TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 186, December 1973

ERGODICITY OF THE CARTESIAN PRODUCT

BY

ELIAS G. FLYTZANIS

ABSTRACT. h_1 is an ergodic conservative transformation on a σ-finite measure space and h_2 is an ergodic measure preserving transformation on a finite measure space. We study the point spectrum properties of $h_1 \times h_2$. In particular we show $h_1 \times h_2$ is ergodic if and only if $h_1 \times h_2$ have no eigenvalues in common other than the eigenvalue 1. The conditions on h_1, h_2 stated above are in a sense the most general for the validity of this result.

1. Introduction. All measure spaces (S, Σ, m) will be σ-finite in the sense of Halmos [7]. We consider transformations $h: S \to S$ that are measurable (A in Σ implies $h^{-1}(A)$ in Σ), nonsingular ($m(A) = 0$ implies $m(h^{-1}(A)) = 0$). h is said to be conservative if $h^{-1}(A) \subseteq A$ implies $h^{-1}(A) = A$. We note that all statements about sets are made modulo sets of measure zero. Sometimes we replace the condition of being conservative on a σ-finite measure space by the stronger condition of being measure preserving ($m(h^{-1}(A)) = m(A)$) on a probability space. A complex number c is said to be an eigenvalue of h if the equation $f(h(\cdot)) = cf(\cdot)$ a.e. has a solution in $L_\infty(S)$. h is said to be ergodic if $h^{-1}(A) = A$ implies $m(A) = 0$ or $m(S - A) = 0$. In general the measurable sets that are invariant under h (modulo sets of measure 0) form a sub-σ-field and h is ergodic if and only if that field is trivial.

Lemma 1. *If h is conservative then the eigenvalues of h form a subgroup of the circle group. If h is also ergodic then every eigenfunction is a constant a.e. in absolute value, and two eigenfunctions corresponding to the same eigenvalue differ only by a multiplicative constant. (It is not known whether the group of eigenvalues has to be countable.)*

H denotes a separable Hilbert space and T a contraction in H, i.e. a bounded linear operator with $\|T\| \leq 1$. In § 2 we consider the eigenoperator equation

$$TX(h(\cdot)) = X(\cdot) \quad \text{a.e.,} \tag{1}$$

where $X: S \to H$ is a measurable function. This type of equation was studied for the first time in [1]. We complete the solution by using some results from [8].

Received by the editors February 13, 1973.
AMS (MOS) subject classifications (1970). Primary 18A65.
Key words and phrases. Ergodic transformation, cartesian product, eigenoperation, Hilbert space, unitary operator.

In §3 we apply the results of §2 to the following problem. b_1 is a conservative transformation on a σ-finite measure space (S_1, Σ_1, m_1) and b_2 is a measure preserving transformation on a probability space (S_2, Σ_2, m_2). We consider the eigenvalues of the cartesian product $b_1 \times b_2$ defined on the measure space $(S_1 \times S_2, \Sigma_1 \times \Sigma_2, m_1 \times m_2)$. $b_1 \times b_2$ is also conservative and hence the eigenvalues have norm 1. The eigenfunction equation $f(b_1(s_1), b_2(s_2)) = cf(s_1, s_2)$, where f is in $L_\infty(S_1, S_2)$ can be put in the form (1).

In particular we consider the ergodicity of $b_1 \times b_2$, and Theorem 2(ii) generalizes previous results ([3], [9], [12]). Theorem 2 allows us to construct (Example 1) conservative ergodic transformations that do not accept a finite invariant measure and have any given countable subgroup of the circle as their set of eigenvalues and whose restriction to a sub-σ-field is isomorphic to any given measure preserving transformation on a probability space. Also we show by an example taken from [10] that Theorem 2(iii) is not valid if we weaken the conditions on our transformations. However, a partial result still holds if b_2 is a measure preserving on an infinite σ-finite measure space (Lemma 7). We note here that the results obtained are valid also for flows as well as more general group actions with the appropriate definitions.

The author would like to thank the referee for pointing out [10].

2. **Eigenoperator equation.** b is a conservative transformation on a σ-finite measure space (S, Σ, m). H and T are as above, and $X: S \to H$ is a measurable solution to the eigenoperator equation (1). The essential range $R(X) \subseteq H$ of X is defined by "x in $R(X)$ if for every neighborhood N of x we have $m(X^{-1}(N)) > 0$" We used the fact that if b is conservative then given A in Σ, $m(A) > 0$, there exists a positive integer n so that $m(b^{-n}(A) \cap A) > 0$. Using this, we can prove the next lemma in the same way as the corresponding result in [1].

Lemma 2. (i) *$R(X)$ is nonempty closed and invariant under T.*

(ii) *For a.a. s in S, $X(s)$ in $R(X)$.*

(iii) *For each x in $R(X)$ there exists a sequence of positive integers $\{n\}$ such that $T^n x \to x$.*

Lemma 2(iii) characterizes T completely in the subspace spanned by $P(x) = \{T^i x, i = 0, 1, 2, \cdots\}$ [8, pp. 100–114]. We have

Lemma 3. (i) *$T \mid \overline{\text{span}}\,(P(X))$ is a unitary operator having discrete spectrum.*
(ii) *$T \mid \overline{\text{span}}\,(R(X))$ is a unitary operator having discrete spectrum.*

We note that (ii) follows from (i), using Lemma 2(i) and the property $\|T\| \le 1$.

Let v_i, $i = 1, 2, \cdots$, denote an orthonormal basis of eigenvectors in $\overline{\text{span}}\, R(X)$ given by the last lemma and c_i, $i = 1, 2, \cdots$, the corresponding eigenvalues.

We set

$$f_i(\cdot) = \langle v_i, X(\cdot)\rangle, \quad i = 1, 2, \cdots.$$

Lemma 4. *f_i is an eigenfunction of h corresponding to the eigenvalue c_i for each $i = 1, 2, \cdots$.*

Proof.

$$\bar{c}_i f_i(h(\cdot)) = \bar{c}_i \langle v_i, X(h(\cdot))\rangle = \langle T^* v_i, X(h(\cdot))\rangle = \langle v_i, TX(h(\cdot))\rangle = \langle v_i, X(\cdot)\rangle \quad \text{a.e.}$$

Hence $f_i(h(\cdot)) = c_i f_i(\cdot)$. We show that f_i is not the trivial function. Since v_i lies in $\overline{\text{span}}\,(R(X))$ we have $\langle v_i, z\rangle \neq 0$ for some z in $R(X)$. Hence v_i defines a nonzero continuous function on $R(X)$. It follows from the definition of $R(X)$ that f_i is not the trivial function. Q.E.D.

Theorem 1. (i) *h is a conservative transformation on a σ-finite measure space (S, Σ, m). H is a separable Hilbert space and T is a contraction operator in H. Then the solutions to the eigenoperator equation,*

$$TX(h(\cdot)) = X(\cdot) \quad a.e. \text{ in } S,$$

where $X: S \to H$ is a measurable function are all the functions of the form

$$X(\cdot) = \sum \bar{g}_i(\cdot) v_i \quad a.e. \text{ in } S,$$

where v_i is any orthonormal set of eigenvectors of T and if $Tv_i = c_i v_i$ then $g_i(h_i(\cdot)) = c_i g(\cdot)$. Also $\sum |g_i(\cdot)|^2 < \infty$ a.e. in S and the convergence is pointwise.

(ii) *If h is also ergodic then $\|X\|$ is constant a.e., and the convergence is in the essential sup norm.*

The theorem above is a generalization of Lemma 1 under the assumption $\|T\| \leq 1$.

3. **The cartesian product.** h_1 is a conservative transformation on a σ-finite measure space (S_1, Σ_1, m_1) and h_2 is a measure preserving transformation on a probability space (S_2, Σ_2, m_2). We consider the transformation $h_1 \times h_2$ defined on the cartesian product $(S_1 \times S_2, \Sigma_1 \times \Sigma_2, m_1 \times m_2)$ by $h_1 \times h_2(s_1, s_2) = (h_1(s_1), h_2(s_2))$. We show first that $h_1 \times h_2$ is conservative. We need the following

Lemma 5. *h is conservative if and only if $f(h(\cdot)) \leq f(\cdot)$ a.e. for a real valued measurable (or characteristic) function implies $f(h(\cdot)) = f(\cdot)$ a.e.*

Lemma 6. *$h_1 \times h_2$ is conservative.*

Proof. Let $f(s_1, s_2)$ be a characteristic function and $f(h_1(s_1), h_2(s_2)) \leq f(s_1, s_2)$ a.e. By [3, p. 194] we have

(i) $$\int f(h_1(s_1), h_2(s_2))ds_2 \leq \quad f(s_1, s_2)ds_2,$$

where both sides are m_1 measurable. Setting $X(\cdot) = \int f(s_1, s_2)ds_2$ and using the measure preserving character of h_2, relation (i) becomes

$$X(h(s_1)) \leq X(s_1) \quad \text{a.e. in } S_1.$$

By Lemma 5 we obtain $X(h(s_1)) = X(s_1)$ a.e. and (i) becomes also an equality. By [3, p. 194] again, we have $f(h_1(s_1), h_2(s_2)) = f(s_1, s_2)$ a.e. in $S_1 \times S_2$. The result follows from Lemma 5. Q.E.D.

Hence $h_1 \times h_2$ has all its eigenvalues on the unit circle. Let c be an eigenvalue of $h_1 \times h_2$ and $f(h_1(s_1), h_2(s_2)) = cf(s_1, s_2)$ where f is in $L_\infty(S_1 \times S_2)$. The function $X: S_1 \to L_2(S_2)$ is m_1-measurable by [2, p. 196, Lemma 6b] and the equation becomes

$$TX(h_1(\cdot)) = X(\cdot) \quad \text{a.e. in } S_1,$$

where T is the isometry in $L_2(S_2)$ defined by $Tg(\cdot) = \bar{c}g(h_2(\cdot))$ a.e. in S_2. By Theorem 1 we obtain

Theorem 2. *h_1, h_2 are as above. $C = \{c_i\}$, $D = \{d_j\}$ are the corresponding groups of eigenvalues. We consider the groups $C \cap D$ and CD. Then*

(i) *CD is the group of eigenvalues of $h_1 \times h_2$.*

(ii) *The sub-σ-field of $\Sigma_1 \times \Sigma_2$ of the invariant sets of $h_1 \times h_2$ is that generated by the functions of the form $\bar{u}(s_1) \cdot v(s_2)$, where u, v, are eigenfunctions of h_1, h_2 respectively corresponding to the same eigenvalue in $C \cap D$.*

(iii) *$h_1 \times h_2$ is ergodic if and only if h_1, h_2 are both ergodic and $C \cap D = \{1\}$.*

Example 1. In [2] or [4, Example 6.8] a general method is given for constructing ergodic transformations that do not accept a finite invariant measure and in [11] or [4, Example 6.8] a conservative and ergodic measure preserving transformation on a probability space is constructed having no eigenvalues except 1. It is possible to combine these two techniques and obtain an ergodic transformation that does not accept a finite invariant measure and also that has no eigenvalues except 1. Denoting this transformation by h_1, we can take for h_2 any ergodic m.p.t. on a probability space and then $h_1 \times h_2$ will be always ergodic, it will not accept a finite measure space and its restriction to a sub-σ-field will be isomorphic to the given h_2. Also it will have the same eigenvalues as h_2.

Remarks. Concerning the validity of Theorem 2 (iii) under more general conditions on the transformations h_1, h_2 we note the following. Clearly the result is not valid if we do not assume h_1 to be conservative, as can be seen by taking h_1 to be the shift transformation on the integers. In fact for invertible transformations the only ergodic transformation that is not conservative is exactly this. Concerning the conditions on h_2 we note the following

Example 2. In [10] we are given an example of an ergodic transformation h which is measure preserving on an infinite σ-finite measure space and for which

(i) $h \times h$ is ergodic,

(ii) $h \times h \times h$ is not ergodic.

It follows from (i) that h has no eigenvalues. Setting $h_1 = h \times h$ and $h_2 = h$ we note that they have no eigenvalues in common and yet $h_1 \times h_2$ is not ergodic. In fact for the case where h_2 is ergodic and measure preserving on an infinite σ-finite measure space we note that the induced transformation in $L_2(S_2)$ has no eigenvalues and using the same technique as above we obtain

Lemma 7. *h_1 is a conservative ergodic transformation on a σ-finite measure space and h_2 is an ergodic measure preserving transformation on an infinite σ-finite measure space. Then $h_1 \times h_2$ is conservative; and if A is an invariant subset of $h_1 \times h_2$ then the function on S_1 defined by m_2 $(s_2 : (s_1, s_2)$ in $A)$ has infinite value over a set of nonzero measure in S_1. In particular, there are no invariant sets of finite measure.*

The result above must be rather limited as we use the isometry induced by h_2 in $L_2(S_2)$ and even the ergodicity of h_2 is not an invariant of this isometry. We could obtain complete information for the general cartesian product problem if we could solve the eigenoperator equation where T is assumed only bounded though not necessarily an isometry. It can be shown that in this case the essential range of $X(\cdot)$ consists of nonwandering points of T. If the Hilbert space is finite dimensional this makes T totally bounded and reduces to the case considered. The solution is not known if the Hilbert space is infinite dimensional.

REFERENCES

1. A. Beck, *Eigenoperators of ergodic transformations*, Trans. Amer. Math. Soc. 94 (1960), 118–129. MR 27 #2606.

2. R. V. Chacon, *A class of linear transformations*, Proc. Amer. Math. Soc. 15 (1964), 560–564. MR 29 #2362.

3. N. Dunford and J. T. Schwartz, *Linear operators. I: General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

4. N. A. Friedman, *Introduction to ergodic theory*, Van Nostrand, Princeton, N. J., 1970.

5. H. Fürstenberg, *Disjointness in ergodic theory, minimal sets and a problem in Diophantine approximations*, Math. Systems Theory 1 (1967), 1–49. MR 35 #4369.

6. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc. Japan, no. 3, Math. Soc. Japan, Tokyo, 1956; reprint, Chelsea, New York, 1960. MR 20 #3958.

7. ———, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR 11, 504.

8. K. Jacobs, *Lecture notes on ergodic theory*. Vol. 1, Mathematisk Institut, Aarhus Universitet, Aarhus, 1962/63, pp. 1–207. MR 28 #3138.

9. B. Kronfeld, *Ergodic transformations and independence*, Glasnik Math. Ser. III 5(25) (1970), 221–226. MR 43 #2189.

10. S. Kakutani and W. Parry, *Infinite measure preserving transformations with mixing*, Bull. Amer. Math. Soc. 69 (1963), 752–756. MR 27 #3776.

11. D. S. Ornstein, *On invariant measures*, Bull. Amer. Math. Soc. 66 (1960), 297–300. MR 26 #3872.

12. W. Parry and F. Hahn, *Some characteristic properties of dynamical systems with quasi-discrete spectra*, Math. Systems Theory 2 (1968), 179–190. MR 37 #6435.

DEPARTMENT OF MATHEMATICS, PIERCE COLLEGE, ATHENS, GREECE