

REPRESENTATIONS OF JORDAN TRIPLES

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ABSTRACT. Some standard results on representations of quadratic Jordan algebras are extended to Jordan triples. It is shown that the universal envelope of a finite-dimensional Jordan triple is finite-dimensional, and that it is nilpotent if the Jordan triple is radical. A permanence principle and a duality principle are proved which are useful in deriving identities.

Introduction. A *Jordan triple* is a module V over a commutative ring k together with a composition $(x, y) \mapsto P(x)y$ which is quadratic in x and linear in y and satisfies certain identities (see (1)–(3) below). A typical example is the space of $p \times q$ -matrices over k with $P(x)y = x('y)x$. If J is a quadratic Jordan algebra with quadratic operators U_x then J is also a Jordan triple with $P(x)y = U_x y$. Thus Jordan triples are a natural generalization of quadratic Jordan algebras. For a systematic theory of Jordan triples see [2] and [5].

In this note, we extend to Jordan triples certain standard results from the representation theory of quadratic Jordan algebras (see [4]). Our main results concern the case where V is finite-dimensional over a field k . Then the universal envelope of V is also finite-dimensional (Theorem 2.4), and it is nilpotent in case V is radical (Theorem 3.3). The latter result is due to C.T. Anderson in case $\text{char } k \neq 2$. We also prove a permanence principle and a duality principle which are useful in deriving identities.

In [6], K. Yamaguti also defines representations of Jordan triple systems. However, his concept of Jordan triple system is different from ours (the Jordan triple systems of type II considered in [6] are a generalization of our Jordan triples).

1. Representations.

1.1. Jordan triples. Let k be a commutative ring with unit and let V and W be unital k -modules. A map $P: V \rightarrow W$ is called *quadratic* if $P(\alpha x) = \alpha^2 P(x)$ for all $\alpha \in k$, $x \in V$, and if $P(x, y) = P(x + y) - P(x) - P(y)$ is bilinear in x and y . If R is any commutative associative k -algebra then there is a unique quadratic map $P_R: V \otimes_k R \rightarrow W \otimes_k R$ of R -modules making the diagram

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$$\begin{array}{ccc}
 V \otimes_k R & \xrightarrow{P_R} & W \otimes_k R \\
 \uparrow & & \uparrow \\
 V & \xrightarrow{P} & W
 \end{array}$$

commutative (see [1]). In case $W = \text{End}_k V$, we denote the composition $V \otimes_k R \rightarrow (\text{End}_k V) \otimes_k R \rightarrow \text{End}_R(V \otimes_k R)$ also by P_R .

Let now $P: V \rightarrow \text{End}_k V$ be a quadratic map. We set

$$\{xyz\} = L(x, y)z = P(x, z)y.$$

Then $(x, y, z) \mapsto \{xyz\}$ is a k -trilinear map from $V \times V \times V$ into V such that $\{xyz\} = \{zyx\}$ and $\{xyx\} = 2P(x)y$. The pair (V, P) is called a *Jordan triple* if the identities

$$(1) \quad L(x, y)P(x) = P(x)L(y, x) = P(P(x)y, x),$$

$$(2) \quad L(P(x)y, y) = L(x, P(y)x),$$

$$(3) \quad P(P(x)y) = P(x)P(y)P(x)$$

hold in V and in all scalar extensions (V_R, P_R) of (V, P) (equivalently, if all linearizations of (1)–(3) hold in V).

A k -linear map $f: V \rightarrow W$ of Jordan triples is called a *homomorphism* if $f(P(x)y) = P(f(x))f(y)$ for all $x, y \in V$. An *ideal* of V is a k -submodule I satisfying $P(I)V + P(V)I + \{VVI\} \subset I$. For the general theory of Jordan triples see [2], [5].

1.2. *Identities.* By linearizing (1) we obtain

$$\begin{aligned}
 (4) \quad L(x, y)P(x, z) + L(z, y)P(x) &= P(x, z)L(y, x) + P(x)L(y, z) \\
 &= P(\{xyz\}, x) + P(P(x)y, z).
 \end{aligned}$$

We apply this to an element $u \in V$, regard it as a function of z and change u to z . Then we have

$$(5) \quad L(x, y)L(x, z) + L(P(x)z, y) = L(x, \{yxz\}) + P(x)P(y, z).$$

We linearize (2) with respect to x and y and obtain

$$(6) \quad L(\{xyz\}, y) = L(z, P(y)x) + L(x, P(y)z),$$

$$(7) \quad L(x, \{yxz\}) = L(P(x)y, z) + L(P(x)z, y).$$

Again we apply this to an element of V and regard it as a function of z and obtain, after a change of notation,

$$(8) \quad L(z, y)L(x, y) = P(x, z)P(y) + L(z, P(y)x),$$

$$(9) \quad P(x, z)L(y, x) = P(P(x)y, z) + L(z, y)P(x).$$

Subtract (9) from (4) to obtain

$$(10) \quad P(x)L(y, z) + L(z, y)P(x) = P(x, \{xyz\}).$$

Addition of (5) and (7) gives

$$(11) \quad L(x, y)L(x, z) = L(P(x)y, z) + P(x)P(y, z).$$

1.3. Definition. Let V be a Jordan triple over k , and let A be a unital associative k -algebra. A *representation* of V in A is a pair (l, p) of maps where $l: V \times V \rightarrow A$ is bilinear and $p: V \rightarrow A$ is quadratic, such that the following identities hold in all scalar extensions.

$$(12) \quad l(x, y)p(x) = p(x)l(y, x) = p(x, P(x)y),$$

$$(13) \quad p(x)l(y, z) + l(z, y)p(x) = p(x, \{xyz\}),$$

$$(14) \quad l(x, y)l(x, z) = l(P(x)y, z) + p(x)p(y, z),$$

$$(15) \quad l(z, x)l(y, x) = l(z, P(x)y) + p(y, z)p(x),$$

$$(16) \quad p(P(x)y) = p(x)p(y)p(x).$$

If A has an involution $a \mapsto a^*$ such that $l(x, y)^* = l(y, x)$ and $p(x)^* = p(x)$ for all $x, y \in V$ then (l, p) is called a **-representation*. In this case, (15) is a consequence of (14).

Example. (a) The *regular representation* (L, P) of V in $\text{End}_k V$.

(b) The *regular *-representation* of V in $E = \text{End}_k V \times (\text{End}_k V)^{\text{op}}$ given by

$$l(x, y) = (L(x, y), L(y, x)) \quad \text{and} \quad p(x) = (P(x), P(x)).$$

The interchange $(f, g) \mapsto (g, f)$ is an involution of E making (l, p) a *-representation.

1.4. Lemma. If (l, p) is a representation of V in A then the following formulas hold.

$$(17) \quad l(P(x)y, y) = l(x, P(y)x),$$

$$(18) \quad p(x, z)l(y, x) = l(z, y)p(x) + p(P(x)y, z),$$

$$(19) \quad l(x, y)p(x, z) = p(x)l(y, z) + p(P(x)y, z).$$

Proof. (17) follows by setting $y = z$ in (14) and (15) and subtracting. We linearize (12):

$$\begin{aligned} l(z, y)p(x) + l(x, y)p(x, z) &= p(x)l(y, z) + p(x, z)l(y, x) \\ &= p(x, \{xyz\}) + p(z, P(x)y), \end{aligned}$$

subtract (13) and obtain (18) and (19).

1.5. *Split null extensions.* If M is a k -module and (l, p) is a representation of V in $\text{End}_k M$ then we say M is a V -module. As in the case of quadratic Jordan algebras (see [4]) we have

Proposition. $V \oplus M$ becomes a Jordan triple with

$$P(x \oplus m)(y \oplus n) = P(x)y \oplus [p(x)n + l(x, y)m]$$

($x, y \in V, m, n \in M$), the split null extension of V by M .

Proof. If we use the fact that any product in $V \oplus M$ containing more than one element from M is zero, as well as the identities (12)–(19), the verification of (1) in $V \oplus M$ amounts to

$$\begin{aligned} p(x)p(y, z) + l(x, \{yxz\}) &= l(x, y)l(x, z) + l(P(x)z, y) \\ &= l(P(x)y, z) + l(x, z)l(x, y). \end{aligned}$$

But this is an easy consequence of (14) and (17). Similarly, (2) follows without difficulty from (15), (17), and (18), and (3) comes down to showing

$$(20) \quad l(P(x)y, z)p(x) = p(x)l(y, P(x)z)$$

and

$$(21) \quad p(x)p(y)l(x, z) + l(x, P(y)P(x)z) = l(P(x)y, z)l(x, y).$$

By (13), (12), and (17) we have

$$\begin{aligned} p(x)l(y, P(x)z) + l(P(x)z, y)p(x) &= p(x, \{x, y, P(x)z\}) \\ &= p(x, P(x)\{yxz\}) = l(x, \{yxz\})p(x) \\ &= l(P(x)y, z)p(x) + l(P(x)z, y)p(x) \end{aligned}$$

which proves (20). For (21), we use (19) and (15) and get

$$\begin{aligned} &p(x)p(y)l(x, z) + l(x, P(y)P(x)z) \\ &= p(x)l(y, x)p(y, z) \\ &\quad - p(x)p(P(y)x, z) + l(x, y)l(P(x)z, y) - p(x)l(z, x)p(y) \\ &= l(x, y)l(x, z)l(x, y) - p(x)p(P(y)x, z) - p(x)l(z, x)p(y) \\ &\hspace{25em} \text{(by (12) and (14))} \\ &= l(P(x)y, z)l(x, y) + p(x)[p(y, z)l(x, y) - p(P(y)x, z) - l(z, x)p(y)] \\ &\hspace{25em} \text{(by (14))} \\ &= l(P(x)y, z)l(x, y) \quad \text{(by (18)).} \end{aligned}$$

We remark that the discussion in [4] concerning the cohomology of quadratic Jordan algebras can be carried over word for word to the Jordan triple case, so we omit the details.

1.6. Let (JT_k) denote the class of all Jordan triples over k . As in [4], we obtain from 1.5 the following

Permanence principle. *If F is any identity in the $L(x, y)$'s and $P(z)$'s which is valid for the regular representation of all $V \in (JT_k)$ then the identity obtained from F by replacing L, P by l, p is valid for all representations of all $V \in (JT_k)$.*

Indeed, to prove F for a representation (l, p) of V in A , consider A as a V -module by composing (l, p) with the left regular representation of A . Then F is valid for the regular representation of the split null extension $V \oplus A$, and by restricting to A and applying F to the unit element of A the assertion follows.

Another useful device in deriving identities is the

Duality principle. *If F is any identity in $l(x, y)$'s and $p(z)$'s which is valid for every representation of all $V \in (JT_k)$ then its dual F^* , obtained by replacing $l(x, y)$ by $l(y, x)$ and reversing the order of the $l(x, y)$'s and $p(z)$'s, is also valid for every representation.*

Indeed, F holds in particular for the regular $*$ -representation of V in $E = \text{End}_k V \times (\text{End}_k V)^{\text{op}}$. Applying the involution of E and projecting onto the first factor $\text{End}_k V$ of E , we see that F^* holds for all regular representations. By the permanence principle, F^* holds for all representations.

1.7. *Homotopes.* Let $a \in V$. With the operations

$$U_x y = P(x)P(a)y, \quad x^2 = P(x)a,$$

the k -module V becomes a quadratic Jordan algebra V_a , the *homotope* of V with respect to a (cf. [5]). Let $\hat{V}_a = k.1 \oplus V_a$ be the unital quadratic Jordan algebra obtained from V_a by adjoining a unit element. Recall that a *unital quadratic representation* of a unital quadratic Jordan algebra J in a unital associative algebra A is a quadratic map $\mu: J \rightarrow A$ satisfying the following identities in all scalar extensions (cf. [4]):

$$(22) \quad \mu(1) = 1,$$

$$(23) \quad \mu(U_x y) = \mu(x)\mu(y)\mu(x),$$

$$(24) \quad \nu(x, y)\mu(x) = \mu(x)\nu(y, x) = \mu(U_x y, x),$$

where $\nu(x, y) = \mu(x, 1)\mu(y, 1) - \mu(x, y)$.

Proposition. *Let (l, p) be a representation of the Jordan triple V in A . Then for every $a \in V$,*

$$\mu(\alpha.1 + x) = \alpha^2.1 + \alpha l(x, a) + p(x)p(a)$$

defines a unital quadratic representation of $\hat{V}_a$ in A .

Proof. Let $\bar{\mu}(\alpha.1 + x) = U(\alpha.1 + x)|V_a = \alpha^2 \text{Id}_V + \alpha L(x, a) + P(x)P(a)$. Since V_a is an ideal of $\hat{V}_a$, this is a unital quadratic representation of $\hat{V}_a$, and the validity of (23) and (24) for $\bar{\mu}$ is equivalent to certain identities in L 's and P 's. By the permanence principle, the same identities hold with L, P replaced by l, p , i.e., for μ . Hence μ satisfies (23) and (24). Since it is obviously quadratic and $\mu(1) = 1$, the proposition follows.

Recall that a pair $(x, a) \in V \times V$ is called quasi-invertible if $1 - x$ is invertible in $\hat{V}_a$ (cf. [5]).

Corollary. *If (x, a) is quasi-invertible then*

$$b(x, a) = 1 - l(x, a) + p(x)p(a)$$

is invertible in A .

Indeed, $b(x, a) = \mu(1 - x)$, and μ maps invertible elements of $\hat{V}_a$ into invertible elements of A .

1.8. Definition. A λ -representation of V in a unital associative algebra A is a bilinear map $l: V \times V \rightarrow A$ such that the identities

$$(17) \quad l(P(x)y, y) = l(x, P(y)x),$$

$$(25) \quad l(x, y)l(u, v) - l(u, v)l(x, y) = l(\{xyu\}, v) - l(u, \{yxv\}),$$

$$(26) \quad \begin{aligned} l(x, y)^4 &= l(x, y)^2 l(P(x)y, y) + l(P(x)y, y) l(x, y)^2 \\ &\quad + l(P(x)y, y)^2 - l(P(P(x)y)y, y) - l(x, P(P(y)x)x) \end{aligned}$$

hold in all scalar extensions.

A π -representation of V in A is a quadratic map $p: V \rightarrow A$ such that the identities

$$(27) \quad p(x + y) = p(x) + p(y),$$

$$(16) \quad p(P(x)y) = p(x)p(y)p(x)$$

hold in all scalar extensions.

Lemma. (a) *If (l, p) is a representation of V then l is a λ -representation.*

(b) *If p is a π -representation then $2p(x) = 0$ for all $x \in V$ and*

$$(28) \quad p(\{xyz\}) = p(x)p(y)p(z) + p(z)p(y)p(x).$$

Proof. (a) (25) follows from the corresponding identity for the regular representation (see [5]) by the permanence principle. If we set $y = z$ in (14) we get $l(x, y)^2 - l(P(x)y, y) = 2p(x)p(y)$. By squaring this and using (16) and (17) we get (26).

(b) By (27), $2p(x) = p(2x) = 4p(x)$, and (28) follows from

$$\begin{aligned}
 p(\{xyz\}) &= p(P(x+z)y - P(x)y - P(z)y) \\
 &= p(P(x+z)y) + p(P(x)y) + p(P(z)y) \\
 &= p(x+z)p(y)p(x+z) + p(x)p(y)p(x) + p(z)p(y)p(z) \\
 &= p(x)p(y)p(z) + p(z)p(y)p(x) + 2p(x)p(y)p(x) + 2p(z)p(y)p(z).
 \end{aligned}$$

2. Universal envelopes.

2.1. We first construct a universal object for quadratic maps. Let V be a k -module, let $q: V \rightarrow X$ be a bijection of V onto a set X , and let F be the free k -module generated by X . We set $q(x, y) = q(x+y) - q(x) - q(y)$, and let R be the submodule of F generated by

$$\begin{aligned}
 q(\alpha x) - \alpha^2 q(x), \quad q(\alpha x, y) - \alpha q(x, y), \\
 q(x+y, z) - q(x, z) - q(y, z),
 \end{aligned}$$

where $\alpha \in k, x, y, z \in V$. We set $V^{\text{II}} = F/R$ and denote the image of $q(x)$ under the canonical map by x^{II} . We also set $\langle x, y \rangle = (x+y)^{\text{II}} - x^{\text{II}} - y^{\text{II}}$. Then $x \mapsto x^{\text{II}}$ is a quadratic map, V^{II} is generated by $\{x^{\text{II}}: x \in V\}$, and for any quadratic map $Q: V \rightarrow W$ there is a unique linear map $f: V^{\text{II}} \rightarrow W$ such that $Q(x) = f(x^{\text{II}})$. Also it is easily seen that V^{II} is functorial in V and compatible with extensions of the ring of scalars.

2.2. *The universal envelope.* Let V be a Jordan triple over k , let $W = V^{\text{II}} \oplus (V \otimes_k V)$, and let $T(W)$ be the tensor algebra over W . The product of two elements $u, v \in T(W)$ is denoted by $u \cdot v$.

Let J be the ideal of $T(W)$ generated by the elements

$$\begin{aligned}
 (x \otimes y) \cdot x^{\text{II}} - x^{\text{II}} \cdot (y \otimes x), \quad (x \otimes y) \cdot x^{\text{II}} - \langle x, P(x)y \rangle, \\
 x^{\text{II}} \cdot (y \otimes z) + (z \otimes y) \cdot x^{\text{II}} - \langle x, \{xyz\} \rangle, \\
 (x \otimes y) \cdot (x \otimes z) - P(x)y \otimes z - x^{\text{II}} \cdot \langle y, z \rangle, \\
 (z \otimes x) \cdot (y \otimes x) - z \otimes P(x)y - \langle y, z \rangle \cdot x^{\text{II}}, \\
 (P(x)y)^{\text{II}} - x^{\text{II}} \cdot y^{\text{II}} \cdot x^{\text{II}},
 \end{aligned}$$

corresponding to (12)–(16). The *universal envelope* of V is $U(V) = T(W)/J$. We define $\tilde{l}: V \times V \rightarrow U(V)$ by $\tilde{l}(x, y) = x \otimes y + J$ and $\tilde{p}: V \rightarrow U(V)$ by $\tilde{p}(x) = x^{\text{II}} + J$.

Proposition. (a) *There is an involution $*$ of $U(V)$ such that $(\tilde{l}, \tilde{p})$ is a $*$ -representation of V . For any representation (l, p) of V in A there is a unique homomorphism $f: U(V) \rightarrow A$ of unital associative algebras such that $p = f \circ \tilde{p}$ and $l = f \circ \tilde{l}$. If (l, p) is a $*$ -representation then f commutes with the involutions of $U(V)$ and A .*

(b) *There is an augmentation $\epsilon: U(V) \rightarrow k$ so that $U(V) = k.1 \oplus U^\circ(V)$ where $U^\circ(V) = \text{Ker } \epsilon$ is the augmentation ideal. Also, $U(V)$ is functorial in V and is compatible with scalar extensions.*

(c) *If I is an ideal of V and $\tilde{I}$ is the ideal of $U(V)$ generated by $\tilde{l}(I, V)$, $\tilde{l}(V, I)$, $\tilde{p}(I, V)$, and $\tilde{p}(I)$ then $U(V/I) \cong U(V)/\tilde{I}$.*

The proof of this proposition follows established lines and is therefore omitted. Let us just indicate how the involution $*$ of $U(V)$ is defined. The k -module $W = V^{\text{II}} \oplus (V \otimes V)$ possesses an endomorphism of period 2 given by $x^{\text{II}} \mapsto x^{\text{II}}$ and $x \otimes y \mapsto y \otimes x$. By the universal property of the tensor algebra, this endomorphism extends to an involution of $T(W)$ leaving J invariant, and thus induces an involution $*$ of $U(V)$ with the desired properties.

2.3. Similarly as in 2.2, we define the *universal λ -envelope* $U_\lambda(V) = T(V \otimes V)/J_\lambda$ where J_λ is the ideal of $T(V \otimes V)$ generated by the elements

$$\begin{aligned} &P(x)y \otimes y - x \otimes P(y)x, \\ &(x \otimes y) \cdot (u \otimes v) - (u \otimes v) \cdot (x \otimes y) - \{xyu\} \otimes v + u \otimes \{yxv\}, \\ &(x \otimes y)^4 - (x \otimes y)^2 \cdot (P(x)y \otimes y) - (P(x)y \otimes y) \cdot (x \otimes y)^2 \\ &\quad - (P(x)y \otimes y)^2 + P(P(x)y)y \otimes y + x \otimes P(P(y)x)x, \end{aligned}$$

corresponding to (17), (25), and (26), and the *universal π -envelope* $U_\pi(V) = T(V^{\text{II}})/J_\pi$ where J_π is the ideal of $T(V^{\text{II}})$ generated by the elements $\langle x, y \rangle = (x + y)^{\text{II}} - x^{\text{II}} - y^{\text{II}}$ and $(p(x)y)^{\text{II}} - x^{\text{II}} \cdot y^{\text{II}} \cdot x^{\text{II}}$. Define $\tilde{l}: V \times V \rightarrow U_\lambda(V)$ by $\tilde{l}(x, y) = x \otimes y + J_\lambda$ and $\tilde{p}: V \rightarrow U_\pi(V)$ by $\tilde{p}(x) = x^{\text{II}} + J_\pi$.

Proposition. (a) $\tilde{l}$ (resp. $\tilde{p}$) is a λ -representation (resp. a π -representation) of V in $U_\lambda(V)$ (resp. in $U_\pi(V)$), and any λ - (resp. π -) representation may be factored via $U_\lambda(V)$ (resp. $U_\pi(V)$).

(b) $U_\lambda(V)$ and $U_\pi(V)$ are functorial in V and compatible with scalar extensions. There are augmentations $U_\lambda(V) \rightarrow k$ and $U_\pi(V) \rightarrow k$ so that $U_\lambda(V) = k.1 \oplus U_\lambda^\circ(V)$ and $U_\pi(V) = k.1 \oplus U_\pi^\circ(V)$.

(c) If I is an ideal of V and if I_λ (resp. I_π) is the ideal of $U_\lambda(V)$ (resp. $U_\pi(V)$) generated by $\tilde{l}(I, V)$ and $\tilde{l}(V, I)$ (resp. $\tilde{p}(I)$) then $U_\lambda(V/I) \cong U_\lambda(V)/I_\lambda$ (resp. $U_\pi(V/I) \cong U_\pi(V)/I_\pi$).

(d) $U_\lambda(V)$ and $U_\pi(V)$ possess involutions $*$ such that $\tilde{l}(x, y)^* = \tilde{l}(y, x)$ and $\tilde{p}(x)^* = \tilde{p}(x)$. Also, $U_\pi(V) = U_\pi^+(V) \oplus U_\pi^-(V)$ is $\mathbb{Z}_2$ -graded, the gradation is invariant under $*$, and $\tilde{p}(x) \in U_\pi^-(V)$ for $x \in V$.

The proof of (a), (b), and (c) is again straightforward. To prove (d), let $*$ denote the involutions of $T(V \otimes V)$ (resp. $T(V^{\text{II}})$) induced by $x \otimes y \mapsto y \otimes x$ (resp. the identity on V^{II}). Then J_λ and J_π are invariant under $*$, and the statement about the involutions follows. Finally, J_π is generated by elements of odd degree (with

respect to the natural gradation of $T(V^{\text{II}})$, and the gradation of $T(V^{\text{II}})$ is invariant under $*$.

2.4. Theorem. *If k is a field and V is finite-dimensional over k then so are $U(V)$, $U_\lambda(V)$, and $U_\pi(V)$.*

Proof. Let $x_1, \dots, x_n$ be a basis of V . First we show that $U_\lambda(V)$ is finite-dimensional. As an algebra, $U_\lambda(V)$ is generated by 1 and $\{\bar{l}(x_i, x_j) : i, j = 1, \dots, n\}$. We number the $\bar{l}(x_i, x_j)$ consecutively: $y_1 = \bar{l}(x_1, x_1)$, $y_2 = \bar{l}(x_1, x_2)$, $\dots$, $y_{n^2} = \bar{l}(x_n, x_n)$.

Lemma 1. $U_\lambda(V)$ is spanned by the monomials $y_{i_1} \cdots y_{i_r}$ where $0 \leq r \leq 3n^2$ and $i_1 \leq i_2 \leq \dots \leq i_r$.

Proof. Let X_r be the subspace of $U_\lambda(V)$ spanned by the monomials $y_{i_1} \cdots y_{i_r}$ where $s \leq r$. Clearly, $X_0 = k \cdot 1$, $X_r \subset X_{r+1}$, and $X_r \cdot X_s \subset X_{r+s}$. We claim that $X_r = X_{r-1}$ if $r > 3n^2$. Indeed, because of (25) we have

$$(29) \quad y_i y_j \equiv y_j y_i \pmod{X_1}.$$

In a monomial $y_{i_1} \cdots y_{i_r}$ where $r > 3n^2$, at least one of the y_{i_s} , say y_{i_1} , occurs at least 4 times. By (29), we get $y_{i_1} \cdots y_{i_r} \equiv y_{i_1}^4 y_{i_2} \cdots y_{i_{r-4}} \pmod{X_{r-1}}$, and (26) shows $y_{i_1}^4 \in X_3$. This proves our assertion. Since $U_\lambda(V)$ is the union of the X_r , we have $U_\lambda(V) = X_{3n^2}$. Finally, the ordered monomials suffice because of (29).

Lemma 2. *The subalgebra U' of $U(V)$ generated by $\bar{l}(V, V)$ and $\bar{p}(V, V)$ is a finite-dimensional ideal of $U(V)$.*

Proof. Let L be the subalgebra of $U(V)$ generated by $\bar{l}(V, V)$. Since $\bar{l}: V \times V \rightarrow U(V)$ is a λ -representation, L is a homomorphic image of $U_\lambda^\circ(V)$ and therefore finite-dimensional by Lemma 1. Let P be the subalgebra of $U(V)$ generated by $\bar{p}(V, V)$, and let P_+ be the subalgebra of P generated by $\{\bar{p}(u, v)\bar{p}(x, y) : u, v, x, y \in V\}$. From (14) we obtain by linearizing

$$\bar{p}(u, v)\bar{p}(x, y) = \bar{l}(v, y)\bar{l}(u, x) + \bar{l}(u, y)\bar{l}(v, x) - \bar{l}(\{uv\}, x)$$

which implies $P_+ \subset L$. Also $P = P_+ + \bar{p}(V, V) + P_+ \bar{p}(V, V)$ shows that P is finite-dimensional. From (13) we get by linearizing

$$\bar{l}(z, y)\bar{p}(x, w) = \bar{p}(x, \{wyz\}) + \bar{p}(w, \{xyz\}) - \bar{p}(x, w)\bar{l}(y, z)$$

which implies by induction

$$(30) \quad LP \subset PL + P.$$

This shows that $U' = L + PL + P$ is finite-dimensional. Finally, it follows from (14), (15), (18) and (19) that U' is an ideal of $U(V)$. Note that $U' = U^\circ(V)$ if $\text{char } k \neq 2$.

Now let $z_i = \bar{p}(x_i) \in U_\pi(V)$.

Lemma 3. $U_r(V)$ is spanned by the monomials $z_{i_1} \cdots z_{i_r}$ where $0 \leq r \leq 2n$.

Proof. Similarly to the proof of Lemma 1, let X_r be the subspace of $U_r(V)$ spanned by the monomials $z_{i_1} \cdots z_{i_r}$ where $0 \leq s \leq r$. If $z_{i_1} \cdots z_{i_r}$ is a monomial with $r > 2n$ then at least one of the z_i , say z_1 , occurs at least 3 times. Because of (28) we have $z_1 z_j z_k \equiv z_k z_j z_1 \pmod{X_1}$. Using this repeatedly, we see that $z_{i_1} \cdots z_{i_r}$ is congruent, modulo X_{r-2} , to a monomial of the form $\cdots z_1^3 \cdots$ or $\cdots z_1 z_i z_1 \cdots$. But $z_1^3 \in X_1$ and $z_1 z_i z_1 \in X_1$ by (16). Hence $X_r = X_{r-2}$ if $r > 2n$ which shows $U_r(V) = X_{2n}$.

Now we finish the proof of the theorem. From the definition of U' it is clear that the map $p: x \mapsto \tilde{p}(x) + U'$ is a π -representation of V in $U(V)/U'$, and that $U(V)/U'$ is generated by 1 and $p(V)$. Hence $U(V)/U'$ is a homomorphic image of $U_r(V)$ and therefore finite-dimensional by Lemma 3. Now $U(V)$ is finite-dimensional by Lemma 2.

3. Nilpotence.

3.1. *The radical.* Let V be a Jordan triple over the ring k . The *radical* of V is

$$\text{Rad } V = \{x \in V: (x, y) \text{ is quasi-invertible, for all } y \in V\}.$$

For the basic properties of the radical we refer to [5]. In particular, $\text{Rad } V$ is an ideal of V , and if $f: V \rightarrow W$ is a surjective homomorphism of Jordan triples then $\text{Rad } W \supset f(\text{Rad } V)$. A Jordan triple with $\text{Rad } V = 0$ is called *semisimple*. Recall also that an *inner ideal* of V is a k -submodule I such that $P(I)V \subset I$, and an *absolute zero divisor* is an element $x \in V$ such that $P(x) = 0$. It is easily seen that an absolute zero divisor belongs to the radical.

The proof of the following proposition can be found in [5].

Proposition. If V satisfies the descending chain condition on inner ideals then V is semisimple if and only if it contains no absolute zero divisors $\neq 0$.

3.2. **Proposition.** Let V be a Jordan triple over the ring k satisfying the descending chain condition on inner ideals. Let R be a commutative associative k -algebra, let $V_R = V \otimes_k R$ be the scalar extension of V , and define $\varphi: V \rightarrow V_R$ by $\varphi(x) = x \otimes 1$. Then $\varphi(\text{Rad } V) \subset \text{Rad } V_R$.

Proof. Let $I = \varphi^{-1}(\text{Rad } V_R)$. Since φ is a homomorphism of Jordan triples over k , this is an ideal of V . Let I_R be the R -submodule of V_R generated by $\varphi(I)$. Then I_R is an ideal of V_R , contained in $\text{Rad } V_R$. Since tensoring with R is a right exact functor, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & V & \longrightarrow & V/I \longrightarrow 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} \\ & & I \otimes_k R & \xrightarrow{i} & V_R & \longrightarrow & (V/I) \otimes_k R \longrightarrow 0 \end{array}$$

Clearly, $I_R = i(I \otimes_k R)$ so that V_R/I_R may be identified with $(V/I) \otimes_k R$. We denote the canonical maps $V \rightarrow V/I$ and $V_R \rightarrow V_R/I_R$ by $x \mapsto \bar{x}$. Assume now that $\bar{x}$ is an absolute zero divisor of V/I . Then $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$ is an absolute zero divisor of V_R/I_R and thus contained in the radical of V_R/I_R which is $(\text{Rad } V_R)/I_R$. This means $\varphi(x) \in \text{Rad } V_R$, i.e., $x \in I$ and therefore $\bar{x} = 0$. Since the descending chain condition is inherited by V/I we have V/I semisimple by 3.1, i.e., $I \supset \text{Rad } V$.

3.3. Theorem. *Let V be a finite-dimensional Jordan triple over the field k and assume that $V = \text{Rad } V$. Then $U^\circ(V)$, $U_\lambda^\circ(V)$ and $U_\ast^\circ(V)$ are nilpotent.*

Proof. By 3.2 and the fact that the universal envelopes are compatible with scalar extensions we may assume k algebraically closed. The crucial fact is

Lemma 1. *A finite-dimensional Jordan triple V over an algebraically closed field with $\text{Rad } V \neq 0$ contains an ideal $I \neq 0$ such that $P(I) = L(I, I) = 0$.*

This is proved in [3]. In case $\text{char } k \neq 2$, see also [2].

From now on, we assume $V = \text{Rad } V$ and k algebraically closed.

Lemma 2. $U_\lambda^\circ(V)$ is nilpotent.

The proof is by induction on $\dim V$. Let $\bar{l}: V \times V \rightarrow U_\lambda(V)$ be the universal λ -representation of V in $U_\lambda(V)$ (cf. 2.3). If $\dim V = 1$, i.e., $V = k.x$, then $P(x)x = 0$, and $U_\lambda^\circ(V)$ is generated by $\bar{l}(x, x)$. By (26), $\bar{l}(x, x)^4 = 0$ and hence $U_\lambda^\circ(V)$ is nilpotent. Now let $\dim V > 1$. Then by Lemma 1, V contains a proper ideal I such that $P(I)V = \{IIV\} = 0$. By induction, $U_\lambda^\circ(I)$ is nilpotent. Let $\mathbf{I}$ be the subalgebra of $U_\lambda(V)$ generated by $\bar{l}(I, I)$. Then $\mathbf{I}$ is a homomorphic image of $U_\lambda^\circ(I)$ and hence is nilpotent. Also, $\mathbf{I}$ is contained in the center of $U_\lambda(V)$ because of (25) and $\{IIV\} = 0$. Therefore $\mathbf{I}$ generates a nilpotent ideal $\mathbf{J} = \mathbf{I} \cdot U_\lambda(V)$ of $U_\lambda(V)$.

Let $\mathbf{I}'$ be the subalgebra of $U_\lambda(V)$ generated by $\bar{l}(I, V)$ and $\bar{l}(V, I)$. Then it follows from (25) and $\{IVI\} = \{IIV\} = 0$ that $\mathbf{I}'/\mathbf{J} \cap \mathbf{I}'$ is commutative. Also by (17), (26), and $P(I) = 0$, we have $\bar{l}(x, y)^4 \equiv 0 \pmod{\mathbf{J}}$ if $x \in V, y \in I$, or $x \in I, y \in V$. By finite-dimensionality, $\mathbf{I}'/\mathbf{J} \cap \mathbf{I}'$ is nilpotent, and hence $\mathbf{I}'$ is nilpotent.

Now let $\mathbf{I}_\lambda = \mathbf{I}' \cdot U_\lambda(V)$. Then (25) implies

$$(31) \quad U_\lambda(V) \cdot \mathbf{I}' \subset \mathbf{I}_\lambda.$$

Hence $\mathbf{I}_\lambda$ is an ideal of $U_\lambda(V)$, in fact, the ideal generated by $\bar{l}(I, V)$ and $\bar{l}(V, I)$. Since $\mathbf{I}'$ is nilpotent so is $\mathbf{I}_\lambda$ by (31). By induction, $U_\lambda^\circ(V/I) = U^\circ(V)/\mathbf{I}_\lambda$ (cf. 2.3) is nilpotent. Hence $U_\lambda^\circ(V)$ is nilpotent.

We denote by $\mathbf{U}'$ the ideal of $U(V)$ generated by $\bar{l}(V, V)$ and $\bar{p}(V, V)$ (see Lemma 2 in 2.4).

Lemma 3. U' is nilpotent.

Proof. Let L , P , and P_+ be as in the proof of Lemma 2 in 2.4. Then L is a homomorphic image of $U_\lambda^\circ(V)$ and therefore nilpotent by Lemma 2. Also P_+ is nilpotent since it is contained in L . Let P_- be the subspace of P spanned by the products of an odd number of elements of $\bar{p}(V, V)$. Then we have $P = P_+ + P_-$, $P_-^2 \subset P_+$, $P_+ P_- = P_- P_+ \subset P_-$, and induction shows $P^{2n} \subset P_+^n + P_+^n P_-$. Hence P is nilpotent.

From (30) we see that $J = PL + P$ is an ideal of U' , and by induction we have $J^n \subset P^n L + P^n$. Thus J is nilpotent. Since $U = L + J$ and $U'/J \cong L/L \cap J$ is nilpotent, U' is nilpotent.

Note. Lemma 2 and Lemma 3 (with a different proof) as well as the concept of λ -representation are due to C.T. Anderson (cf. [2], §8).

Lemma 4. There exists an ideal $N \neq 0$ of V such that $P(N) = L(V, N) = L(N, V) = 0$.

Proof. Let I be the ideal of Lemma 1, and consider the regular representation (L_I, P_I) of V in $\text{End}_k I$, i.e., $L_I(x, y) = L(x, y) | I$, and $P_I(x) = P(x) | I$. Let $f: U(V) \rightarrow \text{End}_k I$ be the induced homomorphism of associative algebras. Then by Lemma 3, $f(U')$ is nilpotent. Thus if $f(U')^n \neq 0$ but $f(U')^{n+1} = 0$ then $N = \{x \in I: f(U') \cdot x = 0\} \supset f(U')^n \cdot I \neq 0$ and N is invariant under $f(U(V))$ since U' is an ideal of $U(V)$. Hence N is an ideal of V with the desired properties.

Lemma 5. $U_\star^\circ(V)$ is nilpotent.

Proof. We may assume $\text{char } k = 2$ since otherwise $U_\star^\circ(V) = 0$. The proof is by induction on $\dim V$. If $V = 0$ there is nothing to prove. Let $V \neq 0$, and let N be as in Lemma 4. From $P(N)V = \{VNV\} = 0$, (16) and (28) we obtain

$$(32) \quad zxz = 0,$$

$$(33) \quad xyz = zyx,$$

for $z \in \bar{p}(N)$ and $x, y \in \bar{p}(V)$. We will show that the ideal $N_\star$ of $U_\star(V)$ generated by $\bar{p}(N)$ is nilpotent. Since $U_\star(V)$ is finite-dimensional, it suffices to show that $N_\star$ is spanned by nilpotent elements. Now $N_\star$ is spanned by the monomials $m = x_1 \cdots x_r$ where $x_i \in \bar{p}(V)$ and at least one of the x_i belongs to $\bar{p}(N)$. Using (33), we see that m^3 equals a monomial of the form $\cdots z^3 \cdots$ or $\cdots zxz \cdots$ where $z \in \bar{p}(N)$ and $x \in \bar{p}(V)$. Thus $m^3 = 0$ by (32), and $N_\star$ is nilpotent. By induction, $U_\star^\circ(V/N) = U_\star^\circ(V)/N_\star$ is nilpotent, and the lemma is proved.

Now we finish the proof of the theorem. $U^\circ(V)/U'$ is a homomorphic image of $U_\star^\circ(V)$ and therefore nilpotent by Lemma 5. Thus $U^\circ(V)$ is nilpotent by Lemma 3.

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