ and $f(u) = \psi(a, a/2)$. Then $f(u) = ug(u)\Omega(u, g(u)) - 25u^3/64$ where $g(u) = b_m^2 = u/6 + (2/3)\sqrt{u^2/16 + 3d_0^2/u}$, and $\Omega(u, v) = 4 - (9/4)u - v/4 + (5/8)u$. Then $f'(u) = g(u)\Omega(u, g(u)) + ug'(u)\Omega(u, g(u)) + ug(u)\{\partial\Omega/\partial u + g'(u)\partial\Omega/\partial v\} - 75u^2/64$.

To estimate $|f'(u)|$, we must estimate g, Ω, and their derivatives. We have $|g(u)| \le 3.64 < 4$ from before and $|g(u)| \ge u/3$, trivially. Hence $|\Omega(u, g(u))| \le$

$4 + 13u/8 + |g(u)/4| \leq 4 + 13/8 + 1 \leq 7$. We also have $|g'(u)| \leq 7$ from before. On the other hand $|\partial\Omega/\partial u| = |13/8| < 2$ and $|\partial\Omega/\partial v| = 1/4$. Putting the estimates together, $|f'(u)| < 4 \times 7 + 7 \times 7 + 4 \times 2 + 5 + 2 = 92$.

We will show that $f(u) \geq d_0^2 + 1/100$ for $4/9 \leq u \leq 1$. Let us suppose that a computing machine has verified that $f(u_i) \geq d_0^2 + 1/50 + \epsilon$ for $4/9 = u_0 < u_1 < \cdots < u_n = 1$, where $|u_{i-1} - u_i| < (50 \times 92)^{-1}$. It then follows that $f(u) \geq d_0^2 + \epsilon$ for $4/9 \leq u \leq 1$. For $u \in [u_{i-1}, u_i]$ for some i, and

$$|f(u)| \geq |f(u_{i-1})| - \left|\int_{u_{i-1}}^{u} f'(t)\,dt\right|$$

$$\geq d_0^2 + 1/50 + \epsilon - 92|u_{i-1} - u_i| \geq d_0 + \epsilon.$$

A computing machine was programmed to find the minimum value of $f(u_i)$ for $1 \leq i \leq 8251$, where $u_i = 4/9 + (i-1)/14850$, and the answer was 3.51822.

Had there been no roundoff error, we could say that $f(u_i) \geq d_0^2 + 1/6$. It is certainly safe to say that $f(u) \geq d_0^2 + 1/50 + 1/100$. Hence $\psi(a, a/2) = f(a^2) > d_0^2 + 1/100$ for $2/3 \leq a \leq 1$.

We shall use the same method for $\psi(a, b_m)$. Let us rename $f(a) = \psi(a, b_m) = a^2b_m^2\{4 - (9/4)a^2 - b_m^2/4 - b_m(b_m - 3a)/2\} - a^2b_m^2(b_m - 3a)^2/4$. It is unfortunate that $\psi(a, b_m)$ cannot be written simply as a function of a^2; all our functions are now functions of a. Let $g_1(a) = b_m^2 = g(a^2)$, and let $h(a) = b_m$. Put $\Phi(x, y, z) = 4 - (9/4)x^2 - y/4 - z(z - 3x)/2$ and $\Theta(x, z) = -x^2z^2(z - 3x)^2/4$. Then

$$f(a) = a^2g_1(a)\Phi(a, g_1(a), h(a)) - \Theta(a, h(a)),$$

and

$$f'(a) = 2ag_1(a)\Phi(a, g_1(a), h(a)) + a^2g_1'(a)\Phi(a, g_1(a), h(a))$$
$$+ a^2g_1(a)\{\partial\Phi/\partial x + (\partial\Phi/\partial y)g_1'(a) + (\partial\Phi/\partial z)h'(a)\} - \partial\Theta/\partial x - (\partial\Theta/\partial z)h'(a).$$

Using some of the estimates from before and making some new ones, we see that $|g_1(a)| = |g(a^2)| < 4$, $|h(a)| = |\sqrt{4 - b_m^2} - a| \leq 1/2$, $|\Phi| \leq 4 + 9/4 + 1 + h^2(a)/2 + 3|h(a)|/2 \leq 1/8 + 28/4 < 8$, $|g_1'(a)| < 14$, $\partial\Phi/\partial x = 9x/2 + 3z/2$, $|\partial\Phi/\partial x| \leq 9/2 + (3/2) \times 1/2 < 6$, $|\partial\Phi/\partial y| = 1/4$, $|\partial\Phi/\partial z| = |-z + 3x/2| \leq 1/2 + 3/2 = 2$, and $|h'(a)| < 13$. By calculation, $\partial\Theta/\partial x = xz^2(z^2 - 9xz + 18x^2)/2$. Taking the maximum of the positive and negative parts, $|\partial\Theta/\partial x| \leq h_m^2/2 \times \max\{h_m^2 + 18a^2, 9h_m\} \leq (1/8)\max\{1/4 + 13, 9/2\} < 3$. Similarly, $\partial\Theta/\partial z = -x^2z(z^2 - 6zx + 9x^2 + z^2 - 3zx)/2$, and $|\partial\Theta/\partial z| \leq (h_m/2)\max\{2h_m^2 + 9, 9h_m\} < 3$. Putting these estimates together, $|f'(a)| \leq 2 \times 4 \times 8 + 14 \times 13 + 4(6 + 14/4 + 2 \times 13) + 3 + 3 \times 13 = 430$. To show that $f(a) \geq d_0^2 + \epsilon$, therefore, it is enough to show that $f(a_i) \geq d_0^2 + 1/50 + \epsilon$ for $2/3 = a_0 < \cdots < a_n = 1$ where $|a_{i-1} - a_i| < 1/25{,}000$.

A computing machine was programmed to find the minimum value of $f(a_i)$ for $1 \leq i \leq 100{,}001$ where $a_i = 2/3 + i/300{,}000$, and the answer was $3.40344 \ldots$.

Had there been no roundoff error we could say that $f(a_i) \geq d_0^2 + 1/15$. It is certainly safe to say that $f(a_i) \geq d_0 + 1/50 + 1/100$ so that $\psi(a, b_m) = f(a) \geq d_0^2 + 1/100$ for $2/3 \leq a \leq 1$. Therefore $\psi(a, b) \geq \min\{\psi(a, a/2), \psi(a, b_m)\} \geq d_0^2 + 1/100$ and $f_2^2 \geq \min\{\psi(a, b), \phi(a, b)\} \geq \min\{d_0^2 + 1/100, p(a)\}$ as claimed. Thus (9) is proved.

We now prove (10). Let $f(t) = 2t^3 - 11t^2 + 12t - d_0^2$. Then $f'(t) = 6t^2 - 22t + 12$ and $f''(t) = 12t - 22 < 0$ for $0 \leq t \leq 1$. Hence $f(t)$ is a concave function and has at most two zeroes in the range $[0, 1]$. In fact, $f(4/9) = 0$, and $f(\alpha^2) = 0$ where $\alpha^2 = 0.90402\ldots$ and $f(t) > 0$ for $4/9 < t < \alpha^2$. Since $f(a^2) = p(a) - d_0^2$, we conclude that $p(a) > d_0^2$ for $2/3 < a < \alpha = 0.950802\ldots$ and (10) is proved.

It now follows from (8), (9) and (10) that $d(\Lambda) \geq d_0$ for $2/3 \leq a \leq 0.9508$ with equality only when $a = 2/3$.

REFERENCES

1. L. Few, Ph. D. Thesis, London, 1953.

2. ———, *Multiple packing of spheres: A survey*, Proc. Colloq. on Convexity (Copenhagen, 1965), Københavns Univ. Mat. Inst., Copenhagen, 1967, 88–93. MR 35 #6036.

3. ———, *The double packing of spheres*, J. London Math. Soc. 28 (1953), 297–304. MR 14, 1115.

4. L. Few and P. Kanagasabapathy, *The double packing of spheres*, J. London Math. Soc. 44 (1969), 141–146. MR 39 #4752.

5. A. Heppes, *Mehrfache gitterförmige Kreislargerungen in der Ebene*, Acta Math. Acad. Sci. Hungar. 10 (1959), 141–148. MR 21 #3812.

6. L. E. Dickson, *Studies in the theory of numbers*, Univ. of Chicago Press, Chicago, Ill., 1939.

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