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ADDENDUM TO "CONVERSE THEOREMS AND EXTENSIONS IN CHEBYSHEV RATIONAL APPROXIMATION TO CERTAIN ENTIRE FUNCTIONS IN $[0, +\infty)$"

BY

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Recently (Trans. Amer. Math. Soc. 170 (1972), 171–185) we have proved among other results the following theorem. The notation and numbering are the same as in [1].

Theorem 7 [1, p. 183]. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function of order* $\rho = 0$ *with* $a_0 > 0$ *and* $a_k \geq 0$ *for all* $k \geq 1$ *such that there exist finite numbers*

$$1 < \rho_l = \Lambda + 1 \equiv \overline{\lim_{r\to\infty}} \frac{\log \log M_f(r)}{\log \log r},$$

and $0 < b_l \leq B_l$ *such that*

$$\overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{(\log r)^{\Lambda+1}} = B_l, \quad \underline{\lim_{r\to\infty}} \frac{\log M_f(r)}{(\log r)^{\Lambda+1}} = b_l. \tag{3.11}$$

Then, there exists a sequence of real polynomials $\{P_n(x)\}_{n=0}^{\infty}$ *with* $P_n \in \pi_n$ *for each* $n \geq 0$ *such that*

$$\overline{\lim_{n\to\infty}} \left\{ \left\| \frac{1}{P_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} = 0. \tag{3.12}$$

Our aim in writing this addendum is to present under the assumptions of Theorem 7, a much sharper result than (3.12), by modifying slightly the proof of Theorem 7.

Lemma [2, Lemma 7]. *If* $f(z)$ *satisfies the assumptions of Theorem 7, then*

$$\overline{\lim_{n\to\infty}} \frac{n^{\Lambda+1}}{[\log |1/a_n|]^{\Lambda}} = \frac{(\Lambda+1)^{(\Lambda+1)}}{\Lambda^{\Lambda}} B_l.$$

Remark. The notation used here is slightly different from the one used in [2].

Theorem 7*. *Under the assumptions of Theorem 7, we get*

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$$(3.12^*)\qquad \overline{\lim_{n\to\infty}}\left\{\left\|\frac{1}{P_n(x)}-\frac{1}{f(x)}\right\|_{L_\infty[0,\infty)}\right\}^{1/n\log n} < e^{-1/\Lambda}.$$

Proof. We have from [1, p. 181],

$$(3.5')\qquad \left|\frac{1}{f(x)}-\frac{1}{P_n(x;r)}\right| \le \frac{2\delta_n(r)}{f^2(0)},\qquad x\in[0,r],$$

and

$$(3.6)\qquad \left|\frac{1}{f(x)}-\frac{1}{P_n(x;r)}\right| \le \frac{2}{f(r)},\qquad x\ge r.$$

From [1, (3.4)] we get

$$(3.4')\qquad \delta_n(r) \le \sum_{k=n+1}^{\infty} a_k r^k.$$

Hence, from (3.4′) and (3.5′), we obtain

$$(3.16)\qquad \left|\frac{1}{f(x)}-\frac{1}{P_n(x;r)}\right| \le \frac{2\sum_{k=n+1}^{\infty} a_k r^k}{a_0^2}.$$

Now from Lemma 1, we have for all $k\ge k_0(\epsilon)$

$$(3.17)\qquad \begin{aligned} |a_k| r^k &\le \left[\exp\left\{\frac{-k\Lambda}{(\Lambda+1)}\left(\frac{k}{(\Lambda+1)(B_l+\epsilon)}\right)^{1/\Lambda}\right\}\right] r^k,\\ &\le \exp\left\{\log r - \frac{\Lambda}{(\Lambda+1)}\left(\frac{k}{(\Lambda+1)(B_l+\epsilon)}\right)^{1/\Lambda}\right\} k. \end{aligned}$$

Set

$$(3.18)\qquad \log r = \frac{\Lambda}{(\Lambda+1)}\left(\frac{(n+1)}{(\Lambda+1)(B_l+\epsilon)}\right)^{1/\Lambda} - \frac{\log(n+1)}{\Lambda}.$$

A simple calculation based on (3.17) and (3.18) gives us for all $n\ge n_0$

$$(3.19)\qquad \begin{aligned} \sum_{k=n+1}^{\infty} a_k r^k &\le \left[\exp\left(\frac{-(n+1)\log(n+1)}{\Lambda}\right)\right]\left(\sum_{k=0}^{\infty}\frac{1}{n^{k/\Lambda}}\right)\\ &\le \left[\exp\left[\frac{-(n+1)\log(n+1)}{\cdot\ \Lambda}\right]\right]\left(\frac{n^{1/\Lambda}}{n^{1/\Lambda}-1}\right). \end{aligned}$$

On the other hand we get from (3.6) and (3.11) for all $r\ge r_0(\epsilon_1)$

$$M_f(r) \geq \exp[(\log r)^{\Lambda+1}(b_l - \epsilon_1)]. \tag{3.20}$$

(3.18) and (3.20) give us for all $n \geq n_0$

$$\begin{aligned} M_f(r) &\geq \exp\left\{\left[\frac{\Lambda}{(\Lambda+1)}\left(\frac{(n+1)}{(\Lambda+1)(B_l+\epsilon)}\right)^{1/\Lambda} - \frac{\log(n+1)}{\Lambda}\right]^{(\Lambda+1)}(b_l - \epsilon_1)\right\} \\ &\geq \exp\left[c_1 n^{(\Lambda+1)/\Lambda}\left(1 - c_2\frac{\log(n+1)}{\Lambda(n+1)^{1/\Lambda}}\right)^{\Lambda+1}\right] \end{aligned} \tag{3.21}$$

where

$$c_1 = \left(\frac{\Lambda}{(\Lambda+1)}\right)^{\Lambda+1}(b_l - \epsilon_1)\left(\frac{1}{(\Lambda+1)(B_l+\epsilon)}\right)^{(\Lambda+1)/\Lambda}, \qquad c_2 = \frac{(\Lambda+1)((\Lambda+1)(B_l+\epsilon))^{1/\Lambda}}{\Lambda}.$$

Now we get from (3.16) and (3.19) for $x \in [0, r]$

$$\left\|\frac{1}{f(x)} - \frac{1}{P_n(x;r)}\right\|_{[0,r]}^{1/n\log n} \leq \exp\left(-\frac{1}{\Lambda}\right). \tag{3.22}$$

On the other hand from (3.6), (3.20) and (3.21) we get

$$\left\|\frac{1}{f(x)} - \frac{1}{P_n(x;r)}\right\|^{1/n\log n} = 0, \quad \text{for all } x \geq r. \tag{3.23}$$

Hence from (3.22) and (3.23) by setting (cf. [1, p. 182]) $P_n(x) \equiv P_n(x; r(n))$, we get (3.12*), i.e.

$$\varlimsup_{n\to\infty}\left\|\frac{1}{f(x)} - \frac{1}{P_n(x)}\right\|_{L_\infty[0,\infty)}^{1/n\log n} \leq \exp\left(-\frac{1}{\Lambda}\right).$$

Remarks. There exist transcendental entire functions of order zero which fail to satisfy the assumptions of the Theorem 7* but satisfy

$$\varlimsup_{n\to\infty}\left\|\frac{1}{f(x)} - \frac{1}{P_n(x)}\right\|^{1/n\log n} < 1. \tag{3.24}$$

For example the function

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{x^n}{2^{\log 2}3^{\log 3}4^{\log 4}\cdots n^{\log n}}$$

is of order zero with $\Lambda = \infty$ and satisfies (3.24).

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By choosing

$$\log r = \frac{\Lambda}{2(\Lambda+1)}\left(\frac{(n+1)}{(\Lambda+1)(B_l+\epsilon)}\right)^{1/\Lambda}$$

instead of (3.18) and adopting the same technique we get instead of (3.12*) the following

$$\text{(3.25)} \qquad \overline{\lim_{n\to\infty}}\left(\left\|\frac{1}{P_n(x)} - \frac{1}{f(x)}\right\|_{L_\infty[0,\infty)}\right)^{1/n^{1+1/\Lambda}} < 1.$$

The result (3.25) is the best possible in the sense that Λ cannot be replaced by $(\Lambda - \epsilon)$ for any $\epsilon > 0$ for all entire functions satisfying the assumptions of Theorem 7*. The details of this will appear separately. In view of (3.25) the following example may be of some interest.

Let $f(x) = \Sigma_0^\infty x^n/e^{2^n}$. For this function $\Lambda = 0$ and

$$\text{(3.26)} \qquad \lim_{n\to\infty} (\lambda_{0,n})^{1/2^{(n+1)}} = \frac{1}{e}.$$

The proof of this is very lengthy. The details of this also will appear separately.

REFERENCES

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2. A. R. Reddy, *Approximation of an entire function*, J. Approximation Theory 3 (1970), 128–137. MR 41 #4091.

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