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STRUCTURE THEOREMS FOR CERTAIN TOPOLOGICAL RINGS

BY

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ABSTRACT. A Hausdorff topological ring B is called centrally linearly compact if the open left ideals form a fundamental system of neighborhoods of zero and B is a strictly linearly compact module over its center. A topological ring A is called locally centrally linearly compact if it contains an open, centrally linearly compact subring. For example, a totally disconnected (locally) compact ring is (locally) centrally linearly compact, and a Hausdorff finite-dimensional algebra with identity over a local field (a complete topological field whose topology is given by a discrete valuation) is locally centrally linearly compact. Let A be a Hausdorff topological ring with identity such that the connected component $\mathfrak{c}$ of zero is locally compact, $A/\mathfrak{c}$ is locally centrally linearly compact, and the center C of A is a topological ring having no proper open ideals. A general structure theorem for A is given that yields, in particular, the following consequences: (1) If the additive order of each element of A is infinite or squarefree, then $A = A_0 \times D$ where A_0 is a finite-dimensional real algebra and D is the local direct product of a family (A_γ) of topological rings relative to open subrings (B_γ), where each A_γ is the cartesian product of finitely many finite-dimensional algebras over local fields. (2) If A has no nonzero nilpotent ideals, each A_γ is the cartesian product of finitely many full matrix rings over division rings that are finite dimensional over their centers, which are local fields. (3) If the additive order of each element of A is infinite or squarefree and if C contains an invertible, topologically nilpotent element, then A is the cartesian product of finitely many finite-dimensional algebras over R, C, or local fields.

1. **Introduction.** Our goal is to identify those properties of a Hausdorff topological ring with identity that allow it to be described as built, in some concrete way, from topological rings that either are finite-dimensional Hausdorff topological algebras over local fields, the real field **R**, or the complex field **C**, or are very similar to such algebras. By a *local field* we mean a topological field whose topology is given by a complete, discrete valuation. Thus a topological field K is locally compact and totally disconnected if and only if K is a local field and the residue field of its valuation ring is finite. Local fields, **R**, and **C** are examples of complete topological fields whose topology is given by a proper absolute value; a finite-dimensional algebra A over such a field admits a unique Hausdorff topology making it a topological vector space, and A is actually a topological algebra for that topology [3, Theorem 2, p. 18; cf.

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Corollary 2, p. 19]. In this introductory section we shall describe the topological rings to be investigated.

We recall that a topological (not necessarily unitary) module is *linearly topologized* if the open submodules form a fundamental system of neighborhoods of zero. A topological module is *linearly compact* if it is Hausdorff, linearly topologized, and if every filter base of cosets of submodules has an adherent point. A linearly compact A-module E is *strictly linearly compact* if every continuous epimorphism from E onto any linearly compact A-module is open, or equivalently, if E/U is artinian for every open submodule U. (We assume familiarity with basic properties of linearly compact and strictly linearly compact modules, discussed in [11] or [4, Exercises 14–22, pp. 108–112].) A topological ring B is (strictly) linearly compact if the left B-module B is (strictly) linearly compact.

Definition 1. A Hausdorff topological ring B is *centrally linearly compact* if the open left ideals of B form a fundamental system of neighborhoods of zero and if B is a strictly linearly compact module over its center. A topological ring A is *locally centrally linearly compact* if A contains an open subring that is centrally linearly compact.

If B is centrally linearly compact and if R is any subring of B that contains the center C_B of B, then B is a strictly linearly compact (left) R-module, for B is R-linearly topologized as the open left ideals form a fundamental system of neighborhoods of zero, and every R-submodule is also a C_B-submodule. In particular, a centrally linearly compact ring is a strictly linearly compact ring. A commutative topological ring is clearly centrally linearly compact if and only if it is strictly linearly compact.

A totally disconnected locally compact ring A is locally centrally linearly compact, for A contains a compact subring B [10, Lemma 4], and the open ideals of B form a fundamental system of neighborhoods of zero [9, Lemma 9]. We shall determine the structure of a Hausdorff topological ring A with identity that has the following three properties: (a) the connected component $\mathfrak{c}$ of zero is locally compact; (b) $A/\mathfrak{c}$ is locally centrally linearly compact; (c) the center of A is a topological ring having no proper open ideals. Thus our results may be applied to obtain the structure of a locally compact ring with identity whose center is a topological ring having no proper open ideals.

Our first step is to identify a very special class of centrally linearly compact rings.

Definition 2. A topological ring B is *basic* if it satisfies the following three properties:

1°. The center C_B of B is a local, one-dimensional noetherian ring (with identity) that is complete for its natural topology (i.e., the $\mathfrak{m}$-adic topology, where

$\mathfrak{m}$ is the maximal ideal of C_B), and the prime ideals of the zero ideal of C_B are all isolated.

2°. B is a noetherian C_B-module, and the given topology of B is its natural topology as a C_B-module (for which $(\mathfrak{m}^n B)_{n\geq 0}$ is a fundamental system of neighborhoods of zero).

3°. Every cancellable element (i.e., non-zero-divisor) of the ring C_B is also a cancellable element of B.

Condition 3° implies that the identity element of C_B is also the identity element of B. By 2° and [18, Theorem 4, p. 254], the topology induced on C_B by the given topology of B is its natural topology. Since C_B is complete for that topology, therefore, B is also complete [18, Theorem 5, p. 256]. By [18, pp. 271–272], C_B is a linearly compact ring. Since $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finitely generated C_B-module and hence a finite-dimensional $(C_B/\mathfrak{m})$-vector space, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an artinian C_B-module for all $n \geq 0$, so by induction $C_B/\mathfrak{m}^k$ is an artinian C_B-module for all $k \geq 1$; thus C_B is a strictly linearly compact ring. As B is a finitely generated C_B-module, B is, for some $r \geq 1$, the image of the C_B-module C_B^r under a continuous homomorphism, and hence B is a strictly linearly compact C_B-module. In sum, we have proved the following theorem.

Theorem 1. *A basic topological ring is centrally linearly compact.*

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the prime ideals of the zero ideal of C_B. By 1° of Definition 2, $\mathfrak{m}, \mathfrak{p}_1, \cdots, \mathfrak{p}_r$ are all the prime ideals of C_B, and $\mathfrak{m} \neq \mathfrak{p}_i$ for all $i \in [1, r]$. The set of zero-divisors of B belonging to C_B is identical with the set of zero-divisors of the ring C_B by 3° of Definition 2, and that set is $\bigcup_{i=1}^r \mathfrak{p}_i$ [17, Corollary 3, p. 214]. Therefore, since $\mathfrak{m} \not\subseteq \bigcup_{i=1}^r \mathfrak{p}_i$ [17, p. 215], $\mathfrak{m}$ contains a cancellable element of B. If c is any cancellable element belonging to $\mathfrak{m}$, then $\mathfrak{m}$ is the only prime ideal of C_B containing $C_B c$, so $C_B c$ is primary for $\mathfrak{m}$ and thus $\mathfrak{m}^k \subseteq C_B c$ for some $k \geq 0$.

Let A be the total quotient ring of B; thus A is the ring of all fractions a/c where $a \in B$ and c is a cancellable element of B belonging to C_B (or equivalently, by 3° of Definition 2, a cancellable element of C_B). Then $(\mathfrak{m}^n B)_{n\geq 1}$ is a fundamental system of neighborhoods of zero for a topology $\mathcal{T}$ on A making A into an additive topological group. Since B is open in A for $\mathcal{T}$, the multiplicative composition of A is continuous at $(0, 0)$, and for each $b \in B$ the functions $x \mapsto bx$ and $x \mapsto xb$ from A into A are continuous at zero. To show that A, equipped with topology $\mathcal{T}$, is a topological ring, therefore, it suffices to show that if c is any cancellable element of B belonging to $\mathfrak{m}$, then $x \mapsto c^{-1}x$ is continuous at zero, or equivalently, $x \mapsto cx$ is an open mapping. But as noted above, $\mathfrak{m}^k \subseteq C_B c$ for some $k \geq 1$, so $\mathfrak{m}^{n+k}B \subseteq c(\mathfrak{m}^n B)$ for all $n \geq 1$; thus

$x \mapsto cx$ is open. Therefore we have proved the following theorem.

Theorem 2. *Let B be a basic topological ring, and let A be the total quotient ring of B. Equipped with the topology for which the open neighborhoods of zero in B form a fundamental system of neighborhoods of zero, A is a topological ring containing B as an open subring; consequently, A is a locally centrally linearly compact ring.*

The total quotient ring A of a basic ring B, topologized as indicated, is called the *topological quotient ring* of B.

In view of the goal we have set ourselves, we need to define a class of topological rings that contains the class $\mathfrak{A}_0'$ of all topological rings that happen to be Hausdorff finite-dimensional topological algebras with identity over local fields but, in some sense, is not much larger than that class. A moment's reflection suggests, however, that $\mathfrak{A}_0'$ is not the class we should consider, for by classical Wedderburn theory, a finite-dimensional algebra with identity over a field K is isomorphic to the cartesian product of finitely many finite-dimensional algebras over K whose centers are local subalgebras (a *local algebra* is an algebra that, regarded as a ring, is a local ring, i.e., is commutative, has an identity, and has only one maximal ideal). Indeed, if C is the center of a finite-dimensional algebra A with identity 1, then $C/\text{Rad}(C)$ is the direct sum of finitely many fields by Wedderburn's theorem; raising idempotents, we obtain orthogonal idempotents $e_1, \cdots, e_n$ in C whose sum is 1 such that each Ce_i is a local algebra [8, Proposition 4, p. 54]; then A is the direct sum of the ideals $Ae_1, \cdots, Ae_n$, and the center of Ae_i is Ce_i. Therefore, instead of $\mathfrak{A}_0'$, we shall seek an appropriate extension of the class $\mathfrak{A}_0$ of all cartesian products of finitely many topological rings that happen to be finite-dimensional Hausdorff topological algebras with identity over local fields; each member of $\mathfrak{A}_0$ is thus the cartesian product of finitely many members of the class $\mathfrak{A}_{0s}$ of all topological rings that happen to be finite-dimensional Hausdorff topological algebras with identity over local fields, the centers of which are local subalgebras.

We contend that the appropriate extension of $\mathfrak{A}_0$ is the class $\mathfrak{A}$ of all cartesian products of finitely many topological quotient rings of basic rings. The analogue of $\mathfrak{A}_{0s}$ in $\mathfrak{A}$ is the class $\mathfrak{A}_s$ of all topological quotient rings of basic rings that are special in the following sense:

Definition 3. A commutative ring with identity is *special* if every zero-divisor is nilpotent, that is, if (0) is a primary ideal. A basic topological ring is *special* if its center is a special ring.

Our contention is based on the following facts: (1) A topological field is a local field if and only if it is locally strictly linearly compact (i.e., has an open, strictly linearly compact subring), or equivalently, if and only if it is the topological quotient field of a basic topological integral domain. (2) More generally, a topological division ring is finite-dimensional over its center and its topology is given by a complete, discrete valuation if and only if it is locally centrally linearly compact, or equivalently, if and only if it is the topological quotient ring of a basic topological ring having no proper zero-divisors. (3) If $A \in \mathfrak{A}$, then A has many of those properties possessed by all members of $\mathfrak{A}_0$, e.g., A is complete, A is artinian and noetherian over its center C, every proper C-submodule (in particular, every proper left or right ideal) of A is closed but not open, and every cancellable element of A is invertible. (4) $\mathfrak{A}_{0s}$ consists precisely of those members of $\mathfrak{A}_s$ that have either zero or prime characteristic. (5) Each member of $\mathfrak{A}$ is the cartesian product of finitely many members of $\mathfrak{A}_s$. (6) $\mathfrak{A}_0$ consists precisely of those members of $\mathfrak{A}$ the additive order of each element of which is either infinite or a squarefree integer.

In §2 we shall prove (3)–(6). (Statements (1) and (2) have essentially been proved in [16, Theorems 1 and 2 and Lemma 2].) The crucial step in establishing (4) is to show that the center C of the topological quotient ring A of a special basic ring of zero or prime characteristic contains a local "coefficient" subfield, i.e., a subfield that is local for its induced topology and is mapped homeomorphically by the canonical epimorphism onto the residue field of C. Essential tools for this step are the theorems of I. S. Cohen on complete local noetherian rings.

The structure of topological rings satisfying the above-listed conditions (a)–(c) is given in §3. The need for taking members of $\mathfrak{A}$ rather than $\mathfrak{A}_s$ as our basic units, despite (5), arises from the fact that our structure theorems involve not only the topological quotient rings of basic rings, but also the basic rings themselves. We obtain several generalizations of the structure theorem of Goldman and Sah [7, Theorem 4.1] for locally compact, commutative, semisimple rings with identity that have no proper open ideals. We also characterize those (not necessarily locally compact) topological rings with identity that are cartesian products of finitely many finite-dimensional Hausdorff topological algebras over **R**, **C**, or a local field; this theorem generalizes a structure theorem [13, Theorem 8] for locally compact rings whose center contains an invertible, topologically nilpotent element.

2. Topological quotient rings of basic rings.

We shall first establish assertion (3) of §1.

Theorem 3. *Let A be the topological quotient ring of a basic ring B, let C_B be the center of B, let $\mathfrak{m}$ be the maximal ideal of C_B, let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be* *the remaining prime ideals (namely, the prime ideals of the zero ideal) of* C_B, *and let* C *be the center of* A.

1°. $C_B = C \cap B$, *and* C *(with its induced topology) is the topological quotient ring of* C_B.

2°. *The only prime ideals of* C *are* $C\mathfrak{p}_1, \cdots, C\mathfrak{p}_r$ *and* $C\mathfrak{p}_i \cap C_B = \mathfrak{p}_i$ *for all* $i \in [1, r]$; *thus* C *is a zero-dimensional noetherian ring and hence is an artinian ring.*

3°. A *is an artinian (noetherian) C-module; in particular,* A *is a (left or right) artinian (noetherian) ring.*

4°. A *is complete.*

5°. *Every C-submodule of* A *is closed (in particular, every left or right ideal of* A *is closed, and every ideal of the topological ring* C *is closed).*

6°. *No proper C-submodule of* A *is open (in particular, no proper left or right ideal of* A *is open), and no proper ideal of the topological ring* C *is open.*

7°. *Every cancellable element of* A *is invertible.*

Proof. Since A is the total quotient ring of B, clearly C is the total quotient ring of C_B and $C \cap B = C_B$. Therefore C_B is open in C, and hence the topology induced on C by that of A makes C the topological quotient ring of C_B, since as we observed after Definition 2, the topology induced on C_B by that of B is its natural topology. Now $C = S^{-1}C_B$ where $S = C_B - \bigcup_{i=1}^{r} \mathfrak{p}_i$; hence C is a noetherian ring whose only prime ideals are $C\mathfrak{p}_1, \cdots, C\mathfrak{p}_r$, and $C\mathfrak{p}_i \cap C_B = \mathfrak{p}_i$ for all $i \in [1, r]$ [17, p. 224]; therefore each $C\mathfrak{p}_i$ is a maximal ideal. Thus C is an artinian ring [17, Theorem 2, p. 203].

To prove 3°, it suffices by 2° to show that A is a finitely generated C-module. But by definition, B is finitely generated over C_B, and it is immediate that if $B = C_B b_1 + \cdots + C_B b_n$, then $A = Cb_1 + \cdots + Cb_n$. Since B is a strictly linearly compact C_B-module by Theorem 1, B is complete, and hence as B is open in A, A is also complete [1, Corollary 2, p. 46]. By [18, Theorem 9, p. 262], every C_B-submodule of B is closed. If M is a C-submodule of A, therefore, $M \cap B$ is closed in B, so M is locally closed at zero, and consequently M is closed [1, Proposition 4, p. 18].

Let c be a cancellable element of C_B contained in $\mathfrak{m}$ (i.e., let $c \in \mathfrak{m} - \bigcup_{i=1}^{r} \mathfrak{p}_i$). Then c is invertible in C, and $\lim c^n = 0$. Thus every open neighborhood of zero in the topological ring C contains the invertible element c^n for some n; hence C is the only open ideal of the topological ring C. Let M be an open C-submodule of A. Given $z \in A$, there exists $n \geq 1$ such that $c^n z \in M$ since $\lim c^n z = 0$, so $z = c^{-n}(c^n z) \in CM \subseteq M$; hence $M = A$.

Finally, let a be a cancellable element of A. As A is a left artinian ring, there exists $n \geq 0$ such that $Aa^n = Aa^{n+1}$, whence $a^n = ba^{n+1}$ for some $b \in A$ and thus $1 = ba$; similarly, as A is a right artinian ring, a is right invertible.

Theorem 4. *Let A be the topological quotient ring of a commutative, special basic ring B, and let $\mathfrak{p}$ be the radical of (0) in B. Then A is a noetherian primary ring whose unique prime ideal is $A\mathfrak{p}$ and the field $A/A\mathfrak{p}$, equipped with its induced topology, is a local field.*

Proof. The first assertion follows from 2° and 3° of Theorem 3. Since $A\mathfrak{p}$ is closed but not open by 5° and 6° of Theorem 3, $A/A\mathfrak{p}$ is a Hausdorff, indiscrete topological ring that algebraically is a field. Since B is strictly linearly compact by Theorem 1, its image in $A/A\mathfrak{p}$ under the canonical epimorphism is an open, strictly linearly compact subring of $A/A\mathfrak{p}$. Therefore $A/A\mathfrak{p}$ is a local field [16, Theorem 1].

Lemma 1. *Let A be the topological quotient ring of a commutative, special basic ring B. If B' is a subring of A that contains B and if B' is a finitely generated B-module, then B' is a special basic ring and A is the topological quotient ring of B'.*

Proof. Since B is open in B', B' is a semilocal noetherian ring, and its induced topology is its natural topology [14, Theorem 5]. As A is complete (4°, Theorem 3) and as B' is open and hence closed in A, B' is complete. Therefore by a theorem of Chevalley [18, Corollary 2, p. 283], B' is the topological direct sum of complete local rings; but as A is local, A has no idempotents other than 0 and 1; hence B' is a local ring, and $\dim B' = 1$ [18, Corollary 3, p. 291]. Since $\mathfrak{m}' \cap B = \mathfrak{m}$, where $\mathfrak{m}$ and $\mathfrak{m}'$ are the maximal ideals of B and B' respectively [17, Complement (2), p. 259], $\mathfrak{m}'$ contains an invertible element of A, so $\mathfrak{m}'$ is not a prime ideal of the zero ideal of B'. Thus B' is a basic ring. Clearly A is the topological quotient ring of B'. Therefore as A is local, B' is special by 2° of Theorem 3.

Lemma 2. *If A is the topological quotient ring of a commutative, special basic ring B, then A is the topological quotient ring of a commutative, special basic ring B' that contains B and is mapped by the canonical epimorphism onto the valuation ring of the residue field of A.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of B, $\mathfrak{p}$ the radical of (0) in B, and ϕ the canonical epimorphism from A onto its residue field $A/A\mathfrak{p}$. Then $\phi(\mathfrak{m})$ is not the zero ideal of $\phi(B)$, for otherwise $\mathfrak{m} \subseteq A\mathfrak{p} \cap B = \mathfrak{p}$, a contradiction. Thus $\dim \phi(B) = 1$, and in particular, $\phi(B)$ is not a field. As ϕ is an open mapping, the induced topology of $\phi(B)$ is its natural topology, and $\phi(B)$ is complete since it is open and thus closed in $A/A\mathfrak{p}$. By [14, 1° of Theorem 6], the integral closure V of $\phi(B)$ in $A/A\mathfrak{p}$ is a finitely generated $\phi(B)$-module, and V is a local noetherian domain whose induced topology is its natural topology; since $\dim V = \dim \phi(B) = 1$ [18, Corollary 3, p. 291], V is a discrete valuation ring [18, Corollary 1, p. 42]. Since V is open in $A/A\mathfrak{p}$, the topology of $A/A\mathfrak{p}$ is given by the discrete valuation associated to V, so V is the valuation ring of $A/A\mathfrak{p}$.

Let $c_1, \cdots, c_n \in A$ be such that $V = \phi(B)[\phi(c_1), \cdots, \phi(c_n)]$, and let $B' = B[c_1, \cdots, c_n]$. Clearly $\phi(B') = V$. As V is a finitely generated $\phi(B)$-module, each $\phi(c_k)$ is integral over $\phi(B)$, whence as the kernel $A\mathfrak{p}$ of ϕ is nilpotent, each c_k is integral over B. Therefore B' is a finitely generated B-module. The assertion consequently follows from Lemma 1.

If V is a discrete valuation ring, then the discrete topology and the natural topology of V are the only Hausdorff linear topologies on V (i.e., Hausdorff linear topologies on the V-module V), for if c generates the maximal ideal of V, $(Vc^n)_{n\geq 0}$ is a sequence consisting of all nonzero ideals of V. Therefore any algebraic isomorphism from one Hausdorff linearly topologized discrete valuation ring onto another is a topological isomorphism if neither topology is discrete.

We recall that a subfield K of a local noetherian equicharacteristic ring A is a *coefficient field* if K is mapped onto the residue field of A by the canonical epimorphism. Moreover, a subring C of a complete local noetherian ring A of characteristic zero whose residue field has prime characteristic p is a *coefficient ring* if C is a complete, discrete valuation ring whose maximal ideal is generated by $p.1$, if the induced topology of C is its natural topology, and if C is mapped onto the residue field of A by the canonical epimorphism.

Theorem 5. *If A is the topological quotient ring of a commutative, special basic ring and if the characteristic of A is either zero or a prime, then there is a subfield K of A that is a local field for its induced topology and is mapped homeomorphically onto the residue field of A by the canonical epimorphism.*

Proof. By Lemma 2, A is the topological quotient ring of a special basic ring B that is mapped onto the valuation ring V of the residue field of A by the canonical epimorphism ϕ. Let $\mathfrak{m}$ be the maximal ideal of B, $\mathfrak{p}$ the other prime ideal of B; then $A/A\mathfrak{p}$ is the residue field of A. The restriction ϕ_B of ϕ to B is thus an epimorphism from B onto V, and it induces an isomorphism $\overline{\phi}_B$ from the residue field $B/\mathfrak{m}$ of B onto the residue field $V/\phi(\mathfrak{m})$ of V such that $\overline{\phi}_B \circ \sigma = \rho \circ \phi_B$, where σ and ρ are respectively the canonical epimorphisms from B onto $B/\mathfrak{m}$ and from V onto $V/\phi(\mathfrak{m})$. Since the maximal ideal $A\mathfrak{p}$ of A is nilpotent, the characteristic of $A/A\mathfrak{p}$ is zero if the characteristic of A is. Therefore B and V have the same characteristic.

Case 1. V is equicharacteristic (and hence also B is equicharacteristic). By Cohen's theorem [6, Theorem 9], [12, (31.1)], B contains a subfield k mapped onto $B/\mathfrak{m}$ by σ. Thus $\overline{\phi}_B \circ \sigma = \rho \circ \phi_B$ maps k isomorphically onto the residue field $V/\phi(\mathfrak{m})$ of V, so the restriction ϕ_k of ϕ to k is an isomorphism from k onto a coefficient field of V. Since V is a complete, regular, local, one-dimensional ring, there is a topological isomorphism F from the ring $k[[X]]$ of formal power series over k, equipped with its natural (i.e., X-adic) topology, onto V that extends ϕ_k [6, Theorem 15], [12, (31.12)]. Let $c \in B$ be such that $F(X) = \phi(c)$. Then $c \in \mathfrak{m}$, so as B is complete for the $\mathfrak{m}$-adic topology, $(\alpha_n c^n)_{n \geq 0}$ is summable for any sequence $(\alpha_n)_{n\geq 0}$ in k; thus S: $\sum_{k=0}^{\infty} \alpha_k X^k \mapsto \sum_{k=0}^{\infty} \alpha_k c^k$ is a continuous epimorphism from $k[[X]]$ onto a k-subalgebra $k[[c]]$ of B. Since F, S, and the restriction $\phi_{k[[c]]}$ of ϕ to $k[[c]]$ are continuous and since $\phi_{k[[c]]} \circ S$ and F agree on k and at X, we conclude that $\phi_{k[[c]]} \circ S = F$. Therefore as F is an isomorphism, S is injective and $\phi_{k[[c]]}$ is surjective, whence $\phi_{k[[c]]} = F \circ S^{-1}$ is an isomorphism from $k[[c]]$ onto V. The induced topologies of both $k[[c]]$ and V are Hausdorff, linear, and not discrete, so $\phi_{k[[c]]}$ is a topological isomorphism from $k[[c]]$ onto V. Moreover, as $\phi_{k[[c]]}$ is injective, $k[[c]] \cap A\mathfrak{p} = (0)$, so as A is local, $k[[c]]$ has a quotient field K in A. Since $\phi(k[[c]]) = V$, clearly $\phi(K) = A/A\mathfrak{p}$. To show that the restriction of ϕ to K is a topological isomorphism from K onto the local field $A/A\mathfrak{p}$, therefore, it suffices to show that $k[[c]]$ is open in K. Since $k[[c]]$ is a discrete valuation ring, $k[[c]]$ is maximal in the set of proper subrings of K [5, Proposition 6, p. 115]; but $B \cap K$ is a proper subring of K containing $k[[c]]$ since c is not invertible in B; therefore $k[[c]] = B \cap K$, so $k[[c]]$ is open in K.

Case 2. The characteristic of V is zero and the characteristic of $V/\phi(\mathfrak{m})$ is a prime p. By Cohen's theorem, B contains a coefficient ring C [6, Theorem 11], [12, (31.1)]. Let $C_0 = \phi(C)$. Since $A\mathfrak{p}$ is nilpotent, $C \cap A\mathfrak{p} = (0)$, so the restriction ϕ_c of ϕ to C is an isomorphism from C onto C_0. Moreover, C_0 is not discrete, since the p-fold of the identity belongs to the maximal ideal of V (and hence is a topological nilpotent) and to C_0. Therefore ϕ_C is a topological isomorphism from C onto C_0; in particular, C_0 is a complete, discrete valuation ring and its induced topology is its natural topology. Consequently, C_0 is a coefficient ring of V, for $V/\phi(\mathfrak{m}) = \rho(V) = \rho(\phi_B(B)) = \overline{\phi}_B(\sigma(B)) = \overline{\phi}_B(\sigma(C)) = \rho(\phi_B(C)) = \rho(C_0)$.

Since $C \cap A\mathfrak{p} = (0)$, C has a quotient field L in A; let L_0 be the quotient field of C_0 in $A/A\mathfrak{p}$. As in Case 1, since C and C_0 are maximal in the set of proper subrings of L and L_0 respectively, $B \cap L = C$ and $V \cap L_0 = C_0$, so C and C_0 are open in L and L_0 respectively; therefore L and L_0 are local fields and the restriction ϕ_L of ϕ to L is a topological isomorphism from L onto L_0.

Since V is a complete, local, one-dimensional noetherian domain, V is a finitely generated C_0-module by a theorem of Cohen [6, Theorem 16], [12, (31.6)]. Let $\beta_1, \cdots, \beta_n \in V$ be such that $V = C_0\beta_1 + \cdots + C_0\beta_n$. Then $A/A\mathfrak{p} = L_0\beta_1 + \cdots + L_0\beta_n$; indeed, if $a \in A$, then $p^m a \in B$ for some $m \geq 0$ since $\lim p^n a = 0$, so $\phi(p^m a) = \gamma_1\beta_1 + \cdots + \gamma_n\beta_n$ for suitable $\gamma_i \in C_0$, whence $\phi(a) = (p^{-m}\gamma_1)\beta_1 + \cdots + (p^{-m}\gamma_n)\beta_n \in L_0\beta_1 + \cdots + L_0\beta_n$. Let K be maximal in the set of all subfields of A containing L. Since the characteristic of A is zero, the coefficient fields of A are precisely its maximal subfields [6, proof of Theorem 9]; thus $\phi(K) = A/A\mathfrak{p}$. Let $b_i \in K$ be such that $\phi(b_i) = \beta_i$. Then $K = Lb_1 + \cdots + Lb_n$, for if $z \in K$, then $\phi(z) = \phi(c_1)\beta_1 + \cdots + \phi(c_n)\beta_n$ for suitable $c_i \in L$, and therefore $z - \sum_{i=1}^n c_i b_i \in K \cap A\mathfrak{p} = (0)$. The induced topology on K thus makes K a finite-dimensional Hausdorff vector space over the local field L; as ϕ_L is a topological isomorphism from L onto L_0, therefore, it follows from [3, Corollary 2, p. 19] that the restriction of ϕ to K is a topological isomorphism from K onto $A/A\mathfrak{p}$.

A *Cohen algebra* is a local algebra whose maximal ideal has codimension one.

Theorem 6. *Let A be a Hausdorff topological ring with identity. The following statements are equivalent*:

1°. *A is the topological quotient ring of a special basic ring, and the characteristic of A is either zero or a prime.*

2°. *A is a finite-dimensional topological algebra over a local field K, and the center of A is a Cohen subalgebra.*

3°. *A is a finite-dimensional topological algebra over a local field K, and the center of A is a local subalgebra.*

Proof. Let C be the center of A. Assume 1°. By 1° of Theorem 3 and by Theorem 5, C contains a subfield K that is local for its induced topology and is mapped homeomorphically onto the residue field of C by the canonical epimorphism. Thus C is a Cohen algebra over K. Since the maximal ideal of C is finitely generated and nilpotent (Theorem 4), C is finitely generated over K. Since A is a finitely generated C-module (3°, Theorem 3), therefore, A is a finite-dimensional K-algebra. Thus 2° holds.

Assume 3°, and let V be the valuation ring of K. As in the proof of [16, Theorem 2], there is a basis $f_1 = 1, f_2, \cdots, f_n$ of the K-vector space A such that $B = Vf_1 + \cdots + Vf_n$ is an open subring, the open ideals of B form a fundamental system of neighborhoods of zero, and B is a strictly linearly compact V-module. The center C_B of B is contained in C; indeed, if $x \in C_B$ and if $y \in A$, there exists a nonzero scalar $\lambda \in V$ such that $\lambda y \in B$ since B is open, so $\lambda xy = x(\lambda y) = (\lambda y)x = \lambda yx$, and therefore $xy = yx$. Consequently, $C_B = C \cap B$.

As V is noetherian, B is a noetherian V-module, so C_B is also a noetherian V-module and hence a noetherian ring; moreover, as C_B is closed in B, C_B is a strictly linearly compact V-module and hence is a strictly linearly compact ring. Therefore by [4, Exercise 5(a), p. 117], C_B is a complete, semilocal noetherian ring, and its induced topology is its natural topology. But since C is local, C has no idempotents other than 0 and 1, so C_B is a local ring by [18, Corollary 2, p. 283]. As C_B is finitely generated and thus integral over V, $\dim C_B = \dim V = 1$, and the maximal ideal $\mathfrak{m}$ of C_B intersects V in its maximal ideal; as each nonzero element of V is invertible in A, therefore, $\mathfrak{m} \cap V$ contains invertible elements of A and hence cancellable elements of C_B, so $\mathfrak{m}$ is not a prime ideal of (0) in C_B. Thus 1° of Definition 2 holds for B.

Since B is a noetherian V-module, B is also a noetherian C_B-module. By [15, Theorem 13, 2° and 7°], the topology of B is stronger than the $\mathfrak{m}$-adic topology, which is Hausdorff [17, p. 253]; since the topology of B is strictly linearly compact and therefore a minimal Hausdorff linear topology, it consequently is the natural $\mathfrak{m}$-adic topology. Thus 2° of Definition 2 holds for B.

Now A is contained in the total quotient ring of B, for if $z \in A$, there is a nonzero scalar $\lambda \in V$ such that $\lambda z \in B$ as B is open, whence $z = \lambda^{-1}(\lambda z)$ belongs to the total quotient ring of B. The same argument establishes that C is contained in the total quotient ring of $C_B = C \cap B$. Therefore if c is a cancellable element of C_B, then c is cancellable in C, so c is invertible in C (and also in A) as C is a finite-dimensional algebra over a field, and in particular, c is cancellable in B. Thus A is the total quotient ring of B, and 3° of Definition 2 holds for B. Moreover, the local ring C is also the topological quotient ring of C_B, so C_B is special by 2° of Theorem 3.

Theorem 7. *If A is the topological quotient ring of a basic ring B, then A is the topological direct sum of finitely many ideals, each the topological quotient ring of a special basic ring.*

Proof. Let C be the center of A, C_B the center of B. By 3° of Theorem 3 applied to C and by [17, Theorem 3, p. 205], there exist orthogonal idempotents $e_1, \cdots, e_r$ in C such that $\Sigma_{i=1}^r e_i = 1$ and each Ce_k is a noetherian primary ring. Thus A is the topological direct sum of the ideals $Ae_1, \cdots, Ae_r$, since the associated projections $p_k: x \mapsto xe_k$ are continuous (and open). We shall show that Be_k is a special basic ring and that Ae_k is the topological quotient ring of Be_k.

Let $\mathfrak{m}$ be the maximal ideal of C_B. The restriction of p_k to C_B is an epimorphism from C_B onto C_Be_k that clearly takes cancellable elements into cancellable elements; hence C_Be_k is a local noetherian ring, $\dim C_Be_k \leq 1$, and the maximal ideal $\mathfrak{m}e_k$ of C_Be_k contains cancellable elements and hence is not a prime ideal of the zero ideal of C_Be_k; therefore dim $C_Be_k = 1$ and the prime ideals of the zero ideal of C_Be_k are all isolated, since $\mathfrak{m}e_k$ is not one of them. Since p_k is continuous and open, Be_k is open in Ae_k and $(\mathfrak{m}^nBe_k)_{n\geq 1}$ is a fundamental system of neighborhoods of zero in Ae_k. But clearly $\mathfrak{m}^nBe_k = \mathfrak{m}^ne_kBe_k = (\mathfrak{m}e_k)^nBe_k$, so the topology of the (C_Be_k)-module Be_k is its $(\mathfrak{m}e_k)$-adic topology. As A is complete, so is Ae_k, and therefore also the open subring Be_k of Ae_k is complete. Moreover, Be_k is a finitely generated (C_Be_k)-module since B is a finitely generated C_B-module. Therefore the topology induced on any (C_Be_k)-submodule of Be_k is its $(\mathfrak{m}e_k)$-adic topology [18, Theorem 4, p. 254], and the submodule is closed and hence complete for that topology [18, Theorem 9, p. 262]. In particular, the topology induced on C_Be_k is its natural $(\mathfrak{m}e_k)$-adic topology, and C_Be_k is complete for that topology. Therefore C_Be_k is a commutative basic ring.

We observed above that if y is a cancellable element of C_B, then ye_k is cancellable in C_Be_k; therefore as C is the total quotient ring of $C_B = C \cap B$, Ce_k is contained in the total quotient ring of C_Be_k. To show that Ce_k is the total quotient ring of C_Be_k, it consequently suffices to show that a cancellable element d_k of C_Be_k is invertible in Ce_k. Let

$$d = d_k + \sum_{j\neq k} e_j \in C.$$

Suppose that $xd = 0$, where $x \in C_B$. Then as $xd = xd_k + \Sigma_{j\neq k} xe_j$, we conclude that $xe_j = 0$ for all $j \neq k$ and $xd_k = xd_ke_k = 0$ since C is the direct sum of $Ce_1, \cdots, Ce_r$. But then $xe_k = 0$ since $xe_kd_k = xd_k = 0$ and since d_k is cancellable in C_Be_k, so consequently $x = 0$. Thus $xd \neq 0$ for all nonzero elements x of C_B; hence $yd \neq 0$ for all nonzero elements y of the total quotient ring C of C_B; therefore d is a cancellable and hence invertible element of C, so $d_k = de_k$ is an invertible element of Ce_k. Consequently, Ce_k is the total quotient ring of C_Be_k. Since the restriction of p_k to C is the projection onto Ce_k, an open continuous epimorphism, C_Be_k is open in Ce_k. Therefore Ce_k is the topological quotient ring of the basic ring C_Be_k. But as Ce_k has only one prime ideal, C_Be_k is special by 2° of Theorem 3, applied to Ce_k and C_Be_k.

The center C_k of Be_k clearly contains C_Be_k and hence is a (C_Be_k)-submodule of Be_k. As Be_k is a noetherian (C_Be_k)-module, therefore, so is C_k. Consequently by Lemma 1, C_k is a special basic ring for its induced topology. But also, as we saw earlier, the induced topology of the (C_Be_k)-module C_k is its $(\mathfrak{m}e_k)$-adic topology. Hence if $\mathfrak{n}_k$ is the maximal ideal of C_k, there exists $s \geq 1$ such that $\mathfrak{n}_k^s \subseteq \mathfrak{m}e_kC_k \subseteq \mathfrak{n}_k$, whence $\mathfrak{n}_k^sBe_k \subseteq \mathfrak{m}Be_k \subseteq \mathfrak{n}_kBe_k$; consequently

$\mathfrak{n}_k^{st}B_k \subseteq \mathfrak{m}^t Be_k \subseteq \mathfrak{n}_k^t Be_k$ for all $t \geq 1$, so the topology of the C_k-module Be_k is its $\mathfrak{n}_k$-adic topology. Thus 1° and 2° of Definition 2 hold for Be_k.

To show that $C_k \subseteq C$, it suffices to show that if $x \in C_k$ and if $y \in B$, then $xy = yx$; but $xy = (xe_k)y = x(ye_k) = (ye_k)x = yx$. Hence $C_k = C_k e_k \subseteq Ce_k$. If y is a cancellable element of C_B, then as we have seen, ye_k is cancellable in $C_B e_k$, thus invertible in the total quotient ring Ce_k of $C_B e_k$ and hence in Ae_k, and in particular cancellable in C_k. Therefore Ae_k is contained in the total quotient ring of Be_k. To complete the proof that 3° of Definition 2 holds for Be_k and that Ae_k is the topological quotient ring of Be_k, it therefore suffices to prove that every cancellable element g of C_k is invertible in Ae_k. Let c_k be a cancellable element of $C_B e_k$ belonging to $\mathfrak{m}e_k$; then as $g \in Ce_k$, as $\lim c_k^m g = 0$, and as $C_B e_k$ is open in Ce_k, there exists $m \geq 1$ such that $c_k^m g \in C_B e_k$. Thus $c_k^m g$ is a cancellable element of $C_B e_k$, so as we proved earlier, $c_k^m g$ is invertible in Ce_k, whence g is invertible in Ce_k and hence also in Ae_k.

The characteristic of a local ring C is either zero or a power of a prime. Indeed, if the characteristic of C were rs where r and s are relatively prime, then $r.1$ and $s.1$ would be zero-divisors and hence would belong to the maximal ideal of C, so the identity element of C would also belong to its maximal ideal since there exist integers m and n such that $1 = mr + ns$. Consequently, from Theorems 6 and 7 we obtain the following theorem.

Theorem 8. *Let A be a topological ring. The following statements are equivalent*:

1°. *A is the topological direct sum of finitely many ideals, each a finite-dimensional Hausdorff topological algebra with identity over a local field.*

2°. *A is the topological direct sum of finitely many ideals, each the topological quotient ring of a basic ring, and the additive order of each element of A is either infinite or a squarefree integer.*

3°. *A is the topological direct sum of finitely many ideals, each a finite-dimensional Hausdorff topological algebra with identity over a local field whose center is a Cohen subalgebra.*

3. **The structure theorems.** To identify those locally centrally linearly compact rings that are topological quotient rings of basic rings, we shall need the following lemma:

Lemma 3. *If B is an open subring of a topological ring A with identity, and if the center C of A is a topological ring having no proper open ideals, then the center C_B of B is $B \cap C$.*

Proof. Clearly $B \cap C \subseteq C_B$. To show that $C_B \subseteq B \cap C$, it suffices to prove that if $c \in C_B$ and if $a \in A$, then $ac = ca$. Since B is open, $V = \{x \in A: xa \in B\}$ is a neighborhood of zero in A. Let W be the annihilator in C of $ac - ca$. Then W contains $V \cap C$ and hence is an open ideal of C, for if $x \in V \cap C$, then $(xa)c = c(xa) = (cx)a = (xc)a$ since $xa \in B$, $c \in C_B$, and $x \in C$. By hypothesis, therefore, $W = C$, so $1 \in W$, and consequently $ca = ac$.

Theorem 9. *Let A be a Hausdorff topological ring with identity, let C be the center of A, and let B be an open subring of A that contains 1. Then B is a basic topological ring and A is the topological quotient ring of B if and only if C is a topological ring that has no proper open ideals and B is a centrally linearly compact subring whose center C_B is a local ring.*

Proof. The condition is necessary by Theorem 1, 6° of Theorem 3, and Definition 2.

Sufficiency. By Lemma 3, $C_B = B \cap C$; in particular, C_B is an open subring of C. Let $\mathfrak{m}$ be the maximal ideal of C_B. (a) C_B is a strictly linearly compact ring, and $\mathfrak{m}$ is open in C_B. Indeed, C_B is a closed submodule of the C_B-module B and hence is a strictly linearly compact ring. As the open ideals of C_B form a fundamental system of neighborhoods of zero for its induced Hausdorff topology, $\mathfrak{m}$ is open in C_B.

(b) $\mathfrak{m}$ is a finitely generated ideal of C_B. As $\mathfrak{m}$ is open in C_B and hence in C, $C\mathfrak{m}$ is an open ideal of C, so $C\mathfrak{m} = C$, and therefore there exist $c_1, \cdots, c_n \in C$ and $b_1, \cdots, b_n \in \mathfrak{m}$ such that $c_1b_1 + \cdots + c_nb_n = 1$. Let $\mathfrak{b}$ be the ideal $C_Bb_1 + \cdots + C_Bb_n$ of C_B; then $\mathfrak{b}$ is open, for if $\mathfrak{b} = \{x \in B: xc_i \in B, 1 \leq i \leq n\}$, then $\mathfrak{b}$ is an open left ideal of B, so $\mathfrak{b} \cap C_B$ is an open ideal of C_B contained in $\mathfrak{b}$ (for if $x \in \mathfrak{b} \cap C_B$, then $xc_i \in C_B$ for each $i \in [1, n]$, whence $x = (xc_1)b_1 + \cdots + (xc_n)b_n \in \mathfrak{b}$). As C_B is strictly linearly compact, $C_B/\mathfrak{b}$ is an artinian C_B-module and hence a commutative artinian ring. Consequently, $C_B/\mathfrak{b}$ is noetherian, so $\mathfrak{m}/\mathfrak{b}$ is finitely generated, i.e., there exist $m_1, \cdots, m_r \in \mathfrak{m}$ such that $\mathfrak{m} = C_Bm_1 + \cdots + C_Bm_r + \mathfrak{b}$. As $\mathfrak{b}$ is finitely generated, therefore, so is $\mathfrak{m}$.

(c) C_B is a complete local noetherian ring whose topology is its natural topology. If $\mathfrak{o}$ is an open ideal of C_B, then $\mathfrak{m}^n \subseteq \mathfrak{o}$ for some $n \geq 1$; indeed, as C_B is strictly linearly compact, $C_B/\mathfrak{o}$ is an artinian C_B-module and hence is a local artinian ring; the maximal ideal $\mathfrak{m}/\mathfrak{o}$ of $C_B/\mathfrak{o}$ is therefore nilpotent, so $\mathfrak{m}^n \subseteq \mathfrak{o}$ for some n, whence $\mathfrak{m}^n \subseteq \mathfrak{o}$. Consequently, the induced topology of C_B is weaker than its natural topology, and $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. The assertion therefore follows from (a), (b), and [15, Theorem 12].

(d) C is a noetherian ring all of whose ideals are closed. Let $\mathfrak{a}$ be an ideal of C. By (c) and [18, Theorem 9, p. 262], $\mathfrak{a} \cap C_B$ is closed in C_B. As C_B is open in C, therefore, $\mathfrak{a}$ is locally closed at zero and hence is a closed ideal

[1, Proposition 4, p. 18]. Moreover, as C has no proper open ideals, $C(\mathfrak{a} \cap C_B) = \mathfrak{a}$ [7, Proposition 1.3]. By (c), there exist $x_1, \cdots, x_n \in \mathfrak{a} \cap C_B$ such that $\mathfrak{a} \cap C_B = C_B x_1 + \cdots + C_B x_n$, whence $\mathfrak{a} = C(\mathfrak{a} \cap C_B) = Cx_1 + \cdots + Cx_n$.

(e) C is an artinian ring. By (d) and [17, Theorem 2, p. 203], it suffices to show that a prime ideal $\mathfrak{p}$ of C is maximal. Let ϕ be the canonical epimorphism from C onto $C/\mathfrak{p}$, a Hausdorff integral domain by (d). Then $\phi(C_B)$ is a strictly linearly compact C_B-module and hence is clearly a strictly linearly compact ring; also $\phi(C_B)$ is local and noetherian by (c); therefore the topology of $\phi(C_B)$ is its natural topology [4, Exercise 5(a), p. 117], and $\phi(C_B)$ is consequently complete for that topology. In particular, $\phi(C_B)$ is not a field, for in the contrary case its topology would be discrete, so $\mathfrak{p} \cap C_B$ would be open in C_B and hence in C, since the canonical isomorphism from $C_B/(\mathfrak{p} \cap C_B)$ onto $\phi(C_B)$ is continuous, and therefore $\mathfrak{p}$ would be open in C, in contradiction to our hypothesis. Moreover, $C/\mathfrak{p}$ has no proper open ideals, for if $\mathfrak{o}$ were a proper open ideal of $C/\mathfrak{p}$, $\phi^{-1}(\mathfrak{o})$ would be a proper open ideal of C. Thus by 1° of [14, Theorem 6], $C/\mathfrak{p}$ is not integral over $\phi(C_B)$, so by 2° of [14, Theorem 6], $C/\mathfrak{p}$ is a field, i.e., $\mathfrak{p}$ is maximal.

(f) There is an invertible element c of C belonging to $\mathfrak{m}$. By (e) and [17, Theorem 3, p. 205], C has only finitely many maximal ideals. Since each maximal ideal of C is closed but not open by (d), the union of the maximal ideals of C is a closed set having no interior point [2, Proposition 1, p. 107], so the set G of invertible elements of C is open and dense. Thus as $\mathfrak{m}$ is open, $G \cap \mathfrak{m} \neq \emptyset$.

(g) B satisfies 1° of Definition 2, i.e., C_B is a basic ring. Clearly $C_B c \subseteq \mathfrak{m}$. Since c is an invertible element of C belonging to $\mathfrak{m}$ and since C_B is open in C, $C_B c$ is an open ideal of C_B, hence $C_B c$ contains $\mathfrak{m}^n$ for some $n \geq 1$ by (c), so $C_B c$ is primary for $\mathfrak{m}$ and thus $\dim C_B \leq 1$. Moreover, c is cancellable in C_B and hence does not belong to a prime ideal of the zero ideal of C_B. Therefore $\dim C_B = 1$ and the prime ideals of the zero ideal of C_B are all isolated since $\mathfrak{m}$ is not among them. Thus by (c), C_B is a basic ring.

(h) A is the total quotient ring of B, and 3° of Definition 2 holds for B. First, A is contained in the total quotient ring of B, for if $z \in A$, then $c^n z \in B$ for some $n \geq 1$ since $\lim c^n z = 0$, so $z = c^{-n}(c^n z)$ is the quotient of an element of B and a cancellable element of C_B. We need only prove, therefore, that a cancellable element d of C_B is invertible in C (and hence invertible in A and cancellable in B). We may assume that d is not invertible in C_B, i.e., that $d \in \mathfrak{m}$. By (g), $\mathfrak{m}$ is the only prime ideal of C_B containing d, so dC_B is primary for $\mathfrak{m}$ and hence is open in C_B and therefore also in C. The ideal dC of C generated by dC_B is consequently open in C, so $dC = C$ by hypothesis, whence d is invertible in C. To complete the proof of the theorem, therefore, we need only prove that 2° of Definition 2 holds for B.

(i) The topology of B is its natural topology as a C_B-module. First we shall show that $\mathfrak{m}^n B$ is open. By (c) and Definition 1, there is an open left ideal $\mathfrak{a}$ of B such that $\mathfrak{a} \cap C_B \subseteq \mathfrak{m}^n$. Then as $\mathfrak{a} \cap C_B$ is open in C, $(\mathfrak{a} \cap C_B)C$ is an open ideal of C, so $(\mathfrak{a} \cap C_B)C = C$, and consequently there exist $a_1, \cdots, a_s \in \mathfrak{a} \cap C_B$, $c_1, \cdots, c_s \in C$ such that $a_1c_1 + \cdots + a_sc_s = 1$. Now $\mathfrak{b} = \{x \in B: c_ix \in B, 1 \leq i \leq s\}$ is an open right ideal of B contained in $\mathfrak{m}^n B$, for if $x \in \mathfrak{b}$, then $x = a_1(c_1x) + \cdots + a_s(c_sx) \in (\mathfrak{a} \cap C_B)B \subseteq \mathfrak{m}^n B$. Therefore $\mathfrak{m}^n B$ is open. On the other hand, if $\mathfrak{o}$ is an open left ideal of B, then $\mathfrak{o} \cap C_B \supseteq \mathfrak{m}^k$ for some $k \geq 1$ by (c), so $\mathfrak{m}^k B = B\mathfrak{m}^k \subseteq B\mathfrak{o} = \mathfrak{o}$. Therefore $(\mathfrak{m}^n B)_{n \geq 1}$ is a fundamental system of neighborhoods of zero.

(j) B is a noetherian C_B-module. By (i), $\bigcap_{n=1}^{\infty} \overline{\mathfrak{m}^n B} = (0)$. Therefore by (a), (b), (i), and [15, Theorem 11], $\mathfrak{m}B$ is a finitely generated C_B-submodule. Also as B is a strictly linearly compact C_B-module, $B/\mathfrak{m}B$ is an artinian C_B-module by (i); regarded as a vector space over $C_B/\mathfrak{m}$, therefore, $B/\mathfrak{m}B$ is artinian, thus finite-dimensional, and hence noetherian. Consequently, $B/\mathfrak{m}B$ is a noetherian C_B-module. As $\mathfrak{m}B$ is a finitely generated module over C_B, a noetherian ring, we conclude that $\mathfrak{m}B$ is also a noetherian C_B-module. Consequently, B itself is a noetherian C_B-module.

Lemma 4. *If A is a locally centrally linearly compact ring with identity and if the center C of A is a topological ring having no proper open ideals, then there is an open, centrally linearly compact subring B of A that contains* 1.

Proof. By hypothesis, A contains an open, centrally linearly compact subring B'. Let B be the subring of A generated by B' and 1. Every left ideal $\mathfrak{a}$ of B' is also a left ideal of B, for $V_{\mathfrak{a}} = \{b \in A: b\mathfrak{a} \subseteq \mathfrak{a}\}$ is a subring of A that contains B' and 1 and hence contains B. Thus every open left ideal of B' is an open left ideal of B, so the open left ideals of B form a fundamental system of neighborhoods of zero. By hypothesis, the ideal of C generated by $B' \cap C$ is C, i.e., $C = (B' \cap C)C$. In particular, there exist $x_1, \cdots, x_n \in B' \cap C$ and $c_1, \cdots, c_n \in C$ such that $1 = x_1c_1 + \cdots + x_nc_n$. Since $B \subseteq V_{B'}$, for any $b \in B$ we have $b = (bx_1)c_1 + \cdots + (bx_n)c_n \in B'c_1 + \cdots + B'c_n$. Since B' is a strictly linearly compact module over its center $C_{B'}$, $B'c_1 + \cdots + B'c_n$ is also a strictly linearly compact $C_{B'}$-module, for it is continuous homomorphic image of the $C_{B'}$-module B'^n. Consequently, as B is an open and thus closed $C_{B'}$-submodule of $B'c_1 + \cdots + B'c_n$, B is strictly linear compact over $C_{B'}$. As the open left ideals of B form a fundamental system of neighborhoods of zero, therefore, B is also strictly linearly compact over its center. Thus B is an open, centrally linearly compact subring that contains 1.

We recall that if $(A_\gamma)_{\gamma \in \Gamma}$ is a family of topological rings and if, for each

$\gamma \in \Gamma$, B_γ is an open subring of A_γ, the *local direct product* of $(A_\gamma)_{\gamma \in \Gamma}$ relative to $(B_\gamma)_{\gamma \in \Gamma}$ is the subring D of $\Pi_{\gamma \in \Gamma} A_\gamma$ consisting of all (x_γ) such that $x_\gamma \in B_\gamma$ for all but finitely many $\gamma \in \Gamma$, equipped with the topology obtained by declaring the open neighborhoods of zero in $\Pi_{\gamma \in \Gamma} B_\gamma$ (equipped with the cartesian product topology) a fundamental system of neighborhoods of zero in D. It is easy to verify that D is a topological ring, that the canonical injection from A_γ into D is a topological isomorphism from A_γ onto an ideal of D, and that the projection from D onto A_γ is a continuous open epimorphism. It is also apparent that $D = \Pi_{\gamma \in \Gamma} B_\gamma + \bigoplus_{\gamma \in \Gamma} A_\gamma$, where $\bigoplus_{\gamma \in \Gamma} A_\gamma$ is the direct sum of $(A_\gamma)_{\gamma \in \Gamma}$, consisting of all (x_γ) such that $x_\gamma = 0$ for all but finitely many $\gamma \in \Gamma$.

Our structure theorem for a locally centrally linearly compact ring with identity whose center is a topological ring without proper open ideals is the following:

Theorem 10. *Let A be a Hausdorff topological ring with identity. The following statements are equivalent:*

1°. *A is locally centrally linearly compact, and the center C of A is a topological ring having no proper open ideals.*

2°. *A is topologically isomorphic to the local direct product of $(A_\gamma)_{\gamma \in \Gamma}$ relative to $(B_\gamma)_{\gamma \in \Gamma}$, where for each $\gamma \in \Gamma$, B_γ is a basic topological ring and A_γ is the topological quotient ring of B_γ.*

Moreover, if the additive order of each element of A is either infinite or squarefree, then each A_γ is the topological direct sum of finitely many ideals, each a finite-dimensional Hausdorff topological algebra with identity over a local field whose center is a Cohen subalgebra.

Proof. Assume 1°. By Lemma 4, there is an open, centrally linearly compact subring B of A such that $1 \in B$. By Lemma 3, the center of B is $B \cap C$. Thus $B \cap C$ is a strictly linearly compact commutative ring with identity, so $B \cap C$ is topologically isomorphic to the cartesian product of strictly linearly compact local rings [4, Exercise 21(d), p. 112]. Thus $B \cap C$ contains a summable orthogonal family $(e_\gamma)_{\gamma \in \Gamma}$ of idempotents such that $\Sigma_{\gamma \in \Gamma} e_\gamma = 1$ and for each $\gamma \in \Gamma$, $(B \cap C)e_\gamma$ is a local, strictly linearly compact ring. Clearly Ce_γ is the center of the ring Ae_γ, $Be_\gamma = B \cap Ae_\gamma$ and hence Be_γ is open in Ae_γ, and $Be_\gamma \cap Ce_\gamma$ is the local ring $(B \cap C)e_\gamma$. As $x \mapsto xe_\gamma$ is a continuous epimorphism from C onto Ce_γ, the topological ring Ce_γ has no proper open ideals. Therefore by Lemma 3, the center of Be_γ is the local ring $Be_\gamma \cap Ce_\gamma = (B \cap C)e_\gamma$. Since the intersection of an open left ideal of B with Be_γ is an open left ideal of Be_γ, the open left ideals of Be_γ form a fundamental system of neighborhoods of zero for the induced topology of Be_γ. Since the $(B \cap C)$-module Be_γ is a continuous homomorphic image of the $(B \cap C)$-module B, the former module is strictly linearly compact; however, as $e_\gamma \in C$, the $(B \cap C)$-submodules of Be_γ coincide with the $(B \cap C)e_\gamma$-submodules of Be_γ; consequently, the $(B \cap C)e_\gamma$-module Be_γ, with its induced topology, is strictly linearly compact. Therefore Be_γ is an open, centrally linearly compact subring of Ae_γ. By Theorem 9, Be_γ is a basic topological ring, and Ae_γ is the topological quotient ring of Be_γ.

We shall next establish that $F: x \mapsto (xe_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma}(Ae_\gamma)$ is a topological isomorphism from A onto the local direct product D of $(Ae_\gamma)_{\gamma \in \Gamma}$ relative to $(Be_\gamma)_{\gamma \in \Gamma}$. If $\alpha \in \Gamma$ and if $x \in Ae_\alpha$, then $F(x) = (x_\gamma)$, where $x_\alpha = x$ and $x_\gamma = 0$ if $\gamma \neq \alpha$. Thus $F(A) \supseteq \bigoplus_{\gamma \in \Gamma} Ae_\gamma$. For the same reason, $\bigoplus_{\gamma \in \Gamma} Be_\gamma \subseteq F(B) \subseteq \prod_{\gamma \in \Gamma}(Be_\gamma)$. Since F is clearly continuous, $F(B)$ is a linearly compact $(B \cap C)$-module and hence is complete; as $\bigoplus_{\gamma \in \Gamma} Be_\gamma$ is dense in $\prod_{\gamma \in \Gamma}(Be_\gamma)$, therefore, $F(B) = \prod_{\gamma \in \Gamma}(Be_\gamma)$. Consequently, $F(A) \supseteq \prod_{\gamma \in \Gamma}(Be_\gamma) + \bigoplus_{\gamma \in \Gamma} Ae_\gamma = D$. For each $x \in A$, the family $(xe_\gamma)_{\gamma \in \Gamma}$ is summable and $\Sigma xe_\gamma = x\Sigma e_\gamma = x$; consequently, $xe_\gamma \in B \cap Ae_\gamma = Be_\gamma$ for all but finitely many $\gamma \in \Gamma$; therefore $F(x) \in D$. Consequently, $F(A) = D$. Moreover, F is an isomorphism from A onto D, for if $F(x) = 0$, then $xe_\gamma = 0$ for all $\gamma \in \Gamma$, so $x = x\Sigma e_\gamma = \Sigma xe_\gamma = 0$. Since the restriction F_B of F to B is a continuous isomorphism from the $(B \cap C)$-module B onto the $(B \cap C)$-module $\prod_{\gamma \in \Gamma}(Be_\gamma)$, and since the $(B \cap C)$-module B is strictly linearly compact, F_B is a topological isomorphism by the definition of a strictly linearly compact module. Consequently by the definition of the topology of D, F is a topological isomorphism from A to D.

Assume 2°, and let D be the local direct product of $(A_\gamma)_{\gamma \in \Gamma}$ relative to $(B_\gamma)_{\gamma \in \Gamma}$. Let $B = \prod_{\gamma \in \Gamma} B_\gamma$, and let C_γ be the center of A_γ. Clearly the center C of D is $(\prod_{\gamma \in \Gamma} C_\gamma) \cap D$, so $B \cap C = \prod_{\gamma \in \Gamma}(B_\gamma \cap C_\gamma)$. By 6° of Theorem 3, each C_γ is a topological ring having no proper open ideals. Let $\mathfrak{o}$ be an open ideal of C. For each $\gamma \in \Gamma$, the projection $\mathfrak{o}_\gamma$ of $\mathfrak{o}$ on A_γ is an open ideal of C_γ, so $\mathfrak{o}_\gamma = C_\gamma$. Since $C = (\prod_{\gamma \in \Gamma} C_\gamma) \cap D$, therefore, an easy argument establishes that $\mathfrak{o} = C$. Consequently, C is a topological ring having no proper open ideals. By Lemma 3, the center of each B_γ is $B_\gamma \cap C_\gamma$, and the center of B is $B \cap C$. For each $\alpha \in \Gamma$, B_α admits the structure of a $(B \cap C)$-module under the scalar multiplication $(x_\gamma) \cdot y = x_\alpha y$, and the $(B \cap C)$-submodules of B_α are identical with the $(B_\alpha \cap C_\alpha)$-submodules of B_α. Therefore as B_α is a strictly linearly compact $(B_\alpha \cap C_\alpha)$-module, it is also a strictly linearly compact $(B \cap C)$-module. Hence B is a strictly linearly compact $(B \cap C)$-module [11, Theorem 5]. Since the open left ideals of B_γ form a fundamental system of neighborhoods of zero in B_γ for each $\gamma \in \Gamma$, the open left ideals of B clearly form a fundamental system of neighborhoods of zero in B. Thus D is locally centrally linearly compact, and 1° holds.

We wish to extend Theorem 10 to allow for a connected factor. For this, we need the following three lemmas:

Lemma 5. *If E is a locally compact, connected, unitary left [right] topological module over a locally centrally linearly compact ring A with identity whose center is a topological ring having no proper open ideals, then $E = (0)$.*

Proof. By Theorem 10, there are an orthogonal summable family $(e_\gamma)_{\gamma \in \Gamma}$ of central idempotents and an open subring B containing 1 such that $\Sigma e_\gamma = 1$, Be_γ is a basic ring, and Ae_γ is the topological quotient ring of Be_γ. Assume that E is a nonzero left topological A-module. If x is a nonzero element of E, then $x = 1 \cdot x = \Sigma(e_\gamma \cdot x)$, so there exists $\alpha \in \Gamma$ such that $e_\alpha \cdot E \neq (0)$. Decomposing Ae_α in accordance with Theorem 7, we may assume further that Be_α is a special basic ring. Now $e_\alpha \cdot E$ is connected since it is a continuous image of E; $e_\alpha \cdot E$ is also closed in E and hence is locally compact, for if $\mathfrak{F}$ is a filter on $e_\alpha \cdot E$ converging to $x \in E$, then $e_\alpha \cdot \mathfrak{F}$ converges to $e_\alpha \cdot x$, but $e_\alpha \cdot \mathfrak{F} = \mathfrak{F}$, whence $x = e_\alpha \cdot x \in e_\alpha \cdot E$. Consequently, $M = e_\alpha \cdot E$ is a nonzero, connected, locally compact, unitary topological module over the center C_α of Ae_α, and C_α is a local artinian ring (or equivalently, a noetherian primary ring) by 2° of Theorem 3. Let $\mathfrak{m}$ be the maximal ideal of C_α. By Theorem 4, $C_\alpha/\mathfrak{m}$ is a local field. As $\mathfrak{m}$ is nilpotent, there is a smallest natural number r such that $\overline{\mathfrak{m}^r \cdot M}$ is properly contained in M. Thus $M_r = M/\overline{\mathfrak{m}^r \cdot M}$ is a nonzero, locally compact, connected topological vector space over the local field $C/\mathfrak{m}$, for by continuity, $\mathfrak{m} \cdot M = \mathfrak{m} \cdot \overline{\mathfrak{m}^{r-1} \cdot M} \subseteq \overline{\mathfrak{m}^r \cdot M}$. Consequently, M_r is finite dimensional over $C/\mathfrak{m}$ [3, Theorem 3, p. 20], so the connected space M_r is topologically isomorphic to the totally disconnected space $(C/\mathfrak{m})^s$ for some $s > 0$ [3, Theorem 2, p. 18], a contradiction.

We are indebted to John S. Cook for a discussion concerning the proof of the following lemma:

Lemma 6. *Let A be a Hausdorff topological ring, and let $\mathfrak{c}$ be the connected component of zero. If A has a left [right] identity, if $\mathfrak{c}$ is locally compact and not the zero ideal, and if $A/\mathfrak{c}$ is a locally centrally linearly compact ring with identity whose center is a topological ring having no proper open ideals, then $\mathfrak{c}$ is a finite-dimensional topological algebra over* **R**, *the right [left] annihilator of $\mathfrak{c}$ in $\mathfrak{c}$ is (0), and $\mathfrak{c}$ contains a nonzero idempotent.*

Proof. We assume that A has a left identity element e; let ϕ be the canonical epimorphism from A onto $A/\mathfrak{c}$. By the Jacobson-Taussky theorem [13, Theorem 5], $\mathfrak{c}$ contains a connected, compact ideal $\mathfrak{h}$ such that $\mathfrak{c}\mathfrak{h} = (0)$ and $\mathfrak{c}/\mathfrak{h}$ is a finite-dimensional topological algebra over **R**. Since $A/\mathfrak{c}$ is a ring with identity, its identity element is $\phi(e)$. Therefore as $\mathfrak{c}\mathfrak{h} = (0)$, $\mathfrak{h}$ is a topological unitary left module over $A/\mathfrak{c}$. By Lemma 5, $\mathfrak{h} = (0)$; therefore $\mathfrak{c}$ is a finite-dimensional topological algebra over **R**. Let $\mathfrak{a} = \{x \in \mathfrak{c} : \mathfrak{c}x = (0)\}$. Clearly $\mathfrak{a}$

is a closed ideal of A; also $\mathfrak{a}$ is clearly a subspace of the $\mathbf{R}$-algebra $\mathfrak{c}$ and hence is connected. Therefore as $\mathfrak{c}\mathfrak{a} = (0)$, $\mathfrak{a}$ is a topological unitary left module over $A/\mathfrak{c}$, so $\mathfrak{a} = (0)$ by Lemma 5.

Consequently, the right regular representation $R: c \mapsto R_c$ from $\mathfrak{c}$ into $\mathrm{End}_{\mathbf{R}}(\mathfrak{c})$, where R_c is defined by $R_c(x) = xc$ for all $x \in \mathfrak{c}$, is an antimonomorphism. As $\mathfrak{c}$ is a finite-dimensional algebra, its radical is nilpotent, so $\mathfrak{c}$ is not a radical ring as $\mathfrak{a} = (0)$, and consequently $\mathfrak{c}$ contains a nonnilpotent element u. The sequence $(R_u^n(\mathfrak{c}))_{n\geq 1}$ of subspaces of $\mathfrak{c}$ is decreasing, and hence for some m, $R_{u^m}(\mathfrak{c}) = R_u^m(\mathfrak{c}) = R_u^s(\mathfrak{c}) = R_{u^s}(\mathfrak{c})$ for all $s \geq m$. Let $v = u^m$, and let $M = R_v(\mathfrak{c})$. Then M is the range of R_v, $M \neq (0)$ since u is not nilpotent, and $R_v(M) = R_{u^{2m}}(\mathfrak{c}) = M$. Consequently, the restriction v_M of R_v to M is an automorphism of the $\mathbf{R}$-vector space M. Therefore, the characteristic polynomial $X^n + \cdots + \alpha_1 X + \alpha_0$ of v_M has a nonzero constant term α_0. Let $p = -\alpha_0^{-1}(v^n + \cdots + \alpha_1 v)$. Then $p \in \mathfrak{c}$, $R_p(\mathfrak{c}) \subseteq M$, and by the Cayley-Hamilton theorem, the restriction of R_p to M is the identity linear operator. Hence R_p is a projection on M, so as R is an antimonomorphism, p is a nonzero idempotent.

Lemma 7. *Let A and A' be topological rings with identity elements 1 and $1'$, and let ϕ be a continuous homomorphism from A into A' such that $\phi(1) = 1'$. If C is a subring of A containing 1 such that the topological ring C contains no proper open ideals and if C' is a subring of A' containing $\phi(C)$, then the topological ring C' contains no proper open ideals.*

Proof. Let $\mathfrak{d}'$ be an open ideal of C'. There exists an open set U' in A' such that $U' \cap C' = \mathfrak{d}'$. Moreover, $\phi^{-1}(U') \cap C$ is an (open) ideal of C; indeed, if $x \in C$ and $y \in \phi^{-1}(U') \cap C$, then $\phi(x) \in \phi(C) \subseteq C'$ and $\phi(y) \in \phi(\phi^{-1}(U') \cap C) \subseteq U' \cap C' = \mathfrak{d}'$, so $\phi(xy) = \phi(x)\phi(y) \in C'\mathfrak{d}' = \mathfrak{d}'$, whence $xy \in \phi^{-1}(U' \cap C') \cap C \subseteq \phi^{-1}(U') \cap C$; similarly, if $x, y \in \phi^{-1}(U') \cap C$, then $x - y \in \phi^{-1}(U') \cap C$. Consequently, $\phi^{-1}(U') \cap C = C$, so $1 \in C \subseteq \phi^{-1}(U')$, whence $1' = \phi(1) \in U' \cap C' = \mathfrak{d}'$, and therefore $\mathfrak{d}' = C'$.

For convenience, let us call a topological ring A an *S-ring* if A is a Hausdorff topological ring with identity, if the connected component $\mathfrak{c}$ of zero in A is locally compact, and if either $A = \mathfrak{c}$ or $A/\mathfrak{c}$ is locally centrally linearly compact. Thus the topological rings satisfying (a)–(c) of §1 are precisely the S-rings whose centers are topological rings having no proper open ideals. Here is our structure theorem for such rings:

Theorem 11. *Let A be a Hausdorff topological ring with identity, and let $\mathfrak{c}$ be the connected component of zero in A. The following statements are equivalent:*

1°. *A is an S-ring, and the center of A is a topological ring having no proper open ideals.*

2°. *A is an S-ring, and the center of $A/\mathfrak{c}$ is a topological ring having no proper open ideals.*

3°. *A is topologically isomorphic to $A_1 \times \cdots \times A_n \times D$ $(n \geq 0)$, where each A_i is a finite-dimensional Hausdorff topological algebra with identity over* **R** *or* **C** *whose center is a Cohen subalgebra, and where either $D = (0)$ or D is the local direct product of topological rings $(A_\gamma)_{\gamma \in \Gamma}$ relative to $(B_\gamma)_{\gamma \in \Gamma}$, where for each $\gamma \in \Gamma$, B_γ is a basic topological ring and A_γ is the topological quotient ring of B_γ.*

Moreover, the final statement of Theorem 10 *pertains.*

Proof. Assuming that $\mathfrak{c}$ is a proper ideal of A and applying Lemma 7 to the centers C and C' of A and $A/\mathfrak{c}$ respectively and to the canonical epimorphism ϕ from A onto $A/\mathfrak{c}$, we conclude that 1° implies 2°.

To show that 2° implies 3°, we may assume by Theorem 10, that $\mathfrak{c} \neq (0)$. We shall first prove that $\mathfrak{c}$ contains an identity element. The assertion holds by hypothesis if $A = \mathfrak{c}$, so we shall assume that $A/\mathfrak{c} \neq (0)$. By Lemma 6, $\mathfrak{c}$ is a finite-dimensional Hausdorff topological algebra over **R** and the left annihilator of $\mathfrak{c}$ in $\mathfrak{c}$ is (0); therefore the left regular representation $L: c \mapsto L_c$ from $\mathfrak{c}$ into $\mathrm{End}_R(\mathfrak{c})$, where L_c is defined by $L_c(x) = cx$ for all $x \in \mathfrak{c}$, is a monomorphism. Let e be an idempotent in $\mathfrak{c}$ such that $L_e(\mathfrak{c})$ is maximal in the set of all the subspaces $L_p(\mathfrak{c})$ where p is an idempotent of $\mathfrak{c}$.

Suppose that $(1 - e)\mathfrak{c} \neq (0)$. Let ϕ' be the restriction of ϕ to $(1 - e)A$, and let π be the continuous open epimorphism $x \mapsto (1 - e)x$ from the additive topological group A onto the additive topological group $(1 - e)A$. Since $e \in \mathfrak{c}$, $\phi = \phi'\pi$; therefore as both ϕ and π are continuous and open, ϕ' is a continuous open epimorphism from $(1 - e)A$ onto $A/\mathfrak{c}$. The kernel of ϕ' is $(1 - e)A \cap \mathfrak{c} = (1 - e)\mathfrak{c}$; therefore $(1 - e)A/(1 - e)\mathfrak{c}$ is topologically isomorphic to $A/\mathfrak{c}$. Clearly $(1 - e)\mathfrak{c}$ is closed in $\mathfrak{c}$ and hence is locally compact; $(1 - e)\mathfrak{c}$ is the continuous image of a connected set and hence is connected; as the connected component of zero in $(1 - e)A$ is clearly contained in $(1 - e)A \cap \mathfrak{c}$, therefore, we conclude that $(1 - e)\mathfrak{c}$ is the connected component of zero in $(1 - e)A$. By Lemma 6 applied to $(1 - e)A$, which has the left identity $1 - e$, $(1 - e)\mathfrak{c}$ has a nonzero idempotent f. As $f = (1 - e)f$, $ef = 0$. Let $M = L_e(\mathfrak{c})$, $N = L_f(\mathfrak{c})$, and let $g = e + f$. Then $L_g(\mathfrak{c}) \subseteq M + N$; however, as $ef = 0$, $L_g(m - L_f(m)) = m$ for each $m \in M$, and $L_g(n) = n$ for each $n \in N$. The restriction of L_g to $M + N$ is therefore an automorphism of the **R**-vector space $M + N$. Applying the Cayley-Hamilton theorem as in the proof of Lemma 6, we conclude that there is an idempotent h in $\mathfrak{c}$ such that L_h is a projection on $M + N$. Since $M \cap N = (0)$ (as $ef = 0$) and since $N \neq (0)$, we thus obtain a contradiction.

Therefore $(1 - e)\mathfrak{c} = (0)$, so L_e is the identity linear operator on $\mathfrak{c}$, and consequently e is the identity element of $\mathfrak{c}$. Moreover, e is a central idempotent of A, for if $x \in A$, then ex and xe belong to $\mathfrak{c}$, so $ex = (ex)e = e(xe) = xe$. Consequently, A is the topological direct sum of the ideals $Ae = \mathfrak{c}$ and $A(1 - e)$, which is topologically isomorphic to $A/\mathfrak{c}$. Therefore 3° holds by Theorem 10 and a remark of §1 concerning finite-dimensional algebras with identity (for if A is a finite-dimensional local $\mathbf{R}$-algebra, by the proof of Cohen's theorem [6, Theorem 9], $\mathbf{R} \cdot 1$ is contained in a coefficient field that is a finite-dimensional topological $\mathbf{R}$-algebra and hence is topologically isomorphic to $\mathbf{R}$ or $\mathbf{C}$).

Since the center of a cartesian product of rings is the cartesian product of their centers, it is easy to see that 3° implies 1° by Theorem 10.

To apply Theorem 11 to locally compact rings, we need the following lemma:

Lemma 8. *Let A be the topological quotient ring of a basic ring B. The following statements are equivalent:*

1°. *A is locally compact.*

2°. *B is compact.*

3°. *The residue field of the center of B is finite.*

Furthermore, if B is special, the following statement is equivalent to 1°–3°:

4°. *The residue field of the center C of A is a locally compact field.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of the center C_B of B. If B is compact, then the residue field $C_B/\mathfrak{m}$ of C_B is compact and discrete and hence is finite. Conversely, assume that $C_B/\mathfrak{m}$ is finite. Then for each $n \geq 1$, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is finite since it is a finite-dimensional $(C_B/\mathfrak{m})$-vector space. By induction, $C_B/\mathfrak{m}^r$ is finite for all $r \geq 1$; hence C_B is precompact. As C_B is complete, therefore, C_B is compact. Thus 2° and 3° are equivalent.

Clearly 2° implies 1°. If 1° holds, then m^nB is compact for some $n \geq 1$, so $\mathfrak{m}^nB/\mathfrak{m}^{n+1}B$ is compact and discrete and hence is finite. But since B is not discrete, $\mathfrak{m}^nB/\mathfrak{m}^{n+1}B$ is a nonzero vector space over $C_B/\mathfrak{m}$, so $C_B/\mathfrak{m}$ is finite. Thus 1° implies 3°.

Finally, assume that B is special. Clearly 1° implies 4°. Assume 4°, and let ϕ be the canonical epimorphism from C onto its residue field K, a local field by Theorem 4. By Lemma 2, $\phi(C_B)$ is contained in the valuation ring V of K, which is compact by hypothesis. Let $\mathfrak{M}$ be the maximal ideal of V, and let $\mathfrak{n} = \phi^{-1}(\mathfrak{M}) \cap C_B$. Then $C_B/\mathfrak{n}$ is isomorphic to a subring of $V/\mathfrak{M}$, a finite field, so $C_B/\mathfrak{n}$ is a finite field; hence $\mathfrak{n} = \mathfrak{m}$ and 3° holds.

Theorem 12. *Let A be a Hausdorff topological ring with identity. The following statements are equivalent:*

1°. *A is a locally compact ring whose center is a topological ring having no proper open ideals.*

2°. *A is topologically isomorphic to* $A_1 \times \cdots \times A_n \times D$ $(n \geq 0)$, *where each* A_i *is a finite-dimensional Hausdorff topological algebra over* **R** *or* **C** *whose center is a Cohen subalgebra, and where either* $D = (0)$ *or D is the local direct product of topological rings* $(A_\gamma)_{\gamma \in \Gamma}$ *relative to* $(B_\gamma)_{\gamma \in \Gamma}$, *where for each* $\gamma \in \Gamma$, B_γ *is a compact basic ring and* A_γ *is the topological quotient ring of* B_γ.

Moreover, if the additive order of each element of A is either infinite or squarefree, then each A_γ *is the topological direct sum of finitely many ideals, each a finite-dimensional locally compact algebra with identity over a totally disconnected, indiscrete, locally compact field.*

Proof. If A is locally compact and satisfies 3° of Theorem 11 then each A_γ is the image of A under a continuous open homomorphism and hence is locally compact, so each B_γ is compact by Lemma 8. Thus 1° implies 2°. Conversely, if each B_γ is compact, then D is clearly locally compact by Tihonov's theorem, so A is also locally compact. By Theorem 11, therefore, 1° and 2° are equivalent; the final statement follows from Theorem 11 and Lemma 8.

Theorem 13. *Let A be a Hausdorff topological ring with identity, and let C be the center of A. The following statements are equivalent:*

1°. *A is an S-ring, and C is a topological ring having no proper open ideals or nonzero nilpotents.*

2°. *A is topologically isomorphic to* $A_1 \times \cdots \times A_n \times D$ $(n \geq 0)$, *where each* A_i *is a central, finite-dimensional Hausdorff topological algebra over* **R** *or* **C**, *and where either* $D = (0)$ *or D is the local direct product of topological rings* $(A_\gamma)_{\gamma \in \Gamma}$ *relative to* $(B_\gamma)_{\gamma \in \Gamma}$, *where for each* $\gamma \in \Gamma$, A_γ *is the topological direct sum of finitely many ideals, each a central, finite-dimensional Hausdorff topological algebra over a local field, and where* B_γ *is a basic ring of which* A_γ *is the topological quotient ring.*

Proof. A commutative local artinian ring without nonzero nilpotents is a field. In particular, a finite-dimensional local **R**-algebra without nonzero nilpotents is a finite-dimensional field extension of **R** and hence is **R** or **C**; and if the center of the topological quotient ring of a special basic ring has no nonzero nilpotents, then it is a local field by 2° of Theorem 3 and Theorem 4. The equivalence of 1° and 2° therefore follows from Theorems 11 and 7.

Theorem 14. *Let A be a Hausdorff topological ring with identity, and let C be the center of A. The following statements are equivalent:*

1°. *A is an S-ring that has no nonzero nilpotent ideals, and the topological ring C has no proper open ideals.*

2°. *A is topologically isomorphic to* $A_1 \times \cdots \times A_n \times D$ $(n \geq 0)$, *where each* A_i *is the topological ring of all square matrices of some order over* **R**, **C**, *or the division ring* **H** *of quaternions, and where either* $D = (0)$ *or* D *is the local direct product of topological rings* $(A_\gamma)_{\gamma \in \Gamma}$ *relative to* $(B_\gamma)_{\gamma \in \Gamma}$, *where for each* $\gamma \in \Gamma$, A_γ *is the topological direct sum of finitely many ideals, each the topological ring of all square matrices of some order over a division ring finite dimensional over its center, which is a local field, and where* B_γ *is a basic ring of which* A_γ *is the topological quotient ring.*

Proof. Statements 1° and 2° imply the corresponding statements of Theorem 13. We observe that, with the notation of Theorem 13, A has no nonzero nilpotent ideals if and only if each A_i and each A_γ have no nonzero nilpotent ideals, or equivalently, by 3° of Theorem 3, are semisimple. In particular, 2° implies 1°. On the other hand, 1° implies 2° by Theorem 13 and Wedderburn's theorem on finite-dimensional semisimple algebras.

Specializing either Theorem 12 or Theorem 13 to the commutative, locally compact, semisimple case, we obtain the structure theorem of Goldman and Sah [7, Theorem 4.1].

Theorem 15. *Let A be a Hausdorff topological ring with identity, and let C be the center of A. The following statements are equivalent:*

1°. *A is an S-ring, C is a topological ring that has no proper open ideals, and any one and hence all of the following conditions hold:*

(a) *C contains an invertible element c such that* $\lim c^n = 0$.

(b) *A has only finitely many maximal ideals.*

(c) *Every ideal [left ideal, right ideal] of A is closed.*

(d) *A is a (left or right) noetherian [artinian] ring.*

2°. *A is topologically isomorphic to* $A_1 \times \cdots \times A_n$, *where each* A_i *is either a finite-dimensional Hausdorff topological algebra over* **R** *or* **C** *whose center is a Cohen subalgebra or the topological quotient ring of a special basic ring.*

Proof. To show that 1° implies 2°, we may assume by Theorem 11 that A is the local direct product of $(A_\gamma)_{\gamma \in \Gamma}$ relative to $(B_\gamma)_{\gamma \in \Gamma}$, where each B_γ is a basic ring and A_γ is the topological quotient ring of B_γ. By Theorem 7, it suffices to show that Γ is finite. Assume first that $c = (c_\gamma)$ is an invertible element of C such that $\lim c^n = 0$. Replacing c with a power of c, if necessary, we may assume that $c \in (\Pi_{\gamma \in \Gamma} B_\gamma) \cap C = \Pi_{\gamma \in \Gamma}(B_\gamma \cap C_\gamma)$, where C_γ is the center of A_γ. Now $\lim c_\gamma^n = 0$ for each $\gamma \in \Gamma$, so c_γ belongs to the maximal ideal of the center $B_\gamma \cap C_\gamma$ of B_γ (1° of Theorem 3). Thus $c_\gamma^{-1} \notin B$ for all $\gamma \in \Gamma$. But as $c^{-1} = (c_\gamma^{-1}) \in A$, $c_\gamma^{-1} \notin B_\gamma$ for only finitely many $\gamma \in \Gamma$, so Γ is finite.

If $\mathfrak{M}_\beta$ is a maximal ideal of A_β, then $\mathfrak{M}'_\beta = \{(x_\gamma) \in A: x_\beta \in \mathfrak{M}_\beta\}$ is clearly a maximal ideal of A whose projection on A_β is $\mathfrak{M}_\beta$ and whose projection on every other A_γ is A_γ. Therefore (b) implies that Γ is finite. Since $\bigoplus_{\gamma \in \Gamma} A_\gamma$ is a dense ideal of A, (c) also implies that Γ is finite.

Finally, let $\mathfrak{F}(\Gamma)$ be the set of all finite subsets of Γ. For each $\Delta \in \mathfrak{F}(\Gamma)$, let $A_\Delta = \{(x_\gamma) \in A: x_\gamma = 0$ for all $\gamma \notin \Delta\}$ and let $A'_\Delta = \{(x_\gamma) \in A: x_\gamma = 0$ for all $\gamma \in \Delta\}$. If Γ is infinite, then the set of all A_Δ [respectively, A'_Δ] where $\Delta \in \mathfrak{F}(\Gamma)$ is a family of ideals having no maximal [minimal] member. Thus (d) implies that Γ is finite.

To show that 2° implies 1°, by Theorem 11 we need only establish (a)–(d), and for that we may assume $n = 1$. If A is the topological quotient ring of a special basic ring, then (a) holds by 1° and 2° of Definition 2, and (b)–(d) hold by Theorem 3. If A is a finite-dimensional Hausdorff topological algebra over **R** or **C** with identity element e, then clearly (c) and (d) hold, (b) holds since $A/\mathrm{Rad}(A)$ is the direct sum of simple rings by Wedderburn's theorem, and finally (a) holds, for if $0 < |\lambda| < 1$, then $\lambda \cdot e$ is an invertible element of C such that $\lim (\lambda \cdot e)^n = 0$.

Theorem 16. *Let A be a Hausdorff topological ring with identity. The following statements are equivalent:*

1°. *A is the topological direct sum of finitely many ideals, each a finite-dimensional Hausdorff topological algebra over* **R**, **C**, *or a local field.*

2°. *The additive order of each element of A is either infinite or squarefree, A is an S-ring, and the center of A contains an invertible element c such that* $\lim c^n = 0$.

3°. *A is the topological direct sum of finitely many ideals, each a finite-dimensional Hausdorff topological algebra over* **R**, **C**, *or a local field, the center of which is a Cohen subalgebra.*

Proof. If c is an invertible element of the center C such that $\lim c^n = 0$, then any open ideal of C contains the invertible element c^n for some $n \geq 1$ and hence is C. Thus 1° implies 2° and 2° implies 3° by Theorems 15 and 6.

By Theorem 15, we may replace "the center of A contains an invertible element c such that $\lim c^n = 0$" by the conjuction of "the center of A is a topological ring that has no proper open ideals" and any one of (b), (c), (d) of Theorem 15.

In view of Theorem 12, specializing Theorem 16 to the locally compact case yields [13, Theorem 8].

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