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# EXISTENCE OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH GENERALIZED BOUNDARY CONDITIONS

BY

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ABSTRACT. An investigation of the existence of solutions of the nonlinear boundary value problem $x' = f(t, x, y)$, $y' = g(t, x, y)$, $AV(a, x(a), y(a)) + BW(a, x(a), y(a)) = C_1$, $CV(b, x(b), y(b)) + DW(b, x(b), y(b)) = C_2$, is made. Here we assume $g, f: [a, b] \times R^p \times R^q \to R^p$ are continuous, and $V, W: [a, b] \times R^p \times R^q \to R$ are continuous and locally Lipschitz. The main techniques used are the theory of differential inequalities and Lyapunov functions.

1. **Introduction.** We investigate the existence of solutions of the nonlinear boundary value problem

$$x' = f(t, x, y), \qquad y' = g(t, x, y), \tag{1.1}$$

$$AV(a, x(a), y(a)) + BW(a, x(a), y(a)) = C_1, \tag{1.2}$$

$$CV(b, x(b), y(b)) + DW(b, x(b), y(b)) = C_2, \tag{1.3}$$

where $f: [a, b] \times R^p \times R^q \to R^p$ is continuous, $g: [a, b] \times R^p \times R^q \to R^q$ is continuous, $V, W: [a, b] \times R^p \times R^q \to R$ are continuous and locally Lipschitz in $x$ and $y$. For the special case in which $x$ and $y$ are scalar, $f(t, x, y) = y$, $A = C = 1$, $B = D = 0$, $V = x$, and $W = y$ we have the classical two point boundary value problem

$$x'' - g(t, x, x') = 0, \qquad x(a) = C_1, \qquad x(b) = C_2. \tag{1.4}$$

This problem has been shown to have a unique solution when $g$ satisfies a Lipschitz condition (see Picard [7]). Many others have since refined the estimates on the interval of existence in terms of the Lipschitz constants. Another more recent approach has been to show that the uniqueness of solutions of (1.4) implies the existence of solutions (see Lasota and Opial [5] and Jackson [1]).

In a very recent paper Waltman [8] used these two previous methods to show the existence and uniqueness of solutions of (1.1), (1.2), and (1.3), when

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$A = D = 1$ and $B = C = 0$. A principal tool in his first approach is the comparison theorem of Perov ([3], [6]). Here he assumes that both $f$ and $g$ as well as $V$ and $W$ satisfy a global Lipschitz condition. In his second approach he shows existence from uniqueness. Thus in both cases he obtains simultaneously the existence and uniqueness of solutions of boundary value problems.

We are interested in obtaining the existence of solutions without any global Lipschitz conditions as well as without an explicit uniqueness assumption on boundary value problems. Using the Perov comparison theorem and the theory of Lyapunov functions and differential inequalities we obtain the existence of solutions of (1.1) satisfying the more general boundary conditions (1.2) and (1.3). We do not assume uniqueness of initial value problems for (1.1) nor of a comparison system associated with (1.1). This extends the work of Perov who considered the special case in which $V \equiv x$ and $W \equiv y$. He also assumed uniqueness of solutions of a comparison system associated with (1.1).

Examples are provided with nonlinear boundary conditions to exhibit the utility of our approach. Using our results we are able to ascertain the existence of solutions of two examples provided by Waltman [8] without having to determine whether solutions of boundary value problems are unique. By the proper choice of a comparison system we are able to obtain the existence of solutions of (1.1), (1.2), and (1.3) for any $a$ and $b$.

2. **Results.** Let $R^n$ denote the Euclidean $n$ space and $\|\cdot\|$ any convenient norm in $R^n$. We shall consider the system of differential equations

$$x' = f(t, x, y), \qquad y' = g(t, x, y), \tag{S}$$

where $f\colon [a, b] \times R^p \times R^q \to R^p$ is continuous, and $g\colon [a, b] \times R^p \times R^q \to R^q$ is continuous. We assume throughout that solutions of all initial value problems of (S) exist on $[a, b]$.

Let there exist two functions $W(t, x, y)$, $V(t, x, y)$ such that $W, V\colon [a, b] \times R^p \times R^q \to R$ are continuously differentiable. Assume that

(2.1) for each $t$ the range of $V$ and $W$ is all of $R$ and the set $(W^{-1}(c_1) \cap V^{-1}(c_2)) \neq \varnothing$ (the empty set) for all real numbers $c_1$ and $c_2$;

(2.2) $|V(t, x, y)| + |W(t, x, y)| \to \infty$ as $\|x\| + \|y\| \to \infty$ uniformly for $t \in [a, b]$.

The derivative of $V$ along solutions of (S) is

$$V'(t, x, y) = \frac{\partial V}{\partial t}(t, x, y) + \left(\frac{\partial V}{\partial x}(t, x, y)\right)(f(t, x, y)) + \cdot \left(\frac{\partial V}{\partial y}(t, x, y)\right)(g(t, x, y)).$$

A similar expression holds for the derivative of $W$ along solutions of (S). Let $x(t)$ and $y(t)$ be a solution of (S). We now consider the polar angle $\phi(t)$ and the polar radius $\rho(t)$ in the $V$-$W$ plane. Thus

$$\rho(t) = \sqrt{W^2(t, x(t), y(t)) + V^2(t, x(t), y(t))},$$

$$(2.3) \qquad \phi(t) = \sin^{-1}\left(\frac{W(t, x(t), y(t))}{\rho(t)}\right), \quad \text{and}$$

$$\tan \phi(t) = \frac{W(t, x(t), y(t))}{V(t, x(t), y(t))}.$$

We thus obtain from (2.3),

$$\phi'(t) = \frac{V(t, x(t), y(t))W'(t, x(t), y(t)) - W(t, x(t), y(t))V'(t, x(t)y(t))}{V^2 + W^2}.$$

The following assumptions on $V'$ and $W'$ will play a major role in applying the comparison theorems:

$$(2.4) \qquad \begin{aligned} \delta_3(t, V, W) + F_2(t, V, W) &\le V' \le F_1(t, V, W) + \delta_1(t, V, W), \\ \delta_4(t, V, W) + G_2(t, V, W) &\le W' \le G_1(t, V, W) + \delta_2(t, V, W), \end{aligned}$$

where $F_i$, $G_i : [a, b] \times R^2 \to R$ are continuous and positively homogeneous in $V$ and $W$ (that is $F_i(t, CV, CW) = CF_i(t, V, W)$, $G_i(t, CV, CW) = CG_i(t, V, W)$ for all $C > 0$); and $\delta_i : [a, b] \times R^2 \to R$ are continuous such that

$$(2.5) \qquad \delta_i(t, V, W)/(|V| + |W|) \to 0 \text{ as } |V| + |W| \to \infty, \quad i = 1, 2, 3, 4, \quad \text{uniformly for } t \in [a, b].$$

Define

$$\begin{aligned} P_2(t, V, W) &= G_1(t, V, W) \quad \text{if } V \ge 0, \\ &= G_2(t, V, W) \quad \text{if } V < 0; \\ P_1(t, V, W) &= F_2(t, V, W) \quad \text{if } W \ge 0, \\ &= F_1(t, V, W) \quad \text{if } W < 0; \\ Q_2(t, V, W) &= G_2(t, V, W) \quad \text{if } V \ge 0, \\ &= G_1(t, V, W) \quad \text{if } V < 0; \\ Q_1(t, V, W) &= F_1(t, V, W) \quad \text{if } W \ge 0, \\ &= F_2(t, V, W) \quad \text{if } W < 0. \end{aligned}$$

From (2.4) an easy computation yields

$$(2.6) \qquad \begin{aligned} VQ_2(t, V, W) - WQ_1(t, V, W) + \delta^0(t, V, W) &\le VW' - WV' \\ &\le VP_2(t, V, W) - WP_1(t, V, W) + \delta^1(t, V, W), \end{aligned}$$

where

$$\begin{aligned}\delta^1(t, V, W) &= V\delta_2(t, V, W) - W\delta_3(t, V, W), && V \geq 0, W \geq 0,\\ &= V\delta_2(t, V, W) - W\delta_1(t, V, W), && V \geq 0, W < 0,\\ &= V\delta_4(t, V, W) - W\delta_3(t, V, W), && V < 0, W \geq 0,\\ &= V\delta_4(t, V, W) - W\delta_1(t, V, W), && V < 0, W < 0,\end{aligned}$$

and

$$\begin{aligned}\delta^0(t, V, W) &= V\delta_4(t, V, W) - W\delta_1(t, V, W), && V \geq 0, W \geq 0,\\ &= V\delta_4(t, V, W) - W\delta_3(t, V, W), && V \geq 0, W < 0,\\ &= V\delta_2(t, V, W) - W\delta_1(t, V, W), && V < 0, W \geq 0,\\ &= V\delta_2(t, V, W) - W\delta_3(t, V, W), && V < 0, W < 0.\end{aligned}$$

In view of (2.6), we now define our comparison systems

(2.7) $$A' = Q_1(t, A, B), \qquad B' = Q_2(t, A, B) \quad \text{and}$$

(2.8) $$S' = P_1(t, S, T), \; T' = P_2(t, S, T).$$

Observe that the systems (2.7) and (2.8) may have a discontinuity along the lines $A = 0$, $B = 0$ and $S = 0$, $T = 0$ respectively. We assume that all solutions of (2.7) and (2.8) exist on $[a, b]$ and are differentiable everywhere except along the previously mentioned lines. We also observe that $P_1$, $P_2$, $Q_1$ and $Q_2$ are positively homogeneous in their last two variables.

Let $\theta(t)$ and $\psi(t)$ be the polar angles corresponding to (2.7) and (2.8) respectively; that is $r^2(t) = A^2(t) + B^2(t)$ and $r(t)\sin\theta(t) = B(t)$, $r(t)\cos\theta(t) = A(t)$. (A similar definition holds for $\psi(t)$.) Differentiating the expression $\tan\theta(t) = B(t)/A(t)$, we obtain from the homogeneity of $Q_1$ and $Q_2$

(2.9)
$$\begin{aligned}\theta'(t) &= \frac{r\cos\theta\, Q_2(t, r\cos\theta, r\sin\theta) - r\sin\theta\, Q_1(t, r\cos\theta, r\sin\theta)}{r^2}\\ &= Q_2(t, \cos\theta, \sin\theta)\cos\theta - Q_1(t, \cos\theta, \sin\theta)\sin\theta\\ &\overset{\text{def}}{\equiv} F(t, \theta).\end{aligned}$$

Similarly

(2.10)
$$\begin{aligned}\psi'(t) &= P_2(t, \cos\psi, \sin\psi)\cos\psi - P_1(t, \cos\psi, \sin\psi)\sin\psi\\ &\overset{\text{def}}{\equiv} G(t, \psi)\end{aligned}$$

Let us now consider the boundary value problem written in polar form

(S) $$x' = f(t, x, y), \qquad y' = g(t, x, y),$$

$$V(a,\ x(a),\ y(a))\sin\alpha - W(a,\ x(a),\ y(a))\cos\alpha = 0,$$
(2.11)
$$V(b,\ x(b),\ y(b))\sin\beta - W(b,\ x(b),\ y(b))\cos\beta = 0,$$

where $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$.

Before stating our main existence result we will need conditions which essentially guarantee that solutions of (2.7) and (2.8) do not satisfy the boundary conditions (2.11). This assumption is essential in order to use the shooting method to solve (S) and (2.11). Denote by $\theta^+(t)$ and $\theta^-(t)$ the polar angles of (2.7) satisfying $\theta^+(a) = \alpha$, $\theta^-(a) = \alpha + \pi$. Similarly define $\psi^+(t)$ and $\psi^-(t)$ to be the polar angles of (2.8) satisfying $\psi^+(a) = \alpha$ and $\psi^-(a) = \alpha + \pi$. We use the following hypotheses concerning the solutions of (2.9) and (2.10):

($H_1$) The maximum solutions $\psi_M^+(t)$ and $\psi_M^-(t)$ of (2.10) such that $\psi_M^+(a) = \alpha$, $\psi_M^-(a) = \alpha + \pi$ satisfy

$$\psi_M^+(b) < \beta + (k+1)\pi \quad \text{and} \quad \psi_M^-(b) < \beta + (k+2)\pi$$

for some integer $k$.

($H_2$) The minimum solutions $\theta_m^-(t)$ and $\theta_m^+(t)$ of (2.9) with the initial condition $\theta_m^+(a) = \alpha$, $\theta_m^-(a) = \alpha + \pi$ satisfy

$$\theta_m^-(b) > \beta + (k+1)\pi, \qquad \theta_m^+(b) > \beta + k\pi$$

for the same integer $k$.

We are now able to state our main existence result.

**Theorem 1.** *For any* $V(t, x, y)$ *and* $W(t, x, y)$ *satisfying* (2.1), (2.2), *and* (2.4) *there exists a solution of the boundary value problem* (S), (2.11) *if* ($H_1$) *and* ($H_2$) *hold.*

**Remarks.** For the special case in which $V(t, x, y) = x$, $W(t, x, y) = y$, $F_1(t, x, y) = F_2(t, x, y)$, $G_1(t, x, y) = G_2(t, x, y)$, $\delta_3(t, x, y) = \delta_1(t, x, y)$, $\delta_4(t, x, y) = \delta_2(t, x, y)$ and $f(t, x, y) = F_1(t, x, y) + \delta_1(t, x, y)$, $g(t, x, y) = G_1(t, x, y) + \delta_2(t, x, y)$ we obtain the results of Perov. Perov assumed the added restriction that solutions of

$$x' = F_1(t,\ x,\ y), \qquad y' = G_1(t, x, y)$$

are uniquely determined by the initial conditions. Then since systems (2.9) and (2.10) are uniquely determined by initial conditions we see that ($H_1$) and ($H_2$) are equivalent to the fact that there exist no solutions of (2.7) and (2.8) satisfying the boundary conditions (2.11). (That is solutions starting on the line with slope $\tan\alpha$ at $t = a$ never hit the line with slope $\tan\beta$ at $t = b$.)

**Corollary 1.** *Let the hypotheses of Theorem* 1 *hold. Then there exists a solution of* (S) *satisfying the boundary conditions*

$$(2.12)\qquad \begin{aligned} &R_1 V(a, x(a), y(a)) + R_2 W(a, x(a), y(a)) = C_1, \\ &R_3 V(b), x(b), y(b)) + R_4 W(b, x(b), y(b)) = C_2, \end{aligned}$$

*for any real numbers* $R_i$, $i = 1, 2, 3, 4$, *and* $C_i$, $i = 1, 2$.

3. **Proofs.** We will need the following lemmas to prove Theorem 1.

**Lemma 1.** *Let* $F: R^n \overset{\text{onto}}{\rightarrow} R$ *be continuous. Then for any closed bounded interval* $I = [c, d]$ *there exists a connected subset* $S \subset R^n$ *such that* $F(S) = I$.

**Proof:** We first show the result is true for the case where $F: R \overset{\text{onto}}{\rightarrow} R$. Let $I = [c, d]$ be any closed bounded interval. Pick any $x_0 \in F^{-1}(c)$ and define the sets $\tilde{S} = \{F^{-1}(c)\}$ and $S = \{F^{-1}(d)\}$. We may assume without loss of generality that $S \cap [x_0, \infty) \neq \emptyset$ (otherwise we would look at $S \cap (-\infty, x_0]$ and proceed in a similar manner). Define $S_1 = S \cap [x_0, \infty)$; then $S_1$ is closed and $x_0 \cap S_1 = \emptyset$. Let $y_0 = \inf\{x: x \in S_1\}$ and let $x_1 = \sup\{x: x \in \tilde{S} \cap [x_0, y_0)\}$. From our construction we obtain

$$F([x_1, y_0]) = [c, d],$$

thus proving the case where $n = 1$. Assume now $F: R^n \overset{\text{onto}}{\rightarrow} R$. Then there exist sequences $\{x_k\}, \{y_k\}, x_k, y_k \in R^n$ such that $F(x_k) \to +\infty$ and $F(y_k) \to -\infty$. Construct an arc in $R^n$ by linearly connecting the points $x_0$ to $y_0$, $x_i$ to $x_{i+1}$ and $y_i$ to $y_{i+1}$ for $i = 1, 2 \cdots$. Then $F$ maps this arc onto the real line. Since the arc is homeomorphic to the real line we have reduced the problem to the case where $F: R \overset{\text{onto}}{\rightarrow} R$. This concludes the proof of Lemma 1.

The next lemma is essentially Knesser's funnel theorem [2]. Recall we are always assuming all solutions of (S) exist on $[a, b]$.

**Lemma 2.** *Let* $(x(t), y(t))$ *be a solution of* (S) *satisfying the initial condition* $x(a) = x_0$, $y(a) = y_0$. *Let* $T$ *be a compact, connected set in* $R^p \times R^q$. *Then the set*

$$\bigcup_{(x_0, y_0) \in T;\ t \in [a,b]} (x(t, a, x_0), y(t, a, y_0))$$

*is compact and connected.*

The next lemma follows from Lemma 2 and can be found in [3, p. 172] for the scalar case.

**Lemma 3.** *Let* $x(t), y(t)$ *be a solution of* (S) *such that* $x(a) = x_0$ *and* $y(a) = y_0$. *Then there exists a function* $m(u)$ *fulfilling*

$$\|x(t, a, x_0)\| + \|y(t, a, y_0)\| \geq m(\|x_0\| + \|y_0\|)$$

*where* $\lim_{u \to \infty} m(u) = \infty$.

The next lemma may essentially be found in [9, p. 22].

**Lemma 4.** *Assume all solutions* $r(t)$ *of*

$$r' = K(t, r), \tag{3.1}$$

*where* $K: [a, b] \times R \to R$ *is continuous, exist on* $[a, b]$. *Then for each* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that for each solution* $\phi(t)$ *of* $r' = K(t, r) + \delta(t)$, *where* $\delta(t)$ *is continuous and* $|\delta(t)| < \delta$, *there exists a solution* $r(t)$ *of* (3.1) *with* $r(a) = \phi(a)$ *satisfying* $|r(t) - \phi(t)| < \epsilon$ *for all* $t \in [a, b]$.

We are now able to prove Theorem 1.

**Proof.** For any constant $c$ let $(x(a, c), y(a, c))$ be a point in $R^p \times R^q$ satisfying the boundary condition (2.11) at $t = a$ such that

$$V(a, x(a, c), y(a, c)) = c \cos \alpha, \qquad W(a, x(a, c), y(a, c)) = c \sin \alpha. \tag{3.2}$$

The existence of such a point for each $c$ is due to (2.1). Let $(x(t, c), y(t, c))$ be any solution of (S) through $(x(a, c), y(a, c))$. Let

$$W(t, c) \equiv W(t, x(t, c), y(t, c)) \quad \text{and} \quad V(t, c) \equiv V(t, x(t, c), y(t, c)).$$

As before $\phi(t, c)$ and $r(t, c)$ are the polar angle and polar radius, that is $W(t,c) = r(t, c) \sin \phi(t, c)$, $V(t, c) = r(t, c) \cos \phi(t, c)$, and $r^2(t, c) = V^2(t, x(t, c), y(t, c)) + W^2(t, x(t, c), y(t, c))$. Thus

$$\phi(a, c) = \begin{cases} \alpha & \text{if } c > 0. \\ \alpha + \pi & \text{if } c < 0. \end{cases}$$

Hence from (2.3) and (2.6) we have

$$\begin{aligned} \phi'(t, c) &\leq [r^2 \cos \phi \, P_2(t, \cos \phi, \sin \phi) - r^2 \sin \phi \, P_1(t, \cos \phi, \sin \phi)]/r^2 \\ &\quad + \delta^1(t, r \cos \phi, r \sin \phi)/r^2 \\ &= G(t, \phi) + \delta^1(t, r \cos \phi, r \sin \phi)/r^2. \end{aligned} \tag{3.3}$$

Define

$$\delta(t, c) \equiv \frac{|\delta^1(t, V(t, c), W(t, c))|}{V^2(t, c) + W^2(t, c)} + \frac{|\delta^0(t, V(t, c), W(t, c))|}{V^2(t, c) + W^2(t, c)}.$$

From (2.5) it follows that for each $\delta^* > 0$ there exists $p_0 > 0$ such that whenever $V^2(t, c) + W^2(t, c) > p_0$ then $|\delta(t, c)| < \delta^*$ for all $t \in [a, b)$.

From (2.2) there exists an $r_0 > 0$ such that $\|x\| + \|y\| \geq r_0$ implies $V^2(t, x, y) + W^2(t, x, y) > p_0$ for all $t \in [a, b]$. From Lemma 3 there exists an $r_1 \geq r_0$ such that if $\|x_0\| + \|y_0\| \geq r_1$ then $\|x(t, a, x_0)\| + \|y(t, a, y_0)\| \geq r_0$ for all $t \in [a, b]$. Thus $V^2(t, x(t, a, x_0), y(t, a, y_0)) + W^2(t, x(t, a, x_0), y(t, a, y_0)) \geq p_0$.

Let

$$N^2 > \sup_{\|x\|+\|y\| \le r_1;\, t\in[a,b]} (V^2(t, x, y) + W^2(t, x, y)).$$

We may assume without loss of generality that $\cos\alpha \neq 0$. Thus for any point $(x_0, y_0)$ such that $V(a, x_0, y_0) = \pm N$ ($(x_0, y_0)$ exists from (2.1)) we have $\|x_0\| + \|y_0\| > r_1$ and hence $|\delta(t, c)| < \delta^*$.

From (3.3)

$$\phi'(t, c) \le G(t, \phi(t, c)) + \delta(t, c). \tag{3.4}$$

Let $\gamma_M^+(t)$ and $\gamma_M^-(t)$ be the maximum solutions of

$$\gamma' = G(t, \gamma) + \delta(t, c) \tag{3.5}$$

such that $\gamma_M^+(a) = \alpha$ and $\gamma_M^-(a) = \alpha + \pi$.

From $(H_1)$ we may pick $\epsilon > 0$ so small that

$$\psi_M^+(b) < \beta + (k+1)\pi - \epsilon, \qquad \psi_M^-(b) < \beta + (k+2)\pi - \epsilon. \tag{3.6}$$

From Lemma 4 there exists $\bar\delta(\epsilon) > 0$ and solutions $\psi^+(t), \psi^-(t)$ of (2.10) such that for $|\delta(t, c)| < \bar\delta(\epsilon)$

$$|\gamma_M^+(t) - \psi^+(t)| < \epsilon \quad \text{and} \quad |\gamma_M^-(t) - \psi^-(t)| < \epsilon. \tag{3.7}$$

Now (3.7) holds for any $(x_0, y_0)$ such that $V(a, x_0, y_0) = \pm N$, or equivalently for $c = \pm N/\cos\alpha$, since we may pick $N$ so large that $\delta^* < \bar\delta(\epsilon)$.

We now apply the theory of differential inequalities to (3.4) and (3.5) since these systems are continuous everywhere except for those polar angles which are multiples of $\pi/2$. The usual theory will thus go through (see [4]) and we obtain

$$\phi(t, c) \le \gamma_M^+(t) \quad \text{if } c > 0 \quad \text{and}$$
$$\phi(t, c) \le \gamma_M^-(t) \quad \text{if } c < 0. \tag{3.8}$$

Thus, for $c = +N/\cos\alpha$, we obtain from (3.6), (3.7) and (3.8) at $t = b$

$$\phi(b, c) \le \gamma_M^+(b) < \psi^+(b) + \epsilon < \psi_M^+(b) + \epsilon < \beta + (k+1)\pi.$$

For $c = -N/\cos\alpha$

$$\phi(b, c) \le \gamma_M^-(b) < \psi^-(b) + \epsilon < \psi_M^-(b) + \epsilon < \beta + (k+2)\pi.$$

By applying this previous analysis to the minimum solutions of (2.9) we get using $(H_2)$ that for $c = +N/\cos\alpha$, $\phi(b, c) > \beta + k\pi$ and for $c = -N/\cos\alpha$ $\phi(b, c) > \beta + (k+1)\pi$. We thus conclude that for $c = +N/\cos\alpha$

$$\beta + k\pi < \phi(b, c) < \beta + (k+1)\pi \tag{3.9}$$

and for $c = -N/\cos\alpha$

$$\beta + (k+1)\pi < \phi(b, c) < \beta + (k+2)\pi. \tag{3.10}$$

From (3.9) and (3.10) we see for sufficiently large $N$ that the points

$$V\left(b, x\left(b, \frac{+N}{\cos\alpha}\right), y\left(b, \frac{+N}{\cos\alpha}\right)\right), W\left(b, x\left(b, \frac{+N}{\cos\alpha}\right), y\left(b, \frac{N}{\cos\alpha}\right)\right)$$

and

$$V\left(b, x\left(b, \frac{-N}{\cos\alpha}\right), y\left(b, \frac{-N}{\cos\alpha}\right)\right), W\left(b, x\left(b, \frac{-N}{\cos\alpha}\right), y\left(b, \frac{-N}{\cos\alpha}\right)\right)$$

lie on different sides of the straight line $V \sin\beta - \cos\beta = 0$ in the $V$-$W$ plane.

Applying Lemma 1 at $t = a$ we have the existence of a connected set $S \subset R^p \times R^q$ such that $V(a, s) = [-N, N]$. By Lemma 2 the set

$$\bigcup_{(x_0, y_0)\in S} \{x(t, a, x_0), y(t, a, y_0)\}$$

is a connected set for each $t \in [a, b]$. Since $V$ and $W$ are continuous functions the set

$$\bigcup_{(x_0, y_0)\in S} \{V(b, x(b, a, x_0), y(b, a, y_0)), W(b, x(b, a, x_0), y(b, a, y_0))\}$$

is connected in $R^2$. Thus there exists a point $(x_0^*, y_0^*) \in S$ such that

$$V(a, x_0^*, y_0^*) = N^* \quad \text{and} \quad W(a, x_0^*, y_0^*) = N^* \tan\alpha$$

for some $N^* \in (-N, N)$, and the solution $x(t, a, x_0^*)$, $y(t, a, t_0^*)$ of (S) satisfies

$$V(b, x(b, a, x_0^*), y(b, a, x_0^*)) \sin\beta - W(b, x(b, a, x_0^*), y(b, a, x_0^*)) \cos\beta = 0.$$

This completes the proof of Theorem 1.

**Proof of Corollary 1.** For $C_1 = C_2 = 0$ (2.12) and (2.11) are equivalent. Observe that $V + k_1$ and $W + k_2$ will satisfy the hypotheses (2.1) and (2.2) whenever $V$ and $W$ satisfy them.

Moreover the homogeneity of $F_i$ implies that $|F_i(t, V, W)| \le \sqrt{V^2 + W^2}$ for all $t \in [a, b]$ and for some $A \ge 0$. Thus

$$|F_i(t, V_1 - k_1, W_1 - k_2) - F_i(t, V_1, W_1)|/(V^2 + W^2) \to 0$$

as $V^2 + W^2 \to \infty$. A similar result holds for $G_i$. Hence if $V$ and $W$ satisfy (2.4) then $V_1 = V + k_1$ and $W_1 = W + k_2$ satisfy the same inequality where now for example $V_1' \le F_1(t, V_1, W_1) + \overline{\delta}_1(t, V_1, W_1)$, where $\overline{\delta}_1(t, V_1, W_1) = \delta_1(t, V_1 - k_1, W_1 - k_2) + |F_1(t, V_1 - k_1, W_1 - k_2) - F(t, V_1, W_1)|$ and thus satisfies (2.5). Hence (2.4) holds for $V_1$ and $W_1$ and the hypotheses of Theorem 1 are verified.

## 4. Examples and concluding remarks.

The assumption that $V$ and $W$ be continuously differentiable can be weakened to continuous in $t$, $x$, and $y$ and locally Lipschitz in $x$ and $y$. To see this observe that if $V$ is locally Lipschitz, we can define the derivative of $V$ along solutions of (S) as

$$D^+V(t, x, y) = \limsup_{h\to 0^+} h^{-1}V(t+h, x+hf(t, x, y), y+hg(t, x, y)) - V(t, x, y),$$

and moreover [see 9] it follows that

$$D^+V(t, x, y) = \limsup_{h\to 0^+} h^{-1}V(t+h, x(t+h), y(t+h)) - V(t, x, y).$$

Similarly we can define $D_+V$ to be

$$D_+V(t, x, y) = \liminf_{h\to 0^+} h^{-1}V(t+hf(t, x, y), y+gh(t, x, y)) - V(t, x, y)$$

$$= \liminf_{h\to 0^+} h^{-1}V(t+h, x(t+h), y(t+h)) - V(t, x, y).$$

Similar definitions hold for $D^+W$ and $D_+W$. As in the proof of Theorem 1 if we define $\phi(t) = \tan^{-1}(W/V)$ then we may compute $D^+\phi(t)$ in terms of $D^+W$, $D_+W$, $D^+V$, $D_+V$. As before $D^+\phi(t)$ will be continuous everywhere except for multiples of $\pi/2$. The proof then proceeds exactly as in Theorem 1 where now in (2.4) we replace $V'$ and $W'$ with the appropriate Dini derivative depending on the sign of $V$ and $W$.

The following examples will, in general, have nonlinear boundary conditions. By proper selection of the comparison system we will often not have to restrict the time interval.

**Example 1.** Consider the boundary value problem

$$x' = x + f(y), \qquad y' = 2y + g(x);$$
$$x^2(a) - y(a) = C_1, \qquad x(b) = C_2. \tag{4.1}$$

We will show there exist solutions for all $a$ and $b$. Assume $f, g: R \to R$ are continuous and

$$|f(y)| \to 0 \quad \text{as } |y| \to \infty, \qquad |g(x)|/|x| \to 0 \quad \text{as } |x| \to \infty. \tag{4.2}$$

Define $V = x$, $W = x^2 - y$. We are thus interested in finding solutions of (4.1) subject to $W(a) = C_1$, $V(b) = C_2$. We notice that $V$ and $W$ satisfy (2.1) since the level curves $V = k_1$ and $W = k_2$ intersect in $R^2$ for every $k_1$ and $k_2$. It is is not difficult to see that (2.2) is also verified. Now

$$V' = x' = x + f(y) = V + f(y),$$

$$W' = 2xx' - y' = 2x^2 + 2xf(y) - 2y - g(x)$$

$$= 2y - 2W + 2xf(y) - 2y - g(x) = -2W + 2xf(y) - g(x).$$

Let $F_1 \equiv F_2 \equiv V$, $\delta_1 = \delta_3 = f(y)$, $G_1 = G_2 = -2W$, and $\delta_4 = \delta_2 = 2xf(y) - g(x)$. In view of (4.2) $|2xf(y) - g(x)|/(|x^2 - y| + |x|) \to 0$ as $|x^2 - y| + |x| \to \infty$ and thus (2.5) holds. Define $P_1 \equiv Q_1 \equiv V$ and $Q_2 \equiv P_2 \equiv -2W$. Hence systems (2.7) and (2.8) are the same. Because solutions of (2.9) and (2.10) are unique (due to the linearity of $P_i$ and $Q_i$), then from the remarks following Theorem 1 and from Corollary 1 it is sufficient to show there exist no solutions of

$$(4.3) \qquad V' = V, \quad W' = -2W,$$

satisfying

$$(4.4) \qquad W(a) = 0, \quad V(b) = 0.$$

It is clear that there exist no solutions of (4.3) and (4.4) for any $a$ and $b$.

We now apply our results to a third order system.

**Example 2.** Consider the boundary value problem

$$(4.5) \qquad x' = x + \sin z, \quad y' = x + y, \quad z' = \tfrac{1}{2} z,$$

subject to the boundary conditions

$$(4.6) \qquad x(a) = 1, \quad y(b) - z^2(b) = 0.$$

Define $V = x - 1$, $W = v - z^2$. A little computation yields (2.2) and (2.1) since the level curves of $V$ and $W$ always intersect. Observe also that

$$V' = x' = V + 1 + \sin z \quad \text{and} \quad W' = y' - 2zz' = y + x - z^2 = W + V + 1.$$

Now let $F_1 = F_2 = V$, $G_1 = G_2 = W + V$, $\delta_3 = \delta_1 = 1 + \sin z$, and $\delta_4 = \delta_2 = 1$. A simple computation shows that $\delta_i$ $i = 1, 2, 3, 4$ satisfy (2.5). Thus in order to show the existence of solutions of (4.5) and (4.6) it is sufficient to show there exist no nontrivial solution of

$$(4.7) \qquad V' = V, \quad W' = W + V, \quad V(a) = 0, \quad W(b) = 0.$$

This follows immediately from the fact that (4.7) yields $V' = 0$ and $W' = W$. Clearly no nontrivial solution of (4.7) exists for any $a$ and $b$.

In [8] Waltman considered two types of existence theorems for the nonlinear boundary value problem. In one result he assumed that $f$, $g$, $V$, and $W$ all satisfy a global Lipschitz condition while in another result he needed uniqueness of the boundary value problem in order to get existence. Our theorem will apply to those examples given by Waltman. We will neither require the boundary conditions to satisfy a global Lipschitz condition nor require the uniqueness of the boundary value problem in order to show existence. Of course in general there may be many solutions which solve the problem.

**Example 3 (Waltman [8]).** Consider the boundary value problem

$$x' = \sin y, \quad y' = x - y.$$

$$x(a) - y(a) = 0, \quad x(b) + y(b) + x(b)/(1 + x^2(b)) = 0.$$

Let

$$V = x - y, \quad W = x + y + x/(1 + x^2).$$

Clearly $V$ and $W$ satisfy (2.1) and (2.2). Now

$$V' = x' - y' = \sin y - (x - y) = -V + \sin y,$$

and

$$W' = x' + y' + \frac{1 - x^2}{(1 + x^2)^2} \sin y = V + \frac{(1 - x^2)}{(1 + x^2)^2} \sin y + \sin y.$$

We may let $\delta_1 = \delta_3 = \sin y$ and $\delta_2 = \delta_4 = (1 - x^2)/(1 + x^2)^2 \sin y + \sin y$. It follows that $\delta_i$ satisfies (2.5). Thus we have reduced the problem to showing that no nontrivial solution of

$$V' = -V, \qquad W' = V; \tag{4.8}$$

with $V(a) = 0$, $W(b) = 0$ exists. This follows immediately since the point $V = 0$ is an equilibrium point. Thus there exists a solution to the boundary value problem for any $a$ and $b$. For this problem our proof is much shorter since we do not have to verify the uniqueness of boundary value problems.

We again consider another example due to Waltman.

**Example 4 (Waltman [8]).** Consider the problem

$$x' = x + 2y \sin^2 x, \qquad y' = \frac{v}{2} + \sin^2 x,$$
$$y(a) = C, \qquad x(b) - y^2(b) = C_2.$$

Let $V = y - C$, $W = x - y^2 - C_2$. Then

$$V' = y' = \frac{y}{2} + \sin^2 x = \frac{V}{2} + \sin^2 x + \frac{C}{2};$$
$$W' = x' - 2yy' = x + 2y \sin^2 x - y^2 - 2y \sin^2 x = x - y^2 = W + C_2.$$

Thus, since $V$ and $W$ satisfy (2.1) and (2.2), and (2.5) is satisfied for $\delta_1 = \delta_3 = \sin^2 x + C/2$, $\delta_2 = \delta_4 = C_2$ it is sufficient to show there exist no nontrivial solutions of $V' = V/2$, $W' = W$, satisfying $V(a) = 0$ and $W(b) = 0$. This follows immediately. Once again it is not necessary to determine whether there exists uniqueness of boundary value problems.

**Example 5.** Consider the boundary value problem

$$x' = 1, \qquad y' = |y|^{1/2}, \tag{4.9}$$

where

$$x(a) = 0, \qquad x^2(b) - y(b) = 0. \tag{4.10}$$

If we let $V = x$, $W = x^2 - y$, then

$$V' = 1, \qquad W' = 2xx' - \tfrac{1}{2}|y|^{-1/2}|y|^{1/2} \operatorname{sgn} y = 2x - \tfrac{1}{2} \operatorname{sgn} y.$$

Our comparison system is

$$V' = 0, \qquad W' = 2V \tag{4.11}$$

with $V(a) = 0$ and $W(b) = 0$. Obviously there exist no nontrivial solutions of (4.11). Moreover $V$ and $W$ satisfy (2.1) and (2.2). Thus there exist solutions of (4.9). For this example Waltman's result does not apply since it can be shown that there exists more than one solution of the boundary value problem (4.9), (4.10).

In the previous examples our comparison systems have essentially been linear. In our next example the comparison systems are not linear and in fact have a discontinuity along a certain curve. Of course, we do require the positive homogeneity of our system.

**Example 6.** Consider the boundary value problem

$$x' = \begin{cases} y^2 - \sqrt{(y^2 - x)^2 + y^2}, & y^2 - x > 0, \\ y^2, & y^2 = x, \\ y^2 + \sqrt{(y^2 - x)^2 + y^2}, & y^2 - x < 0, \end{cases}$$

$$y' = \frac{y}{2} + f(x),$$

$$y^2(a) - x(a) = 0, \qquad y(b) = 0.$$

Assume $f\colon R \to R$ is continuous and $|f(x)| \to 0$ as $|x| \to \infty$. It is not difficult to see that solutions exist and are continuously differentiable except at the curve $y^2 = x$.

If we let $V = y^2 - x$, $W = y$, then

$$V' = 2yy' - x' = \begin{cases} 2yf(x) - \sqrt{(y^2 - x)^2 + y^2}, & y^2 - x > 0, \\ 2yf(x), & y^2 = x, \\ 2yf(x) + \sqrt{(y^2 - x)^2 - y^2}, & y^2 - x < 0, \end{cases}$$

$$W' = y' = \frac{y}{2} + f(x).$$

Hence

$$V' = \begin{cases} -\sqrt{V^2 + W^2} + 2yf(x), & V > 0, \\ 0, & V = 0, \\ +\sqrt{V^2 + W^2} + 2yf(x), & V < 0, \end{cases}$$

$$W' = \frac{W}{2} + f(x).$$

Thus, since $V$ and $W$ satisfy (2.1) and (2.2), and (2.5) is satisfied for $\delta_1 = \delta_3 = 2yf(x)$, $\delta_2 = \delta_4 = f(x)$, it is sufficient to show that there exists no nontrivial solution of

$$V' = \begin{cases} -\sqrt{V^2 + W^2}, & V > 0, \\ 0, & V = 0, \\ +\sqrt{W^2 + W^2}, & V < 0, \end{cases}$$

$$W' = \frac{W}{2},$$

satisfying $V(a) = 0$, $W(b) = 0$. This follows immediately by considering the two cases $W(a) > 0$ and $W(a) < 0$. Once again there are no restrictions on the length of the interval $[a, b]$.

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