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DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC EQUATIONS[1]

BY

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ABSTRACT. Let L_0 be a degenerate second order elliptic operator with no zeroth order term in an m-dimensional domain G, and let $L = L_0 + c$. One divides the boundary of G into disjoint sets $\Sigma_1, \Sigma_2, \Sigma_3$; Σ_3 is the noncharacteristic part, and on Σ_2 the "drift" is outward. When c is negative, the following Dirichlet problem has been considered in the literature: $Lu = 0$ in G, u is prescribed on $\Sigma_2 \cup \Sigma_3$. In the present work it is assume that $c \leq 0$. Assuming additional boundary conditions on a certain finite number of points of Σ_1, a unique solution of the Dirichlet problem is established.

Introduction. Consider the second order degenerate elliptic operator with smooth coefficients

$$Lu = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} \tag{0.1}$$

in a smoothly bounded domain G in R^m. The Dirichlet problem for the equation $Lu + c(x)u = 0$ in G has been treated by many authors (see [5] and the references contained therein). In all of these approaches, the boundary ∂G is decomposed as follows:

$$\Sigma_3 = \left\{x \in \partial G;\ \sum_{i,j=1}^{m} a_{ij}(x)\nu_i\nu_j > 0\right\} \qquad ((\nu_i) = \text{outward normal}),$$

$$\Sigma_2 = \left\{x \in \partial G \setminus \Sigma_3;\ \sum_{i=1}^{m}\left(b_i(x) - \frac{1}{2}\sum_{j=1}^{m}\frac{\partial a_{ij}}{\partial x_j}(x)\right)\nu_i > 0\right\},$$

$$\Sigma_1 = \left\{x \in \partial G \setminus \Sigma_3;\ \sum_{i=1}^{m}\left(b_i(x) - \frac{1}{2}\sum_{j=1}^{m}\frac{\partial a_{ij}}{\partial x_j}(x)\right)\nu_i \leq 0\right\}.$$

A typical result of these theories asserts that the equation $Lu + c(x)u = 0$ has a unique solution in some function space when data are prescribed on

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$\Sigma_2 \cup \Sigma_3$ and when it is further required that $c(x) \leq c_0 < 0$ in G. In case $c_0 = 0$, Stroock and Varadhan [7] have shown that the Dirichlet problem in R^m has a unique solution with data prescribed on $\Sigma_2 \cup \Sigma_3$ provided that the paths of the associated diffusion process exit G "sufficiently fast." We intend to show that, if this last condition fails, we can still find a unique solution *provided that we assign data also on a certain portion of* Σ_1. This possibility was never considered before, to the best of our knowledge. Our method is entirely probabilistic, inspired by our previous work [3], [4] on the stability properties of stochastic differential equations.

The points of Σ_1 at which we must assign data are precisely the points to which the associated Markov process converges when $t \to \infty$. These "distinguished boundary points" are defined in terms of the normal *and* tangential behavior of the diffusion process, in contrast to $\Sigma_2 \cup \Sigma_3$ which only depends on the normal components of diffusion and drift. For technical reasons we will only consider cases where there exist a finite number of distinguished boundary points, together with an arbitrary configuration of $\Sigma_2 \cup \Sigma_3$. We denote by Σ_1^- the component of ∂G containing all of the distinguished boundary points.

In §§2–5 we consider the case $m = 2$, and in §6 we consider the case $m \geq 2$.

§§1 and 2 contain preliminary results on the boundary behavior of solutions $x(t)$ of the stochastic equations

$$dx_i = \sum_{r=1}^{n} \sigma_{ir}(x)\, dw^r + b_i(x)dt \qquad (1 \leq i \leq 2)$$

in a special domain in the plane. For technical reasons we assume that when σ and b vanish simultaneously on Σ_1^-, they do not vanish faster than a linear function. In §3 we consider a general domain in the plane and show that either $x(t)$ attains $\Sigma_2 \cup \Sigma_3$ in finite time or else converges to some distinguished boundary point while remaining inside G for all $t < \infty$. In §4 we prove the differentiability (as a function of the starting point) of the probability that the process will converge to a given distinguished boundary point. Finally in §5 we consider the Dirichlet problem in a general domain in the plane, combining the results of the previous sections with known results [6] on the behavior of the diffusion process near $\Sigma_2 \cup \Sigma_3$.

The results of §§2–5 can be extended to $m \geq 3$; this is briefly discussed in §6. However, the main result of §6 is a theorem which even for $m = 2$ is not included in §§2–5.

1. Boundary behavior of stochastic solutions in annular domains. Consider a system of two stochastic differential equations

$$dx_i = \sum_{s=1}^{n} \sigma_{is}(x)dw^s + b_i(x)dt \qquad (i = 1, 2) \tag{1.1}$$

where $w^1(t), \cdots, w^n(t)$ are independent Brownian motions. Let $a = (a_{ij}) = \sigma\sigma^*$ where $\sigma = (\sigma_{is})$, $\sigma^* =$ transpose of σ. We assume

(A) The functions $\sigma_{is}(x)$, $b_i(x)$ and their first two derivatives are continuous and bounded in R^2.

Let G be a bounded domain in R^2. For simplicity we first take

$$G = \{x;\ 1 < |x| < 2\}.$$

Denote by ∂G the boundary of G. We shall assume

(B) On ∂G,

$$\sum_{i,j=1}^{2} a_{ij}\nu_i\nu_j = 0, \tag{1.2}$$

$$\sum_{i=1}^{2}\left[b_i - \frac{1}{2}\sum_{j=1}^{2}\frac{\partial a_{ij}}{\partial x_j}\right]\nu_i \leq 0 \tag{1.3}$$

where ν is the outward normal to ∂G (with respect to G).

Let $R(x)$ be a positive C^2 function in G, which coincides with $\operatorname{dist}(x, \partial G)$ when the latter is sufficiently small. Let

$$\mathfrak{A} = \sum_{i,j=1}^{2} a_{ij}\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j},$$

$$\mathfrak{B} = \sum_{i=1}^{2} b_i\frac{\partial R}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 R}{\partial x_i \partial x_j}, \qquad Q = \frac{1}{R}\left(\mathfrak{B} - \frac{\mathfrak{A}}{2R}\right).$$

(C) For some $\mu > 0$ sufficiently small,

$$Q(x) \leq -\theta_0 < 0 \quad \text{if } 1 < |x| \leq 1 + \mu \qquad (\theta_0 \text{ constant}),$$

$$Q(x) \geq \theta_0 > 0 \quad \text{if } 2 - \mu \leq |x| < 2,$$

$$\mathfrak{A}(x) > 0 \quad \text{if } 1 + \mu \leq |x| \leq 2 - \mu,\ \nabla_x R(x) \neq 0,$$

$$\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \quad \text{if } 1 + \mu \leq |x| \leq 2 - \mu,\ \nabla_x R(x) = 0.$$

By Theorem 1.1 of [4] and by slightly modifying the proof of Theorem 2.1 of [4] we get

Lemma 1.1. *If* (A)–(C) *hold then, for any solution* $x(t)$ *of* (1.1) *with* $x(0) \in G$,

$$P\{x(t) \in G \text{ for all } t > 0\} = 1, \qquad P\{|x(t)| \to 1 \text{ as } t \to \infty\} = 1.$$

It is actually sufficient to assume that σ_{ij} are continuously differentiable in R^2 and twice continuously differentiable in a neighborhood of ∂G.

We shall now analyze the limit set of $x(t)$ on $|x| = 1$ (as $t \to \infty$). For this, we introduce polar coordinates (r, ϕ) as in [4]. We find that

$$(1.4)\quad dr = \sum_{s=1}^{n} \tilde{\sigma}_s(r, \phi)dw^s + \tilde{b}(r, \phi)dt, \qquad d\phi = \sum_{s=1}^{n} \tilde{\tilde{\sigma}}_s(r, \phi)dw^s + \tilde{\tilde{b}}(r, \phi)dt$$

where

$$\tilde{\sigma}_s(r, \phi) = \sigma_{1s}\cos\phi + \sigma_{2s}\sin\phi, \qquad \tilde{b}(r, \phi) = b_1\cos\phi + b_2\sin\phi + \frac{1}{2r}\langle a(x)\lambda^{\perp}, \lambda^{\perp}\rangle,$$

$$\tilde{\tilde{\sigma}}_s(r, \phi) = -\frac{\sin\phi}{r}\sigma_{1s} + \frac{\cos\phi}{r}\sigma_{2s}, \qquad \tilde{\tilde{b}}(r, \phi) = -\frac{\sin\phi}{r}b_1 + \frac{\cos\phi}{r}b_2 - \frac{1}{r^2}\langle a(x)\lambda, \lambda^{\perp}\rangle;$$

here

$$\lambda = (\cos\phi, \sin\phi), \qquad \lambda^{\perp} = (-\sin\phi, \cos\phi)$$

and

$$\langle a(x)\mu, \nu\rangle = \sum a_{ij}(x)\mu_i\nu_j \qquad (\mu = (\mu_1, \mu_2),\ \nu = (\nu_1, \nu_2)).$$

Thus, if $(r(t), \phi(t))$ is a solution of (1.4) and if we define $x_1(t) = r(t)\cos\phi(t)$, $x_2(t) = r(t)\sin\phi(t)$, then $x(t) = (x_1(t), x_2(t))$ is a solution of (1.1).

The system (1.4) can also be written in the form (see [4])

$$(1.5)\quad \begin{aligned} dr &= r\left[\sum_{s=1}^{n} \tilde{\sigma}_s(\phi)dw^s + \tilde{b}(\phi)dt\right] + \left[\sum_{s=1}^{n} R_s dw^s + R_0 dt\right] \\ d\phi &= \left[\sum_{s=1}^{n} \tilde{\tilde{\sigma}}_s(\phi)dw^s + \tilde{\tilde{b}}(\phi)dt\right] + \left[\sum_{s=1}^{n} \Theta_s dw^s + \Theta_0 dt\right] \end{aligned}$$

where $R_s = o(r)$, $\Theta_s = o(1)$ $(0 \le s \le n)$ as $r \to 1$, uniformly with respect to ϕ. It is useful to compare $\phi(t)$ with the solution of the single equation

$$(1.6)\quad d\phi = \sigma(\phi)dw + b(\phi)dt$$

where $\sigma(\phi) = \{\Sigma_{s=1}^{n} [\tilde{\tilde{\sigma}}_s(\phi)]^2\}^{1/2}$, $b(\phi) = \tilde{\tilde{b}}(\phi)$.

In case $\sigma(\phi) \neq 0$ for all ϕ and $\int_0^{2\pi} b(z)/\sigma^2(z)dz \neq 0$, it was proved in [4] that the algebraic angle $\phi(t)$ (i.e., the component $\phi(t)$ of the solution $(r(t), \phi(t))$ of (1.4)) satisfies

$$(1.7)\quad P\left\{\lim_{t\to\infty} \frac{\phi(t)}{t} = c\right\} = 1 \quad (c \text{ constant} \neq 0).$$

Suppose $\sigma(z)$ is degenerate, but it has only a finite number of zeros $\alpha_1, \cdots, \alpha_k$ $(k \geq 1)$ in the interval $[0, 2\pi)$. Then the conclusion (1.7) is still valid [4] provided $b(\alpha_j) > 0$ for all j, or $b(\alpha_j) < 0$ for all j, and provided the following condition holds:

(i) For some $\bar{\epsilon} > 0$,

$$\sum_{s=1}^{n} [\tilde{\tilde{\sigma}}_s(r, \phi)]^2 = \sum_{s=1}^{n} [\tilde{\tilde{\sigma}}_s(\phi)]^2[1 + \eta(r, \phi)] \qquad (1 \le r \le 1 + \bar{\epsilon})$$

where $\eta(r, \phi) \to 0$ if $r \to 1$, uniformly with respect to ϕ.

We shall now consider the degenerate case in situations where the $b(\alpha_j)$ may vanish. The condition (i) will not be assumed in the sequel.

Our basic assumptions are:

(D) If $b(\alpha_j) = 0$ for some j $(1 \le j \le k)$ then there is a simple C^3 curve Δ_{α_j} given by $r = r^*(t)$, $\phi = \phi^*(t)$ $(t_1 \le t \le t_2)$ such that $(r^*(t_1), \phi^*(t_1)) = (1, \alpha_j)$, $(r^*(t_2), \phi^*(t_2))$ lies outside $\bar{G}$, and $(r^*(\tilde{t}), \phi^*(\tilde{t}))$, for some $t_1 < \tilde{t} < t_2$, is a point on ∂G different from $(1, \alpha_j)$, and such that

(i) a part $\{(r^*(t), \phi^*(t));\ t_1 \le t \le t_1 + \epsilon_1\}$ coincides with the segment $1 \le r \le 1 + \bar{\epsilon}$, $\phi = \alpha_j$, and

(ii) the following relations hold at each point of Δ_{α_j}:

$$\sum_{i,j=1}^{2} a_{ij}\nu_i\nu_j = 0, \qquad \sum_{i=1}^{2} \left[b_i - \frac{1}{2} \sum \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i = 0 \tag{1.8}$$

where (ν_1, ν_2) is the normal to Δ_{α_j}.

Finally, if $b(\alpha_j) = b(\alpha_h) = 0$ and $b(\alpha_i) \ne 0$ for all α_i between α_j and α_h, then the points (r, ϕ) with $r = 1 + \bar{\epsilon}$, ϕ in the interval (α_j, α_h) cannot be connected (in G) to points (r, ϕ) with $r = 1 + \bar{\epsilon}$, ϕ outside the interval (α_j, α_h), without crossing either Δ_{α_j} or Δ_{α_h}.

Note that the conditions in (1.8) along the ray $1 \le r \le 1 + \epsilon$, $\phi = \alpha_j$ hold if and only if

$$\tilde{\tilde{\sigma}}_s(r, \alpha_j) = 0 \quad (1 \le s \le n), \qquad \tilde{\tilde{b}}(r, \alpha_j) = 0 \quad \text{for } 1 \le r \le 1 + \bar{\epsilon}.$$

(E) If $b(\alpha_j) = 0$ for some j $(1 \le j \le k)$, then $b(z)$, $\sigma(z)$ vanish at $z = \alpha_j$ to the first order only, and

$$Q_{\alpha_j} \equiv \lim_{z \to \alpha_j} \frac{2(z - \alpha_j) b(z)}{\sigma^2(z)} \ne 1.$$

Note that the limit exists since $b(z)$, $\sigma(z)$ are continuously differentiable.

We first consider the case where $b(z)$ vanishes at two consecutive points, say $\alpha = \alpha_h$, $\beta = \alpha_{h+1}$. Introduce straight segments

$$l_\alpha = \{(r, \alpha);\ 1 \le r \le 1 + \bar{\epsilon}\}, \qquad l_\beta = \{(r, \beta);\ 1 \le r \le 1 + \bar{\epsilon}\}.$$

Denote by m_η $(\eta \ge 0)$ the curve $\{(r, \phi);\ r = 1 + \eta,\ \alpha \le \phi \le \beta\}$, and by Ω_η $(\eta > 0)$ the domain bounded by m_0, m_η, l_α, l_β. Denote by $A_{\alpha\beta}$ the set of points in the probability space for which $\alpha \le \phi(t) \le \beta$ for a sequence of t's converging to ∞. By (D), and the proof of Theorem 1.1 (or rather Theorem 1.1) of [4], if $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$ then the solution $(r(t), \phi(t))$ never intersects $\Delta_\alpha \cup \Delta_\beta$. It follows (by the last part of (D)) that on the set $A_{\alpha\beta}$

$$\alpha < \phi(t) < \beta \tag{1.9}$$

for all t sufficiently large.

Lemma 1.2. *Let* (A)–(E) *hold and let* $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$. *If* $Q_\alpha > 1$, $Q_\beta < 1$ *then, a.s. on the set* $A_{\alpha\beta}$, $\phi(t) \to \beta$ *if* $t \to \infty$.

Proof. Consider the function

$$g(x) = \int_{\alpha^*}^{x} \frac{1}{\beta(y)}\left[\int_{y}^{\alpha^*} \frac{2\beta(z)}{\sigma^2(z)}\,dz\right]dy \quad \text{in } (\alpha+\epsilon_0, \beta-\epsilon_0) \tag{1.10}$$

for some small $\epsilon_0 > 0$, where

$$\beta(y) = \exp\left\{\int_{\alpha^*}^{y} \frac{2b(z)}{\sigma^2(z)}\,dz\right\}.$$

It is easily seen that

$$\hat{L}g \equiv \tfrac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) = -1 \quad \text{in } (\alpha+\epsilon_0, \beta-\epsilon_0).$$

Further,

$$\begin{aligned} g'(x) &> 0 \quad \text{at } x = \alpha+\epsilon_0, \\ &< 0 \quad \text{at } x = \beta-\epsilon_0 \end{aligned} \tag{1.11}$$

if $\alpha + \epsilon_0 < \alpha^* < \beta - \epsilon_0$;

$$g'(x) > 0 \quad \text{at } x = \alpha+\epsilon_0,\ x = \beta-\epsilon_0 \tag{1.12}$$

if $\alpha < \alpha^* < \alpha + \epsilon_0$;

$$g'(x) < 0 \quad \text{at } x = \alpha+\epsilon_0,\ x = \beta-\epsilon_0 \tag{1.13}$$

if $\beta - \epsilon_0 < \alpha^* < \beta$.

Set

$$f(x) = \begin{cases} -A_1 \log(x-\alpha) + B_1 & \text{in } (\alpha, \alpha+\epsilon_0), \\ A_2 \log(\beta - x) + B_2 & \text{in } (\beta-\epsilon_0, \beta), \\ g(x) & \text{in } [\alpha+\epsilon_0, \beta-\epsilon_0]. \end{cases}$$

If $A_1 > 0$, $A_2 > 0$ then, by using the assumptions $Q_\alpha > 1$, $Q_\beta < 1$ we find that $\hat{L}f(x) \le -\nu < 0$ in $(\alpha, \alpha+\epsilon_0)$, $(\beta-\epsilon_0, \beta)$ where ν is a positive constant, provided ϵ_0 is sufficiently small. Choose α^* so that (1.13) holds, and determine the constants A_i, B_i in such a way that $f(x)$ is continuously differentiable at $x = \alpha + \epsilon_0$, $x = \beta - \epsilon_0$. We then find that $A_1 > 0$, $A_2 > 0$.

Let $\{F_m\}$ be a sequence of continuous functions which approximate f'' in the following manner:

$$F_m(x) = f''(x) \quad \text{if } |x - (\alpha+\epsilon_0)| > 1/m,\ |x - (\beta-\epsilon_0)| > 1/m,$$

and $F_m(x)$ connects linearly $f''(\alpha+\epsilon_0 - 1/m)$ to $f''(\alpha+\epsilon_0+1/m)$ and $f''(\beta-\epsilon_0-1/m)$ to $f''(\beta-\epsilon_0+1/m)$. Let

$$f_m(x) = f(\alpha^*) + f'(\alpha^*)(x - \alpha^*) + \int_{\alpha^*}^{x} \int_{\alpha^*}^{y} F_m(z)\, dz\, dy.$$

Then $f_m(x) - f(x) \to 0$, $f_m'(x) - f'(x) \to 0$ uniformly in the interval (α, β), and $f_m''(x) - f''(x) = 0$ outside the intervals with centers $\alpha + \epsilon_0$, $\beta - \epsilon_0$ and length $2/m$.

Denote by $f_m^\delta(x)$, $f^\delta(x)$ any C^2 (2π)-periodic functions of $x \in R^1$ which coincide, respectively, with $f_m(x)$ and $f(x)$ in $(\alpha + \delta, \beta - \delta)$; δ is any positive number smaller than ϵ_0. Denote by $R(r)$ any C^2 function satisfying

$$R(r) = 1 \quad \text{if } 1 < r < 1 + \eta_0, \qquad R(r) = 0 \quad \text{if } 1 + \eta_1 < r < \infty,$$

where $0 < \eta_0 < \eta_1$ and $\eta_1 < \bar{r}$. Let

$$\tilde{\tilde{\sigma}}_s^\epsilon(r, \phi) = \{[\tilde{\tilde{\sigma}}_s(r, \phi)]^2 + \epsilon\}^{1/2} \qquad (\epsilon > 0).$$

Denote by $(r^\epsilon, \phi^\epsilon)$ the solution of (1.4) when $\tilde{\tilde{\sigma}}_s$ is replaced by $\tilde{\tilde{\sigma}}_s^\epsilon$. Denote by L_ϵ the elliptic generator corresponding to the process $(r^\epsilon, \phi^\epsilon)$. Set

$$\Phi_m^\delta(r, \phi) = R(r) f_m^\delta(\phi), \qquad \Phi^\delta(r, \phi) = R(r) f^\delta(\phi).$$

By Itô's formula,

$$\Phi_m^\delta(r^\epsilon(t), \phi^\epsilon(t)) - \Phi_m^\delta(r^\epsilon(0), \phi^\epsilon(0))$$

$$(1.14) \qquad = \int_0^t \nabla\Phi_m^\delta(r^\epsilon(\tau), \ \sigma^\epsilon(\tau)) \cdot \sigma^\epsilon(r^\epsilon(\tau), \phi^\epsilon(\tau))\, dw(\tau) + \int_0^t L_\epsilon \Phi_m^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau))\, d\tau$$

where $\nabla\Phi$ is the gradient of Φ and $\sigma^\epsilon(r, \phi)$ is the matrix corresponding to $\tilde{\sigma}_s(r, \phi)$, $\tilde{\tilde{\sigma}}_s^\epsilon(r, \phi)$. Since L_ϵ is uniformly elliptic, with bounded and uniformly Hölder continuous coefficients, the corresponding parabolic operator has a fundamental solution (see [2]). We can therefore go to the limit with $m \to \infty$ in (1.14) (cf. [4]) and conclude that

$$R(r(t)) f^\delta(\phi(t)) - R(r(0)) f^\delta(\phi(0))$$

$$(1.15) \qquad = \int_0^t \nabla\Phi^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau)) \cdot \sigma^\epsilon(r^\epsilon(\tau), \phi^\epsilon(\tau))\, dw(\tau) + \int_0^t L_\epsilon \Phi^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau))\, d\tau.$$

We shall now consider the behavior of $(r^\epsilon(\tau), \phi^\epsilon(\tau))$ on the set $A_{\alpha\beta}$. Given $0 < \eta < \eta_0$, let T_η be the last time $(r(t), \phi(t))$ is outside $\Omega_{\eta/2}$. With t fixed, and a.s. in $A_{\alpha\beta}$,

$$(r^\epsilon(\tau), \phi^\epsilon(\tau)) \to (r(\tau), \phi(\tau)) \quad \text{uniformly in } \tau, \ 0 \leq \tau \leq t,$$

for a subsequence $\epsilon = \epsilon' \searrow 0$. Hence, if $T_\eta \leq \tau \leq t$, $(r^\epsilon(\tau), \phi^\epsilon(\tau)) \in \Omega_\eta$ for all $\epsilon = \epsilon' \leq \epsilon^*(\omega)$. Given $\delta \leq \delta^*(\omega)$ sufficiently small, we have for any $\epsilon = \epsilon'$ sufficiently small and for any $\gamma > 0$,

$$\int_{T_\eta}^t L_\epsilon \Phi^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau))\,d\tau = \int_{T_\eta}^t L_\epsilon \Phi(r^\epsilon(\tau), \phi^\epsilon(\tau))\,d\tau$$

$$= \int_{T_\eta}^t (L_\epsilon f)(r^\epsilon(\tau), \phi^\epsilon(\tau))\,d\tau$$

$$= \int_{T_\eta}^t \left\{\left[\frac{1}{2}\sum_{s=1}^n [\tilde{\tilde{\sigma}}_s(r^\epsilon, \phi^\epsilon)]^2 + \epsilon\right] f''(\phi^\epsilon) + \tilde{\tilde{b}}(r^\epsilon, \phi^\epsilon) f'(\phi^\epsilon)\right\} d\tau$$

$$= \int_{T_\eta}^t \left\{\left[\frac{1}{2}\sigma^2(\phi^\epsilon) + \epsilon\right] f''(\phi^\epsilon) + b(\phi^\epsilon) f'(\phi^\epsilon) + \theta\gamma\right\} d\tau$$

$$\le -(t - T_\eta)\nu + \int_{T_\eta}^t \epsilon f''(\phi^\epsilon)\,d\tau + \bar{\theta}\gamma(t - T_\eta)$$

where $|\theta| \le 1$, $|\bar{\theta}| \le 1$, provided $\eta \le \eta^*(\gamma)$. Here ν is any positive number such that $Lf(x) \le -\nu$ for all $x \ne \alpha + \epsilon_0$, $x \ne \beta - \epsilon_0$. It follows that

$$\overline{\lim_{\epsilon=\epsilon'\searrow 0}} \int_{T_\eta}^t L_\epsilon \Phi^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau))\,d\tau \le -(t - T_\eta)\frac{\nu}{2} \tag{1.16}$$

if $\gamma < \nu/2$. Since $(r(\tau), \phi(\tau))$ does not intersect the set $l_\alpha \cup l_\beta$ for $\tau \ge 0$, $f''(\phi^\epsilon(\tau))$ $(0 \le \tau \le T_\eta)$ remains bounded as $\epsilon = \epsilon' \searrow 0$. We conclude that

$$\overline{\lim_{\epsilon=\epsilon'\searrow 0}} \int_0^t L_\epsilon \Phi^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau))\,d\tau \le C - \frac{\nu}{2}t \tag{1.17}$$

where C is a.s. finite valued random variable.

Consider next the stochastic integral in (1.15). If $T_\eta \le \tau \le t$, then the vector

$$h_\epsilon^\delta(\tau) \equiv \nabla\Phi^\delta(r^\epsilon(\tau), \phi^\epsilon(\tau)) \cdot \sigma^\epsilon(r^\epsilon(\tau), \phi^\epsilon(\tau)) \tag{1.18}$$

has components $\{(d/d\phi)f^\delta(\phi^\epsilon(\tau))\}\{[\tilde{\tilde{\sigma}}_s(r^\epsilon(\tau), \phi^\epsilon(\tau))]^2 + \epsilon\}^{1/2}$. If we let $\epsilon = \epsilon' \searrow 0$ through an appropriate subsequence ϵ'', then we obtain a.s. (cf. [4, §2])

$$\begin{aligned} \lim_{\epsilon=\epsilon''\searrow 0} \int_{T_\eta}^t h_\epsilon^\delta(\tau) \cdot dw(\tau) &= \int_{T_\eta}^t h^\delta(\tau) \cdot dw(\tau) = \int_{T_\eta}^t h^0(\tau) \cdot dw(\tau) \\ &= \sum_{s=1}^n \int_{T_\eta}^t f'(\phi(\tau))\tilde{\tilde{\sigma}}_s(r(\tau), \phi(\tau))\,dw^s(\tau) \end{aligned} \tag{1.19}$$

where $h^0(\tau)$ is defined by (1.18) with Φ^δ replaced by Φ and with $\epsilon = 0$.

If $0 \le \tau \le T_\eta$, then as $\epsilon \searrow 0$ through an appropriate subsequence of ϵ'',

$$\int_0^{T_\eta} h_\epsilon^\delta(\tau) \cdot dw(\tau) \to \int_0^{T_\eta} h^\delta(\tau) \cdot dw(\tau) = \int_0^{T_\eta} h^0(\tau) \cdot dw(\tau) \equiv \hat{C}$$

where $h^0(\tau)$ has a more complicated expression than in (1.19) (involving $R(\tau)$

and its first derivative); $\hat{C}$ is a.s. finite. We conclude from this and from (1.15), (1.17), (1.19) that, a.s. on $A_{\alpha\beta}$,

$$f(\phi(t)) - f(\phi(0)) \le C - \frac{\nu}{2}t + \hat{C} + \sum_{s=1}^{n} \int_{T_\eta}^{t} f'(\phi(\tau))\tilde{\tilde{\sigma}}_s(r(\tau), \phi(\tau))\, dw^s(\tau).$$

By Lemma 1.3 of [3], the last integral is $o(t)$. Hence

$$\overline{\lim_{t\to\infty}} \frac{f(\phi(t))}{t} \le -\frac{\nu}{2} < 0 \quad \text{a.s. in } A_{\alpha\beta}.$$

This implies that $\phi(t) \to \beta$ as $t \to \infty$, a.s. in $A_{\alpha\beta}$.

2. Boundary behavior of stochastic solutions (continued). Divide the zeros $\alpha_1, \cdots, \alpha_k$ of $\sigma(z)$ in $[0, 2\pi)$ into blocks

$$B_j = \{\alpha_{j,1}, \cdots, \alpha_{j,k_j}\} \qquad (k_j > 1)$$

where $\alpha_{j,i} < \alpha_{j,i+1}$, $\alpha_{j,k_j} = \alpha_{j+1,1}$. (Here we agree that $\alpha_k < \alpha_1$.) For each block B_j, $b(\alpha_{j,1}) = 0$, $b(\alpha_{j,k_j}) = 0$, and $b(\alpha_{j,i}) \ne 0$ if $2 \le i \le k_j - 1$. Let

$$A_j = \{\omega;\, \alpha_{j,1} < \phi(t) < \alpha_{j,k_j} \text{ for all } t \text{ sufficiently large}\}.$$

In view of the Lemma 1.1 and the fact that $(r(t), \phi(t))$ never crosses the segments $\{(r, \alpha_{j,1});\ 1 \le r \le 1 + \bar{\epsilon}\}$ we conclude that $\sum P(A_j) = 1$.

Consider now a block B_j, and set $\alpha = \alpha_{j,1}$, $\beta = \alpha_{j,k_j}$. Suppose

$$b(\alpha) = 0, \quad b(\beta) = 0, \quad b(\alpha_{j,1}) > 0 \quad (2 \le i \le k_i - 1), \tag{2.1}$$

$$Q_\alpha > 1, \quad Q_\beta < 1. \tag{2.2}$$

Lemma 2.1. *Let* (A)–(E) *and* (2.1), (2.2) *hold. If* $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$, *then a.s. in* A_j, $\phi(t) \to \beta$ *as* $t \to \infty$.

Proof. Let

$$f(x) = \begin{cases} -A_1 \log(x - \alpha) + B_1 & \text{in } (\alpha, \alpha + \epsilon_0), \\ A_2 \log(\beta - x) + B_2 & \text{in } (\beta - \epsilon_0, \beta), \\ g(x) & \text{in } [\alpha + \epsilon_0, \beta - \epsilon_0]; \end{cases}$$

the function $g(x)$ consists of three parts:

$$\begin{array}{ll} A_3 g_1(x) + B_3 & \text{in } [\alpha + \epsilon_0, \alpha_{j,2} - \epsilon'), \\ A_4 g_2(x) + B_4 & \text{in } [\alpha_{j,2} - \epsilon', \alpha_{j,k_j-1} + \epsilon'], \\ g_3(x) & \text{in } (\alpha_{j,k_j-1} + \epsilon', \beta - \epsilon_0] \end{array}$$

where $\epsilon' > 0$ is sufficiently small. The function g_2 is constructed as the function f in the proof of Theorem 3.2 in [3]; thus $Lg_2 \leq -\nu < 0$ in $[\alpha_{j,2} - \epsilon', \alpha_{j,k_j-1} + \epsilon']$ and $g'(x) < 0$ at the endpoints. The function g_1 is defined as the function g in (1.10), (1.13) with β replaced by $\alpha_{j,2} - \epsilon'$. Finally, the function g_3 is defined as the function g in (1.10), (1.13) with α replaced by $\alpha_{j,k_j-1} + \epsilon'$. We can choose the constants A_i, B_i so that $f(x)$ is continuously differentiable; the A_i are all positive.

We can now proceed similarly to the proof of Lemma 1.2.

Suppose now that (2.1), (2.2) are replaced by

$$b(\alpha) = 0, \quad b(\beta) = 0, \quad b(\alpha_{j,i}) > 0 \quad (2 \leq i \leq k_i - 1), \tag{2.3}$$

$$Q_\alpha < 1, \quad Q_\beta > 1. \tag{2.4}$$

Lemma 2.2. *Let* (A)–(E) *and* (2.3), (2.4) *hold. If* $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$, *then a.s. in* A_j, $\phi(t) \to \alpha$ *as* $t \to \infty$.

The proof is similar to the proof of Lemma 2.1. Here one takes $f(x) = A_1 \log(x - \alpha) + B_1$ in $(\alpha, \alpha + \epsilon_0)$, $f(x) = -A_2 \log(\beta - x) + B_2$ in $(\beta - \epsilon_0, \beta)$.

Consider next the cases where

$$b(\alpha) = 0, \quad b(\beta) = 0, \quad Q_\alpha < 1, \quad Q_\beta < 1. \tag{2.5}$$

We further assume that one of the following three conditions holds:

$$b(\alpha_{j,i}) > 0 \quad (2 \leq i \leq k_j - 1), \tag{2.6}$$

$$b(\alpha_{j,i}) < 0 \quad (2 \leq i \leq k_j - 1), \tag{2.7}$$

$$\begin{aligned} &b(\alpha_{j,i}) < 0 \quad (2 \leq i \leq i_0), \\ &b(\alpha_{j,i}) > 0 \quad (i_0 + 1 \leq i \leq k_j - 1). \end{aligned} \tag{2.8}$$

Lemma 2.3. *Let* (A)–(E) *and* (2.5) *hold, and let one of the conditions* (2.6), (2.7), (2.8) *hold. If* $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$, *then a.s. in* A_j, *either* $\lim_{t\to\infty}\phi(t) = \alpha$ *or* $\lim_{t\to\infty}\phi(t) = \beta$.

The proof is similar to the proof of Lemma 2.1. One takes $f(x) = A_1 \log(x - \alpha) + B_1$ in $(\alpha, \alpha + \epsilon_0)$, $f(x) = A_2 \log(\beta - x) + B_2$ in $(\beta - \epsilon_0, \beta)$. In case (2.8) holds one takes g_2 to be the function occurring in the proof of Theorem 4.2 in [3].

The case $b(\alpha) = 0$, $b(\beta) = 0$, $Q_\alpha > 1$, $Q_\beta < 1$ will not be considered in this paper. In this case $\phi(t)$ may oscillate between α and β without having a limit, as suggested by the case of linear equations [3].

3. **Behavior of solutions in general domains.** We shall now extend the results of §2 to a general bounded domain G. A point x_0 on the boundary ∂G of G is said to belong to Σ_3 if $\sum_{i,j} a_{ij}(x_0)\nu_i\nu_j > 0$. It belongs to Σ_2 if (1.2) and

$$\sum_i \left[b_i - \frac{1}{2} \sum_j \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i > 0 \quad \text{at } x_0$$

hold. Finally, x_0 belongs to Σ_1 if (1.2), (1.3) hold at x_0.

Denote by $R(x)$ a continuous function in $\bar{G}$, C^2 and positive in $G \cup \Sigma_2 \cup \Sigma_3$, that coincide with $\operatorname{dist}(x, \Sigma_1)$ when the latter is sufficiently small. With $R(x)$ fixed from now on, we define $\mathfrak{A}$, $\mathfrak{B}$, Q as in §1.

We shall need the following assumption:

(P) ∂G consists of a finite number of curves $\Gamma_1, \cdots, \Gamma_q$. Each curve belongs entirely to either $\Sigma_2 \cup \Sigma_3$ or to Σ_1. A curve Γ_j of $\Sigma_2 \cup \Sigma_3$ is in C^2, and a curve Γ_j of Σ_1 is in C^3. There is a positive constant μ such that if a curve Γ_i belongs to Σ_1 then either (i) $Q(x) \le -\theta_0 < 0$ (θ_0 constant) for all $x \in G$ whose distance to Γ_i is $\le \mu$ [we then say that Γ_i belongs to Σ_1^-], or (ii) $Q(x) \ge \theta_0 > 0$ (θ_0 constant) for all $x \in G$ whose distance to Γ_i is $\le \mu$ [we then say that Γ_i belongs to Σ_1^+]. Finally, Σ_1^- is nonempty.

We shall maintain the assumptions (A), drop the assumption (B), and replace (C) by

(C*) $\mathfrak{A}(x) > 0$ for all $x \in G$ with $\operatorname{dist}(x, \Sigma_1) \ge \mu$, $\nabla_x R(x) \neq 0$;

$$\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \text{ for all } x \in G \text{ with } \operatorname{dist}(x, \Sigma_1) \ge \mu,$$

$$\nabla_x R(x) = 0.$$

By slightly modifying the construction of $R(x)$ in the proof of Lemma 2.1 of [4], one can show that if the exterior boundary of G is not in Σ_1 then there actually exists a function $R(x)$ with $\nabla_x R(x) \neq 0$ everywhere in G.

The proof of Theorem 2.2 of [4] can be modified to yield the following extension of Lemma 1.1.

Theorem 3.1. *Let* (A), (P), (C*) *hold. Then, with probability* 1, *either* (i) $x(t)$ *exits* G *in finite time by crossing* $\Sigma_2 \cup \Sigma_3$, *or* (ii) $x(t) \in G$ *for all* $t > 0$ *and* $\operatorname{dist}(x(t), \Sigma_1^-) \to 0$ *as* $t \to \infty$.

Suppose for definiteness that $\Gamma_1 \subset \Sigma_1^-$, and Γ_1 is not the outer boundary of G. If $x_1 = f(\tau)$, $x_2 = g(\tau)$ are parametric equations for Γ_1 (τ = length parameter), then we can introduce new variables

$$y_1 = (1+\rho)\cos(2\pi\tau/L), \quad y_2 = (1+\rho)\sin(2\pi\tau/L) \qquad (L = \text{length of } \Gamma_1)$$

where the "polar coordinates" ρ, τ are defined by

$$x_1 = f(\tau) + \rho\dot{g}(\tau), \qquad x_2 = g(\tau) - \rho\dot{f}(\tau).$$

As in [4] we can extend this mapping into a diffeomorphism from the exterior of Γ_1 onto the set $\{y: |y| \geq 1\}$. In the new coordinates

$$d\rho = \sum_{s=1}^{n} \tilde{\sigma}_s dw^s + \tilde{b}\,dt,$$

$$d\phi = \sum_{s=1}^{n} \tilde{\tilde{\sigma}}_s dw^s + \tilde{\tilde{b}}\,dt \qquad \left(\phi = \frac{2\pi\tau}{L}\right),$$

and

$$\frac{L}{2\pi}\tilde{\tilde{\sigma}}_s(0, \phi) = \dot{f}\sigma_{1s} + \dot{g}\sigma_{2s},$$

$$\frac{L}{2\pi}\tilde{\tilde{b}}(0, \phi) = (\dot{f}b_1 + \dot{g}b_2) - (\dot{g}, -\dot{f})\begin{pmatrix} \Sigma\sigma_{1s}^2 & \Sigma\sigma_{1s}\sigma_{2s} \\ \Sigma\sigma_{1s}\sigma_{2s} & \Sigma\sigma_{2s}^2 \end{pmatrix}\begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix}.$$

Set $\sigma(\phi) = \{\Sigma_{s=1}^{n} [\tilde{\tilde{\sigma}}_s(0, \phi)]^2\}^{1/2}$, $b(\phi) = \tilde{\tilde{b}}(0, \phi)$.

We now assume:

(D$'$) The condition (D) holds with $r = 1 + \rho$. More precisely: $\sigma(z)$ vanishes at a finite number of points $\alpha_1, \cdots, \alpha_k$ $(k \geq 1)$. If $b(\alpha_j) = 0$ for some j, then there is a simple C^3 curve $\Delta^1_{\alpha_j}$ given by $x = x^*(t)$ $(t_1 \leq t \leq t_2)$ such that $x^*(t_1) = (f(\alpha_j), g(\alpha_j))$, $x^*(t_2)$ lies outside $\bar{G}$, and $x^*(\tilde{t})$, for some $\tilde{t} \in (t_1, t_2)$, lies on ∂G and is different from $(f(\alpha_j), g(\alpha_j))$, and such that

(i) a part $\{x^*(t), t_1 < t \leq t_1 + \epsilon\}$ of $\Delta^1_{\alpha_j}$ lies in G and is nontangential to ∂G at $t = t_1$;

(ii) the relations (1.8) hold along $\Delta^1_{\alpha_j}$.

Finally, if $b(\alpha_j) = b(\alpha_h) = 0$ and $b(\alpha_i) \neq 0$ for all the α_i between α_j and α_h, then points of G corresponding to (ρ, ϕ) with $\rho = \epsilon$, ϕ in the interval (α_j, α_h) [$\epsilon > 0$ small], cannot be connected (in G) to points of G corresponding to (ρ, ϕ) with $\rho = \epsilon$, ϕ outside the interval (α_j, α_h), without crossing either $\Delta^1_{\alpha_j}$ or $\Delta^1_{\alpha_h}$.

(E$'$) The condition (E) holds. Furthermore, $b(\alpha_j) = 0$ for at least one value of j.

Suppose (A), (P), (C*) and (D$'$), (E$'$) hold. Denote by A^1 the set where $x(t) \in G$ for all $t > 0$ and dist $(x(t), \Gamma_1) \to 0$. Let A^1_j be the subset of A^1 for which $\alpha_{j,1} < \phi(t) < \alpha_{j,k_j}$ holds for all t sufficiently large. Suppose a portion of each $\Delta^1_{\alpha_j}$ initiating at $(f(\alpha_j), g(\alpha_j))$ coincides with the normal to ∂G at that point. Then the proof of Lemmas 2.1–2.3 remain valid (in the y-coordinates). Here we use the fact that the diffeomorphism $x \to y$ given above does not affect the condition (D$'$), i.e., the conditions in (1.8) are invariant under a diffeomorphism. If $\Delta^1_{\alpha_j}$ does not contain the normal, then we perform a different local diffeomorphism from the x-space onto the y-space, such that Γ_1 is mapped onto the unit circle and such that the image of a portion of $\Delta^1_{a_j}$ does coincide with the normal to this circle. The new diffeomorphism does not affect the tangential stochastic equation, i.e., the functions $\sigma(\phi)$, $b(\phi)$ remain the same.

We conclude: If $x \notin \bigcup_j \Delta_{a_{j,1}}$ then almost surely on A^1_j, either

(i) $\phi(t) \to \beta$ if (2.1), (2.2) hold; or

(ii) $\phi(t) \to \alpha$ if (2.3), (2.4) hold; or

(iii) $\phi(t) \to \alpha$ or $\phi(t) \to \beta$ if (2.5) and one of the conditions (2.6), (2.7), (2.8) hold.

In what follows we assume:

(Q) For each block B_j, either (2.1), (2.2) or (2.3), (2.4) or (2.5) and one of the conditions (2.6), (2.7), (2.8) hold.

The segment $\{(\rho, \phi);\ \phi = \alpha_{j,1},\ 0 \le \rho \le \bar{\epsilon}\}$ in the y-space is mapped onto an arc l_j in the x-space. l_j initiates at a point γ_{j1} on Γ_1, is nontangential to Γ_1 at γ_{j1}, it is contained in $\Delta^1_{\alpha_{j,1}}$, and it lies in the interior of G (with the exception of its endpoint γ_{j1}). It divides a small G-neighborhood N_j of γ_{j1} into domains: N^+_{j1} and N^-_{j1}.

Definition. The point γ_{j1} is called a *distinguished boundary point* of G if at the corresponding point $\alpha_{j,1}$, $\dot{Q}_{\alpha_{j,1}} < 1$.

If $\Gamma_1 \subset \Sigma^-_1$ and the interior of Γ_1 contains G, then the above considerations remain valid with trivial changes; the assertions (i)–(iii) are unchanged.

Consider now the general case. We index the Γ_i so that

$$\Sigma^-_1 = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_p,$$

$$\Sigma^+_1 = \Gamma_{p+1} \cup \cdots \cup \Gamma_{p+h},$$

$$\Sigma_2 \cup \Sigma_3 = \Gamma_{p+h+1} \cup \cdots \cup \Gamma_q.$$

We assume

(D*) The condition (D′) holds for each Γ_j, $1 \le j \le p$.

(E*) The condition (E′) holds for each Γ_j, $1 \le j \le p$.

(Q*) The condition (Q) holds for each Γ_j, $1 \le j \le p$.

We define distinguished boundary points on $\Gamma_2, \cdots, \Gamma_p$ in the same way as for Γ_1. Denote by ζ_j $(1 \le j \le k)$ the set of all distinguished boundary points on Σ^-_1. With each ζ_i we associate two "half" G-neighborhoods N^+_j, N^-_j of ζ_j, in the same way that we have associated N^+_{j1}, N^-_{j1} with γ_{j1}.

In the condition (D*) there appear curves Δ^l_α $(1 \le l \le p)$ defined analogously to the curves Δ^1_α. Denote these curves by Δ_j $(1 \le j \le l)$ and set $\Lambda_j = \Delta_j \cap \bar{G}$. Each ζ_i is an endpoint of some Λ_j.

We sum up the previous considerations in the following theorem.

Theorem 3.2. *Let the conditions* (A), (P), (C*) *and* (D*), (E*), (Q*) *hold. If* $x \notin \bigcup_{j=1}^{l} \Lambda_j$ *then the probability space is a finite disjoint union*

$$\Omega_0 \cup \left(\bigcup_{j=1}^{k} \Omega_j^+\right) \cup \left(\bigcup_{j=1}^{k} \Omega_j^-\right), \tag{3.1}$$

such that the following holds almost surely: if $\omega \in \Omega_0$, $x(t)$ *exits from* G *in finite time by crossing* $\Sigma_2 \cup \Sigma_3$; *if* $\omega \in \Omega_j^+$ *then* $x(t) \in G$ *for all* $t > 0$, *and* $x(t) \in N_j^+$, $x(t) \to \zeta_i$ *as* $t \to \infty$; *if* $\omega \in \Omega_j^-$ *then* $x(t) \in G$ *for all* $t > 0$, *and* $x(t) \in N_j^-$, $x(t) \to \zeta_i$ *as* $t \to \infty$. *The decomposition* (3.1) *depends on* $x(0)$.

Definition. If $x(t) \in G$ for all $t > 0$, and $x(t) \in N_i^+$, $x(t) \to \zeta_i$ as $t \to \infty$, then we shall write: $x(t) \to \zeta_i^+$ as $t \to \infty$. Similarly we define the concept: $x(t) \to \zeta_i^-$ as $t \to \infty$. We denote by $p_i^+(x)$ ($p_i^-(x)$) the probability that $x(t) \to \zeta_i^+$ ($x(t) \to \zeta_i^-$) as $t \to \infty$, given $x(0) = x \in G$.

Clearly $p_i^+(x) \geq 0$, $p_i^-(x) \geq 0$, $\Sigma_{i=1}^k p_i^+(x) + \Sigma_{i=1}^k p_i^-(x) \leq 1$. If $\Sigma_2 \cup \Sigma_3$ is empty, then the last sum is equal to 1 (by Theorem 3.2) if $x \notin \bigcup_{j=1}^{l} \Lambda_j$.

Definition. Denote by $q_i(x)$ $(1 \leq i \leq p)$ the probability that $x(t) \in G$ for all $t > 0$ and $\text{dist}(x(t), \Gamma_i) \to 0$ as $t \to \infty$, given $z(\theta) = x \in G$.

Theorem 3.3. *Let the conditions* (A), (P), (C*) *hold. Then* $q_i(x) \to 1$ *if* $\text{dist}(x, \Gamma_i) \to 0$ $(1 \leq i \leq p)$.

Proof. For any $\lambda > 0$ sufficiently small, let $\Gamma_{i\lambda}$ be the curve in G parallel to Γ_i at a distant λ. Denote by G_λ the domain bounded by Γ_i, $\Gamma_{i\lambda}$. Denote by L the elliptic operator corresponding to the diffusion process (1.1). Then

$$L[R(x)]^\epsilon = [\epsilon^2 \mathcal{A}/2R^2 + \epsilon Q][R(x)]^\epsilon \qquad (\epsilon > 0).$$

Since $\mathcal{A} = O(R^2)$ in G_λ, $LR \leq 0$ in G_λ provided λ and ϵ are sufficiently small. Denote by τ_λ the exit time from G_λ. Then, by Itô's formula,

$$E[R(x(\tau_\lambda))]^\epsilon - [R(x)]^\epsilon = E\int_0^{\tau_\lambda} L[R(x(\tau))]^\epsilon d\tau \leq 0.$$

Since $x(\tau_\lambda) \in \Gamma_{i\lambda}$, $R(x(\tau_\lambda)) = \lambda$. Hence

$$[1 - q_i(x)]\lambda^\epsilon \leq [R(x)]^\epsilon = [\text{dist}(x, \Gamma_i)]^\epsilon,$$

and the assertion follows.

The above proof is valid also in any number of dimensions.

4. **Regularity of the functions** $p_i^\pm(x)$. Let

$$Lu \equiv \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i}. \tag{4.1}$$

Let $\Lambda'_1, \cdots, \Lambda'_{l'}$ be disjoint C^3 curves (the endpoints are included) lying in G, and set

$$\Lambda = \bigcup_{i=1}^{l} \Lambda_i, \quad \Lambda' = \bigcup_{i=1}^{l'} \Lambda'_i.$$

The elliptic operator L will be allowed to degenerate in G only at the points of $\Lambda \cup \Lambda'$. We shall call Λ_i a "boundary spoke" and Λ'_j an "interior spoke."

Consider a parametric representation for Λ'_j:

$$x_1 = f(s), \quad x_2 = g(s) \quad (0 \le s \le L_j) \tag{4.2}$$

where s is the length parameter. Let

$$b^*_j(s) = (\dot{f} b_2 - \dot{g} b_1) - \frac{1}{2}(\dot{f}\ddot{g} - \dot{g}\ddot{f}) \sum_{r=1}^{n} (\dot{f}\sigma_{1r} + \dot{g}\sigma_{2r})^2 \tag{4.3}$$

where the argument of b_i, σ_{ir} is (x_1, x_2) given by (4.2).

We shall need the following assumption regarding the degeneracy of L in G:

(G) On each Λ'_j, $b^*_j(s) \neq 0$ for $0 \le s \le L_j$. The elliptic operator L may degenerate on each arc Λ_i, and in a sufficiently small δ_0-neighborhood of each Λ'_j; at all the remaining points of G, L is nondegenerate.

The number δ_0 is positive and depends only on upper bounds on the first derivatives of σ_{ik}, b_i, and on a positive lower bound on the $|b^*_j(s)|$. Its precise nature will emerge from the proof of Theorem 4.1 below. Denote by Λ'_{δ_0} the δ_0-neighborhood of Λ'.

Theorem 4.1. *Let the conditions* (A), (P), (C*) *and* (G) *hold. Then* $p_i^{\pm}(x)$ $(1 \le i \le p)$ *are Lipschitz continuous functions in* $G - \Lambda$, *and* C^2 *solutions of* $Lu = 0$ *in* $G - (\Lambda \cup \Lambda'_{\delta_0})$.

Proof. We shall combine classical regularity theorems with the method of Freidlin [1]. Denote by $p_i(x)$ any one of the functions $p_i^{\pm}(x)$. Consider first a point $x_0 \in G$ where L is nondegenerate. Let N be a small disc with center x_0 such that L is nondegenerate in $\bar{N}$. By the strong Markov property, for any $x \in N$,

$$p_i(x) = E_x\{p_i(x_{\tau_N})\} = \int_{\partial N} p_i(y) P_x(x_{\tau_N} \in dS_y) \tag{4.4}$$

where τ_N is the exit time from N, and dS_y is the length element on ∂N. Note, by a standard argument, that $p_i(y)$ is a Borel function on ∂N.

Let A be an interval on ∂N. Denote by I_A the characteristic function of A, and by η, ζ the endpoints of A. By classical theorems [2], there exists a unique solution u of

$$Lu = 0 \text{ in } N, \quad u \in C^2(N),$$

$$u(x) \to I_A(y) \quad \text{if } x \to y \in \partial N, \ y \neq \eta, \ y \neq \zeta,$$

$$u(x) \text{ remains bounded as } x \to \eta, \text{ or } x \to \zeta.$$

We can write u in terms of Green's function [2]

$$u(x) = \int_A \frac{\partial G(x, y)}{\partial \nu_y} dS_y \tag{4.5}$$

where ν_y is the inward normal. Denote by N_ϵ the disc with center x_0 and radius $= r_0 - \epsilon$, where r_0 is the radius of N. Let τ_{N_ϵ} be the exit time from N_ϵ. By Itô's formula,

$$u(x) = E_x\{u(x_{\tau_{N_\epsilon}})\} \qquad (\epsilon > 0). \tag{4.6}$$

Since L is nondegenerate in $\bar{N}$, $x(t)$ exists N at ζ or at η with probability 0. Hence, taking $\epsilon \to 0$ in (4.6), we arrive at the formula

$$u(x) = E_x\{u(x_{\tau_N})\} = E_x\{I_A(x_{\tau_N})\} = P_x(x_{\tau_N} \in A).$$

Comparing this with (4.5), we conclude that

$$P_x(x_{\tau_N} \in A) = \int_A \frac{\partial G(x, y)}{\partial \nu_y} dS_y.$$

This implies that

$$P_x(x_{\tau_N} \in dS_y) = \frac{\partial G(x, y)}{\partial \nu_y} dS_y.$$

Hence (4.4) gives

$$p_i(x) = \int_{\partial N} p_i(y) \frac{\partial G(x, y)}{\partial \nu_y} dS_y. \tag{4.7}$$

This shows that $p_i(x)$ is continuous in N. By decreasing N we may assume that $p_i(x)$ is continuous in $\bar{N}$.

The solution v of $Lv = 0$ in N, $v = p_i$ on ∂N is also given by the right-hand side of (4.7). Hence $v = p_i$ in N. Since v belongs to $C^2(N)$, the same is true of p_i. This completes the proof of the second assertion of Theorem 4.1.

To prove the first assertion, consider first the case of an interior spoke Λ_j' having the form $\phi = \phi_0$, $r_0 \leq r \leq r_1$. Let B_δ be the domain $|\phi - \phi_0| < \delta$, $r_0 < r < r_1$. The condition $b_j^*(s) \neq 0$ reduces to $\tilde{\tilde{b}}(r, \phi) \neq 0$ where $\tilde{\tilde{b}}(r, \phi)$ is defined as in §1. Suppose, for definiteness, that $\tilde{\tilde{b}}(r, \phi) \geq \beta > 0$ inside B_δ. By Itô's formula we have

$$\phi(t) = \phi_0 + \sum_{s=1}^{n} \int_0^t \tilde{\tilde{\sigma}}_s(r, \phi)\,dw^s + \int_0^t \tilde{\tilde{b}}(r, \phi)\,dr,$$

and hence

$$\phi(t \wedge \tau_{B_\delta}) \geq \phi_0 + \sum_{s=1}^{n} \int_0^{t\wedge\tau_{B_\delta}} \tilde{\tilde{\sigma}}_s(r, \phi)\,dw^s + \beta(t \wedge \tau_{B_\delta}).$$

Thus

$$\beta E_x(t \wedge \tau_{B_\delta}) \leq \sup_{\phi \in B_\delta} |\phi - \phi_0| = \delta.$$

It follows that

$$(4.8) \qquad E_x(\tau_{B_\delta}) \leq \delta/\beta = C \qquad (x \in B_\delta).$$

By a standard iteration argument it follows that $P_x(\tau_{B_\delta} \geq nt_0) \leq (C/t_0)^n$ $(n = 1, 2, \cdots)$. Consequently,

$$P_x(\tau_{B_\delta} \geq t) \leq e^{-\alpha t}, \qquad \alpha = -(1/t_0)\log(C/t_0).$$

Taking $t_0 = e$ we get

$$(4.9) \qquad P_x(\tau_{B_\delta} \geq t) \leq e^{-\alpha t}, \qquad \alpha = (1/e)\log(\beta e/\delta).$$

We may choose $\delta > 0$ sufficiently small to apply the following result of Freidlin [1, p. 1349] (which we state only in R^2):

Theorem. Suppose σ_{ij}, b_i *are continuously differentiable in* R^2 *with*

$$\max_{i,j,k} \left\{ \left| \frac{\partial \sigma_{ij}}{\partial x_k} \right|, \left| \frac{\partial b_i}{\partial x_k} \right| \right\} = K.$$

Let $\alpha_1 = 8K^2 + 4K$. *Suppose the boundary is uniformly normally regular, and the boundary function* ψ *is the restriction to* ∂B_δ *of a* C^2 *function in a neighborhood of* ∂B_δ. *Then the function* $E_x\{\psi(x_{\tau B_\delta})\}$ *is Lipschitz continuous in* B_δ, *provided* $\alpha > \alpha_1$.

By choosing δ_0 (in the condition (G)) sufficiently small we can ensure that, for some $\delta > \delta_0$, L is nondegenerate on the boundary of B_δ, and $\alpha > \alpha_1$. The uniform normal regularity of ∂B_δ means that $E_x(\tau_{B_\delta}) \leq C_0|x - x_0|$ for all $x_0 \in \partial B_\delta$, $x \in B_\delta$ where C_0 is a constant. This property is guaranteed by the nonvanishing of the normal diffusion on ∂B_δ (see [1]). Further, since L is nondegenerate on ∂B_δ, $p_i(x)$ is C^2 in a neighborhood of ∂B_δ. Hence we can apply Freidlin's theorem to deduce (upon recalling the first equation of (4.4), which holds for $N = B_\delta$) that $p_i(x)$ is Lipschitz continuous in B_δ.

To handle the case of a general spoke Λ_j', we introduce new coordinates (ρ, s) by the equations

$$x_1 = f(s) + \rho \dot{g}(s), \qquad x_2 = g(s) - \rho \dot{f}(s)$$

where $-\rho_0 < \rho < \rho_0$ (ρ_0 is positive and sufficiently small) and f, g are as in (4.2). The stochastic differentials ds, $d\rho$ are given by the formulas

$$d\rho = \sum_{r=1}^{n} \tilde{\sigma}_r \, dw^r + \tilde{b}\, dt,$$

$$d\phi = \sum_{r=1}^{n} \tilde{\tilde{\sigma}}_r \, dw^r + \tilde{\tilde{b}} dt \quad (\phi = 2\pi s/L_j).$$

Explicit calculation gives (cf. [4]) $\tilde{b}(0, \phi) = 2\pi b_j^*(s)/L_j$ where $b_j^*(s)$ is defined in (4.3). Since $b_j^*(s) \neq 0$, we can repeat the argument given in the previous special case.

Remark. Suppose σ_{ij}, b_i belong to $C^m(R^2)$. Using Theorem 3 of Freidlin [1] (instead of the above quoted theorem of [1]) we conclude that the $p_i^{\pm}(x)$ have $m-1$ Lipschitz continuous derivatives in $G - \Lambda$. Here the constant δ_0 occurring in the condition (G) depends also on m.

Definition. If $x \to \zeta_i$, $x \in N_i^+$ then we write $x \to \zeta_i^+$. Similarly we write $x \to \zeta_i^-$ if $x \to \zeta_i$, $x \in N_i^-$.

Theorem 4.2. *Let the conditions* (A), (P), (C*) *and* (D*), (E*), (Q*) *hold. Then* $p_i^+(x) \to 1$ *if* $x \to \zeta_i^+$, *and* $p_i^-(x) \to 1$ *if* $x \to \zeta_i^-$.

This theorem is of the same type as Theorem 3.3. The method of proof is also the same as for Theorem 3.3.

Proof. It suffices to prove the assertion for $p_i^+(x)$. Consider first the special case where the distinguished boundary point ζ_i lies in some Γ_j, say Γ_1, which is the circle $r = 1$, and G lies in the exterior of Γ_1. Let N be "half G-neighborhood" of ζ_i given by $\zeta_i < \phi < \phi_1$, $1 < r < 1 + \delta$. Consider the function

$$f(R, \phi) = R^\epsilon + (\phi - \zeta_i)^\epsilon \quad \text{in } N$$

where $R = r - 1$, and $\epsilon > 0$ is sufficiently small. It is easily seen that $Lf \leq 0$ if δ and ϵ are sufficiently small. Let τ be the exit time from N. By Itô's formula

$$E\{f(R(t \wedge \tau), \phi(t \wedge \tau))\} \leq f(R(0), \phi(0)) \qquad (t > 0). \tag{4.10}$$

Now, $x(t)$ cannot leave N in finite time by crossing either $R = 0$ or $\phi = \zeta_i$. On the other hand, on the remaining boundary of N, $f(R, \phi)$ is bounded below by some constant $\gamma > 0$ (γ depends on δ, ϵ). Hence, taking $t \to \infty$ in (4.10), we obtain the inequality

$$\gamma P_x(\tau < \infty) \leq f(R(0), \phi(0)) \qquad (x = x(0)).$$

Since $f(R(0), \phi(0)) \to 0$ if $x \to \zeta_i^+$, we conclude that

$$P_x(\tau < \infty) \to 0 \quad \text{if } x \to \zeta_i^+.$$

Since, by Theorem 3.2, $p_i^+(x) = 1 - P_x(\tau < \infty)$, the proof is complete.

Remark. By Theorem 4.2, the $p_i^\pm(x)$ are discontinuous at the points of the boundary spoke initiating at ζ_i, which are in some small neighborhood of ζ_i.

5. The Dirichlet problem.

Lemma 5.1. *Let $x(t) = (x_1(t), \cdots, x_l(t))$ be a solution of a system of l stochastic equations of the form* (1.1), *with uniformly Lipschitz continuous coefficients σ_{ij}, b_i. Let τ be any finite valued random variable. Suppose the range of $x(t)$, $0 \le t \le \tau$, is contained in an open set $D \subset R^l$. Let $f(x)$ be a C^2 function in D. Then Itô's formula holds*:

$$f(x(\tau)) - f(x(0)) = \sum_{i=1}^{l} \sum_{j=1}^{n} \int_0^\tau f_{x_i}(x(s))\sigma_{ij}(x(s))\, dw^j + \int_0^\tau Lf(x(s))\, ds$$

where

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{l} b_i(x) \frac{\partial u}{\partial x_i} \qquad [(a_{ij}) = \sigma\sigma^*].$$

Proof. For any $\delta > 0$, modify f into a function $f^\delta(x)$ in $C^2(R^l)$, coinciding with $f(x)$ if $\operatorname{dist}(x, R^l - D) > \delta$. Apply Itô's formula to $f^\delta(x(t))$, substitute $t = \tau$, and take $\delta \searrow 0$.

Now let the assumptions of Theorem 4.2 hold. Consider the Dirichlet problem

$$Lu = 0 \quad \text{in } G - \Lambda, \tag{5.1}$$

$$u = g \quad \text{on } \Sigma_2 \cup \Sigma_3, \tag{5.2}$$

$$\begin{cases} u(x) \to f_i^+ & \text{if } x \to \zeta_i^+, \\ u(x) \to f_i^- & \text{if } x \to \zeta_i^- \quad (1 \le i \le k). \end{cases} \tag{5.3}$$

Here L is defined by (4.1), g is a given continuous function on $\Sigma_2 \cup \Sigma_3$, and $f_i^\pm$ are given numbers.

If u is continuous in $(G \cup \Sigma_2 \cup \Sigma_3) - \Lambda$ and is in $C^2(G - \Lambda)$, and if it satisfies (5.1)–(5.3), then we call it a *classical solution* of the Dirichlet problem (5.1)–(5.3). Notice that u is not required to be continuous on $\Sigma_1^- \cup \Sigma_1^+ \cup \Lambda$. Since $u(x) \to f_i^\pm$ as $x \to \zeta_i^\pm$, u cannot be continuous at the points of Λ near ζ_i if $f_i^+ \ne f_i^-$.

We shall prove in this section the existence and uniqueness of a classical solution.

Theorem 5.2. *Let* (A), (P), (C*) *and* (D*), (E*), (Q*) *hold. Then there exists at most one classical solution of the Dirichlet problem.*

Proof. Let $G_\epsilon = \{x \in G, \operatorname{dist}(x, \partial G) > \epsilon\}$, $\epsilon > 0$. Denote by τ the exit time from G, and denote by τ_ϵ the exit time from G_ϵ. Let u be a classical solution. Since $x(t)$ (with $x(0) \in G - \Lambda$) remains in $G - \Lambda$ for $0 \le t \le T \wedge \tau_\epsilon$, where $T < \infty$, $\epsilon > 0$, we can apply Lemma 5.1. This gives, after taking the expectation,

$$u(x) = E_x\{u(x(T \wedge \tau_\epsilon))\}.$$

Taking $\epsilon \searrow 0$, $T \nearrow \infty$ and using the continuity of u at $\Sigma_2 \cup \Sigma_3$ and Theorem 3.2, we get

$$u(x) = E_x\{g(x(\tau))I_{(\tau<\infty)}\} + \sum_{i=1}^{k} f_i^+ p_i^+(x) + \sum_{i=1}^{k} f_i^- p_i^-(x) \tag{5.4}$$

where I_A is the indicator function of A. This implies that $u(x)$ is uniquely determined (in $G - \Lambda$).

We shall now prove the existence of a solution.

Theorem 5.3. *Let* (A), (P), (C*), (G), (D*), (E*), (Q*) *hold, and let* g *be continuous on* $\Sigma_2 \cup \Sigma_3$. *Then the function* $u(x)$ *given by* (5.4) *is continuous in* $(G \cup \Sigma_2 \cup \Sigma_3) - \Lambda$, *Lipschitz continuous in* $G - \Lambda$, *and* C^2 *in* $G - (\Lambda \cup \Lambda'_{\delta_0})$, *and it satisfies* (5.1) *in* $G - (\Lambda \cup \Lambda'_{\delta_0})$ *and* (5.2), (5.3).

Proof. In Theorem 4.1 we proved that $p_i^\pm(x)$ is Lipschitz continuous in $G - \Lambda$, and is a C^2 solution of (5.1) in $G - (\Lambda \cup \Lambda'_{\delta_0})$. The same proof works also for the function $E_x\{g(x(\tau))I_{(\tau<\infty)}\}$. Hence, the function u, given by (5.4), is Lipschitz continuous in $G - \Lambda$ and is a C^2 solution of (5.1) in $G - (\Lambda \cup \Lambda'_{\delta_0})$. The assertion $u(x) \to f_i^\pm$ as $x \to \zeta_i^\pm$ follows from Theorem 4.2 and the fact that

$$\sum_{j=1}^{k} p_j^+(x) + \sum_{j=1}^{k} p_j^-(x) + E_x\{I_{(\tau<\infty)}\} = 1.$$

(This is the assertion of Theorem 3.2.) Finally, the assertion that $u(x)$ is continuous at the points of $\Sigma_2 \cup \Sigma_3$ and it satisfies (5.2) follows from Theorem 2 of Pinsky [6].

Remark. The function $u(x)$ is a weak solution of (5.1) in G, in the sense that

$$u(x) = \int_{\partial N} u(y) P_x(x_{\tau_N} \in dS_y)$$

where N is a disc in G, $x \in N$, and τ_N is the exit time from N. The proof is the same as for (4.4).

We shall now strengthen the assumptions of Theorem 5.3 in order to achieve a classical solution.

(G*) The condition (G) holds and σ_{ij}, b_i are in $C^2(\Lambda'_{\delta_0})$. The positive constant δ_0 occurring in the condition (G) will now be smaller; it will be as in the remark following the proof of Theorem 4.1, with $m = 2$.

Theorem 5.4. *Let* (A), (P), (C*), (G*), (D*), (E*), (Q*) *hold, and let* g *be a continuous function on* $\Sigma_2 \cup \Sigma_3$. *Then* (5.4) *is the unique classical solution of the Dirichlet problem* (5.1)–(5.3).

Indeed, we only have to verify that u is in $C^2(G - \Lambda)$ and $Lu = 0$ in $G - \Lambda$. For $p_i^{\pm}(x)$ this follows from the remark following the proof of Theorem 4.1. For $E_x\{g(x(\tau))I_{(\tau<\infty)}\}$ the proof is the same.

Remark. Theorems 5.2–5.4 extend to the Dirichlet problem consisting of

$$Lu + c(x)u = 0 \quad \text{in } G - \Lambda \tag{5.5}$$

and (5.2), (5.3), provided $c(x) \le 0$ in G. Instead of (5.4) we have

$$\begin{aligned} u(x) = E_x\left\{g(x(\tau)) \exp\left[\int_0^\tau c(x(s))\,ds\right] I_{(\tau<\infty)}\right\} \\ + \sum_{i=1}^k f_i^+ E_x\left\{\exp\left[\int_0^\infty c(x(s))\,ds\right] I_{[p_i^+(x)>0]}\right\} \\ + \sum_{i=1}^k f_i^- E_x\left\{\exp\left[\int_0^\infty c(x(s))\,ds\right] I_{[p_i^-(x)>0]}\right\}. \end{aligned} \tag{5.6}$$

If $c(x) \le -c_0 < 0$ then the last two sums vanish, so that no boundary conditions on Σ_1 need to be given. This is in agreement with the treatment in [5], [7] (and the references given there) where c_0 is assumed to be positive.

6. The Dirichlet problem in m-dimensional domains. In subsection 6.1 we prove a theorem for $m \ge 2$ which even when $m = 2$ is not contained in §§2–5. In subsections 6.2, 6.3 we discuss the generalizations of the results of §§2–5 to $m \ge 2$.

6.1. Consider a system of m stochastic equations

$$dx_i = \sum_{r=1}^n \sigma_{ir}(x)\,dw^r + b_i(x)\,dt \qquad (1 \le i \le m) \tag{6.1}$$

and let L, given by (0.1), be the corresponding elliptic operator, i.e., $\sigma\sigma^* = (a_{ij})$. We shall denote the analogs of the conditions (A), (P), (C*) for $m \ge 2$ by (A_m), (P_m), (C_m^*) respectively. Assuming that these conditions hold, the assertion of Theorem 3.1 remains valid.

With G a bounded m-dimensional domain, and $\Gamma_1, \cdots, \Gamma_q$ its boundary hypersurfaces, we index the Γ_i as in §3. Thus, Σ_1^- is made up of $\Gamma_1, \cdots, \Gamma_p$. Denote by Γ_i^ϵ ($\epsilon > 0$) the intersection of G with ϵ-neighborhood of Γ_i. We assume

(R) On each Γ_l $(1 \le l \le p)$ there is a finite number of points ξ_{lj} such that $\sigma_{ir}(\xi_{lj}) = 0$, $b_i(\xi_{lj}) = 0$ for $1 \le i \le m$, $1 \le r \le n$. Let $R_{lj}(x) = |x - \xi_{lj}|$ if $|x - \xi_{lj}| < \epsilon'$ (for some $\epsilon' > 0$), and define $Q_{lj}(x)$ as $Q(x)$ in §1 when $R(x)$ is replaced by $R_{lj}(x)$. Then

$$Q_{lj}(x) \le -\theta_0 < 0 \quad \text{if } |x - \xi_{lj}| < \epsilon', \ x \in G.$$

For any $1 \le l \le p$, let $R_l^*(x)$ be a positive C^2 function for $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $x \ne \xi_{lj}$ (for some $\epsilon_0 > 0$) such that $R_l^*(x) = R_{lj}(x)$ if $|x - \xi_{lj}| < \epsilon'$. We shall assume

(S) For all $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $\min_j |x - \xi_{lj}| > \epsilon'$ $(1 \le l \le p)$,

$$\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial R_l^*}{\partial x_i} \frac{\partial R_l^*}{\partial x_j} > 0 \quad \text{if } \nabla_x R_l^*(x) \ne 0; \tag{6.2}$$

$$\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 R_l^*}{\partial x_i \partial x_j} < 0 \quad \text{if } \nabla_x R_l^*(x) = 0. \tag{6.3}$$

Notice that $R_l^*(x)$ can be constructed such that $\nabla_x R_l^*(x)$ is nonzero if $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $x \ne \xi_{lj}$, and $\nabla_x R_l^*(x)$ is not normal to Γ_l if $x \in \Gamma_l$, $x \ne \xi_{lj}$. Hence, if L is nondegenerate in $\Gamma_l^{\epsilon_0}$ and if the stochastic equations induced by (6.1) on Γ_l have a nondegenerate diffusion matrix [i.e., if the elliptic operator induced by L on Γ_l is nondegenerate] at each $x \ne \xi_{lj}$, then $\nabla_x R_l^*(x) \ne 0$ if $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $x \ne \xi_{lj}$ and (6.2) holds.

So far, assuming (A_m), (P_m) and (C_m^*), we already know that $x(t)$ does not intersect Σ_1 in finite time, and $\operatorname{dist}(x(t), \Sigma_1^-) \to 0$ as $t \to \infty$, provided $x(t) \in G$ for all $t > 0$. We now employ the assumptions (R), (S) to construct a function

$$f(x) = \Phi(R_l^*(x)) \qquad (x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l, \ x \ne \xi_{lj})$$

such that $Lf \le -\nu$ (as in Theorem 2.2 of [4]), and then use it to deduce that, on the set where $\operatorname{dist}(x(t), \Gamma_l) \to 0$, $\min_j |x(t) - \xi_{lj}| \to 0$.

We shall denote the set of all the points ξ_{lj} by $\xi_1, \cdots, \xi_k$, and call them *distinguished boundary points*. We pose the Dirichlet problem

$$Lu = 0 \quad \text{in } G, \tag{6.4}$$

$$u = g \quad \text{on } \Sigma_2 \cup \Sigma_3, \tag{6.5}$$

$$u = f_i \quad \text{at } \zeta_i \quad (1 \le i \le k) \tag{6.6}$$

where g is a given continuous function on $\Sigma_2 \cup \Sigma_3$ and the f_i are given numbers.

Theorem 6.1. *Let the conditions* (A_m), (P_m), (C_m^*) *and* (R), (S) *hold, and let* L *be nondegenerate in* G. *Then there exists a unique solution of the Dirichlet problem* (6.4)–(6.6).

In fact, the solution is given by

$$u(x) = E_x\{f(x(\tau))I_{(\tau<\infty)}\} + \sum_{i=1}^{k} f_i p_i(x) \tag{6.7}$$

where τ is the exit time from G, and $p_i(x)$ is the probability that $x(t) \in G$ for all $t > 0$ and $x(t) \to \zeta_i$ as $t \to \infty$, given $x(0) = x$. The regularity of the terms on the right-hand side of (6.7) can be proved as in the case $m = 2$ (in §4).

Remark 1. Theorem 6.1 can be extended to the case where L may degenerate in a small neighborhood of a finite number of "interior spokes," as in the case $m = 2$. This can be proved by the same method as for $m = 2$.

Remark 2. Theorem 6.1 extends to the Dirichlet problem in which (6.4) is replaced by $Lu + c(x)u = 0$ in G, and $c(x) \leq 0$; cf. the remark at the end of §5.

6.2. Let the conditions (A_m), (P_m), (C_m^*) hold and consider the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } G, \\ u = g & \text{on } \Sigma_2 \cup \Sigma_3, \\ u = f_i & \text{on } \Gamma_i \ (1 \leq i \leq p) \end{cases} \tag{6.8}$$

where the f_i are constants; the Γ_i $(1 \leq i \leq p)$ constitute the Σ_1^- boundary of G. If L is nondegenerate in G, then the function

$$u(x) = E_x\{g(x(\tau))I_{(\tau<\infty)}\} + \sum_{i=1}^{p} f_i q_i(x) \tag{6.9}$$

is the unique classical solution of the Dirichlet problem (6.8). The proof of uniqueness is the same as the proof of Theorem 5.2. The proof that $u(x)$ is in $C^2(G)$ is the same as the corresponding proof for $p_i^{\pm}(x)$. The assertion that $u(x) \to f_i$ is $\operatorname{dist}(x, \Gamma_i) \to 0$ is a consequence of Theorem 3.3 (which holds in any number of dimensions). Finally the assertion that $u(x) \to g(y)$ if $x \to y$, $y \in \Sigma_2 \cup \Sigma_3$ follows from [6, Theorem 2].

6.3. All of the results of §§2–5 can be generalized to the case $m \geq 3$. The conditions needed, however, take a more complicated form. In order to clarify the procedure, we shall describe only a special case, namely, $m = 3$ and G is a shell given by $1 < r < 2$. We further assume that the conditions of Lemma 1.1 hold for $m = 3$ so that the assertion of Lemma 1.1 is valid, with respect to the system in polar coordinates

$$\begin{aligned} dr &= \sum_{j=1}^{n} \tilde{\sigma}_{1j}\,dw^j + \tilde{b}_1\,dt, \\ d\theta &= \sum_{j=1}^{n} \tilde{\sigma}_{2j}\,dw^j + \tilde{b}_2\,dt, \\ d\phi &= \sum_{j=1}^{n} \tilde{\sigma}_{3j}\,dw^j + \tilde{b}_3\,dt. \end{aligned} \tag{6.10}$$

On $r = 1$, this system reduces to

$$d\theta = \sum_{j=1}^{n} \tilde{\sigma}_{2j}(1, \theta, \phi)\,dw^j + \tilde{b}_2(1, \theta, \phi)\,dt,$$

(6.11)

$$d\phi = \sum_{j=1}^{n} \tilde{\sigma}_{3j}(1, \theta, \phi)\,dw^j + \tilde{b}_2(1, \theta, \phi)\,dt.$$

We shall assume

(T_1) Along the closed curve Γ: $(\theta = \theta_0, 0 \le \phi \le 2\pi)$ we have

$$\sum_{i,j=2}^{3} \tilde{a}_{ij}\nu_i\nu_j = 0, \qquad \sum_{i=2}^{3}\left[\hat{b}_i - \frac{1}{2}\sum_{j=1}^{2}\frac{\partial\tilde{a}_{ij}}{\partial\theta_j}\right]\nu_i = 0$$

when $\theta_1 = \theta$, $\theta_2 = \phi$, (ν_2, ν_3) is normal to Γ, and $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^*$.

(T_2) The condition (C*) holds with x replaced by (θ_1, θ_2) and $R(x)$ replaced by a positive function $R^*(\theta_1, \theta_2)$ coinciding with the distance function from Γ when the latter is sufficiently small.

(T_3) Define $Q(\theta_1, \theta_2)$ with respect to (6.11) and $R(\theta_1, \theta_2)$ in the same way that $Q(x)$ was defined with respect to the system (1.2) with respect to $R(x)$. Then $Q(\theta_1, \theta_2) \le -\nu < 0$ $(0 \le \phi \le 2\pi, |\theta - \theta_0| < \epsilon')$ for some $\epsilon' > 0$.

(T_4) No solution of (6.10) crosses the conical surface S: $\theta = \theta_0$. This is the case if and only if $\Sigma_{i,j=1}^3 a_{ij}\nu_i\nu_j = 0$, $\Sigma_{j=1}^3 [b_i - \frac{1}{2}\Sigma_{j=1}^3 \partial a_{ij}/\partial x_j]\nu_i = 0$ on $\theta = \theta_0$, where (ν_i) is the normal to S.

Using the condition (T_1) we can prove (as in [4]) that the solution $(\theta_0(t), \phi_0(t))$ of (6.11) never crosses Γ. Using also the conditions (T_2), (T_3) we can construct a function $V(\theta, \phi)$ for $\theta \ne \theta_0$ such that $LV \le -\nu < 0$ and $V \to -\infty$ if $\theta \to \theta_0$. If we apply Itô's formula to $V(\theta(t), \phi(t))$, where $(r(t), \theta(t), \phi(t))$ is a solution of the system (6.10) with $\theta(0) \ne \theta_0$, then we conclude (as in [4]) that

$$r(t) \to 1, \quad \theta(t) \to \theta_0 \qquad (t \to \infty). \tag{6.12}$$

Next we consider the restriction of (6.11) to $\theta = \theta_0$, i.e.

$$d\phi = \sum_{j=1}^{n} \tilde{\sigma}_{3j}(1, \theta_0, \phi)\,dw^j + \tilde{b}_3(1, \theta_0, \phi)\,dt$$

and assume

(T_5) Conditions analogous to (D), (E) and (Q) hold for (6.11) with respect to Γ.

Thus, through each point where $\tilde{\sigma}_{3j}(1, \theta_0, \phi) = 0$ $(1 \le j \le n)$ and $\tilde{b}_3(1, \theta_0, \phi) = 0$ there passes a "boundary spoke" Λ_j lying in the sphere. Λ_j connects a pair of adjacent α_j's. With the aid of the condition (T_5), we construct a function $f(\phi)$ with $Lf \le -\nu < 0$ as in §§1 and 2. Applying Itô's formula to $f(\phi(t))$ where $(r(t), \theta(t), \phi(t))$ is a solution of (6.10), we can then show that $\phi(t) \to \alpha_i^{\pm}$

$(1 \le i \le k)$ with probability $p_i^{\pm}(x)$ $(\Sigma_{i=1}^k [p_i^+(x) + p_i^-(x)] = 1)$.

The points $\zeta_i = (1, \theta_0, \alpha_i)$ are called *distinguished boundary points.* We can now pose the Dirichlet problem

$$(6.13) \qquad \begin{aligned} &Lu = 0 \quad \text{in } G, \\ &u(x) \to f_i^{\pm}, \quad x \to \zeta_i^{\pm} \quad (1 \le i \le k). \end{aligned}$$

Theorem 6.2. *Let the assumptions* $(T_1 - T_5)$ *hold and let* L *be nondegenerate in* $G - S$*. Then there exists a unique solution of the Dirichlet problem* (6.13). *It is given by the formula*

$$u(x) = \sum_{i=1}^k f_i^+ p_i^+(x) + \sum_{i=1}^k f_i^- p_i^-(x) \qquad (x \in S).$$

This theorem extends easily to general domains G with a $\Sigma_2 \cup \Sigma_3$ boundary component.

A subsequent treatment by one of us (M.P.) shows that condition (D) is superfluous. The details will appear in a forthcoming publication.

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