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UNCOMPLEMENTED $C(X)$-SUBALGEBRAS OF $C(X)$

BY

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ABSTRACT. In this paper, the uncomplemented subalgebras of the Banach algebra $C(X)$ which are isometrically and algebraically isomorphic to $C(X)$ are investigated. In particular, it is shown that if X is a 0-dimensional compact metric space with its ωth topological derivative $X^{(\omega)}$ nonempty, then there is an uncomplemented subalgebra of $C(X)$ isometrically and algebraically isomorphic to $C(X)$.

For each ordinal $\alpha \geq 1$, a class $\mathcal{C}_\alpha$ of homeomorphic 0-dimensional uncountable compact metric spaces is introduced. It is shown that each uncountable 0-dimensional compact metric space contains an open-and-closed subset which belongs to some $\mathcal{C}_\alpha$.

1. **Introduction.** Let X and Y be topological spaces and ϕ be a (continuous) map from X onto Y. The induced linear operator ϕ^0 is the multiplicative isometric isomorphism from $C(Y)$ into $C(X)$ that takes $f \in C(Y)$ into $f\phi$. A major result is

Theorem 4.6. *If X contains an open, 0-dimensional compact metric subspace K with its ωth topological derivative $K^{(\omega)}$ nonempty, then there is a map ϕ of X onto itself such that $\phi^0[C(X)]$ is uncomplemented in $C(X)$.*

Observe that the hypothesis for X is satisfied by all uncountable, 0-dimensional compact metric spaces (e.g., the Cantor set $\mathcal{C}$) and by the space $\Gamma(\alpha)$ of ordinals not exceeding α provided $\alpha \geq \omega^\omega$.

If ϕ is a map from X onto Y then an *averaging operator* for ϕ is a continuous linear operator μ from $C(X)$ into $C(Y)$ satisfying $\mu\phi^0(f) = f$ for $f \in C(Y)$. It is easy to see that ϕ admits an averaging operator for ϕ if and only if there is a projection P of $C(X)$ onto its subalgebra $\phi^0[C(Y)]$ where μ and P are related by $P = \phi^0\mu$ [21, Corollary 2.3]. As with most of the results of this paper, the conclusion to Theorem 4.6 can be stated in terms of averaging operators (i.e., there is a map of X onto itself which does not admit an averaging operator).

The $\mathcal{C}_\alpha$-spaces introduced in §4 are formed by adding rays $\Gamma_0(\alpha) = \{\beta \mid \beta$ is an ordinal, $\beta < \alpha\}$ to the Cantor set $\mathcal{C}$ so that each point in $\mathcal{C}_\alpha$ is the limit of a

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ray $\Gamma_0(\alpha)$. A homeomorphic classification of $\mathcal{C}_\alpha$-spaces similar to the classical homeomorphic classification of the Cantor set is established (Lemma 4.3). The fact that each nondispersed 0-dimensional compact metric space contains an open-and-closed $\mathcal{C}_\alpha$-space for some α provides a new technique for working with these spaces.

In the final section of this paper, two applications of the "uncomplemented $C(X)$-subalgebras of $C(X)$" results are included. Theorem 5.1 and Theorem 5.2 are "uncomplemented" analogoues of A. Pełczyński's "complemented $C(S)$-subspaces" theorems in his paper, *On* $C(S)$*-subspaces of separable Banach spaces* [22].

2. Preliminaries. The notation and terminology is that of Dunford and Schwartz's *Linear operators*. I [14] and Kelley's *General topology* [18] with the following exception: A *decomposition* D of a topological space X is a disjoint collection of *closed* subsets of X such that $X = \bigcup\{A: A \in D\}$ and the quotient space is denoted by X/D. An isomorphism μ between two Banach algebras which is multiplicative (i.e., $\mu(fg) = \mu(f)\mu(g)$) is called an *algebra* isomorphism.

For each subspace S of X let $S^{(1)}$ denote the set of all accumulation points of S which are contained in S. Then $S^{(1)}$ is the complement in S of the set of points which are isolated in the relative topology of S. If λ is an ordinal, the *topological derivative of order* λ of S, denoted $S^{(\lambda)}$, is defined by transfinite induction as follows: $S^{(0)} = S$, $S^{(\lambda)} = (S^{(\alpha)})^{(1)}$ if $\lambda = \alpha + 1$, and $S^{(\lambda)} = \bigcap_{\alpha<\lambda} S^{(\alpha)}$ if λ is a limit ordinal.

The first limit ordinal is denoted by ω and the first uncountable ordinal by Ω. We assume that all maps are continuous and that all topological spaces are Hausdorff.

3. Construction of uncomplemented subalgebras. This section is closely related to the author's work in [4]. Most of the terminology and notations used in that paper are needed in this section and are used without being redefined.

If D is a decomposition of a topological space X and q is the quotient map D, then q^0 is an isometric isomorphism from $C(X/D)$ onto the subalgebra of $C(X)$ consisting of the functions which are constant on each set in D. Frequently, this subalgebra of $C(X)$ is identified with $C(X/D)$ without specific reference to the isomorphism q^0. We follow Arens [3] and write $D = 0$ when D has no plural sets. The abbreviation u.s.c. is used for upper semicontinuous.

If Z is a D-saturated subset of X, then the restriction of the decomposition D to Z is denoted D_Z. If D is u.s.c., then D_Z is u.s.c. and the identity map of Z/D_Z onto $q(Z)$ is a homeomorphism (see [10, I, §5.2, Proposition 4, p. 54]). If Z is normal, then Z/D_Z is normal [15, p. 85] and $q(Z)$ is a normal subset of X/D.

Lemma 3.1 and Proposition 3.2 are basically the same as Lemma 1.2 and Theorem 1.3, respectively, in [4]. The main difference is that the assumption that X is normal is replaced by the assumption that a subspace Z of X is normal.

The proofs of these two results are omitted, since their proofs are similar to the corresponding proofs in [4] and the only additional information needed is contained in the preceding paragraph. The purpose of Lemma 3.1 is to replace Lemma 1.2 of [4] in the proof of Proposition 3.2.

Lemma 3.1. *Let X be a topological space and let D be an u.s.c. decomposition of X. Suppose there is a D-saturated, normal subspace Z of X and a plural set Y of D in* Int (Z) *such that D is contracting at Y and the boundary ∂Y of Y contains at least n points. If P is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geq 3 - 2/n$.*

Moreover, if $\epsilon > 0$, if U is a neighborhood of Y and if $y_1, y_2, \cdots, y_n$ are distinct points in ∂Y, then there exist an i and a neighborhood V of y_i such that for each t in $V \sim Y$ there exists f in $C(X)$ with $f|(X \sim U) = 0$, $\|f\| = f(t) = 1$, and $Pf(t) > 3 - 2/n - \epsilon$.

The following proposition establishes a lower bound for the norms of projections of $C(X)$ onto a $C(Y)$-subalgebra of $C(X)$. This proposition demonstrates that the existence of repeated limits of plural sets in the decomposition that Y induces on X can increase the norm of projections from $C(X)$ onto this $C(Y)$-subalgebra. This result substantially generalizes R. Arens's "$3 - 2/n$ lower bound theorem" [3, Theorem 3.1] and extends a similar result obtained independently by S. Ditor to noncompact spaces [12, Corollary 5.4]. However, Ditor's result is more general in the compact case. The definition of $L_n(m_1, m_2, \cdots, m_n)$ is given in [4].

Proposition 3.2. *Let D be an u.s.c. decomposition of a topological space X and Z a normal subspace of X such that* Int (Z) *is D-saturated. If $D_{\mathrm{Int}(Z)}$ has property $L_n(m_1, m_2, \cdots, m_n)$ and P is a projection of $C(X)$ onto $C(X/D)$, then*

$$\|P\| \geq 2n + 1 - \sum_{i=1}^{n} 2/m_i.$$

Remarks 3.3. The "upper semicontinuous" requirement in Theorem 1.3 in [4] was inadvertently omitted.

Let ϕ be a map of a compact space X onto a compact (Hausdorff) space Y and let Δ_ϕ be the decomposition $\{\phi^{-1}(y) | y \in Y\}$ of X. Then Δ_ϕ is u.s.c., since ϕ is closed. It is interesting to observe that if Δ_ϕ has property $L_n(m_1, m_2, \cdots, m_n)$, then using Ditor's definition in [12], $\Delta_\phi^{(n)}(m_1, m_2, \cdots, m_n) \neq \emptyset$. Therefore, either by Proposition 3.2 or by Corollary 5.4 in [12], an averaging operator U for ϕ has

$$\|U\| \geq 2n + 1 - \sum_{i=1}^{n} 2/m_i = 1 + 2\sum_{i=1}^{n} (1 - 1/m_i).$$

The next lemma is a more general form of the author's Lemma 2.4 in [4]. Both this lemma and the construction given in its proof are essential in the proofs of the theorems of this paper.

Lemma 3.3. (Construction of subspaces of $C(X)$ with high projection norm). *Let X be a topological space and n a positive integer. Suppose Z is a normal subspace of X and S is a subset of* Int (Z) *with $S^{(n)} \neq \emptyset$ such that each point in $S^{(1)}$ has a countable neighborhood base. Then for each positive integer $k > 1$ there exists an u.s.c. decomposition D of X such that*

(1) *Each plural set in D consists of k elements of S.*

(2) *X/D is Hausdorff. Moreover, X/D is, respectively, normal, compact, first-countable, or compact and metrizable, provided X has the corresponding property.*

(3) *If q is the quotient map from X onto X/D, then q^0 is an algebraic isometric isomorphism from $C(X/D)$ into $C(X)$.*

(4) *The decomposition $D^{(j)}$ of X contains a plural set if and only if $j < n$.*

(5) *If P is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geq 2n - 1 - (2n - 2)/k$.*

Proof. Let $x \in S^{(n)}$ and let G be a closed neighborhood of x included in Int (Z). By induction, we select nonempty families $C_1, C_2, \cdots, C_{n+1}$ and $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_{n+1}$ of subsets of Int (G) such that if $1 \leq m \leq n + 1$ (and $C_0 = \emptyset$) then

(a) C_1 consists of a singleton subset of S and each set in C_m for $m > 1$ consists of k elements from $S^{(n-m+1)}$.

(b) If $a \in A$ for some $A \in C_m$ and U is a neighborhood of a, then U includes a set in C_{m-1} [i.e., a is an accumulation point of sets in C_{m-1}].

(c) $\mathfrak{U}_m$ is a family of disjoint, closed subsets such that for each A in C_m, there is a neighborhood U_A of A in $\mathfrak{U}_m$ which does not include any other set in C_m.

(d) If $U \in \mathfrak{U}_m$ then U does not intersect any set in C_j for $1 \leq j < m$.

(e) $\mathfrak{U}_m$ implies $\mathfrak{U}_{m-1}$ for $m > 1$.

(f). The decomposition D_m of X consisting of the plural sets in $(\bigcup_{i=1}^{n-1} C_i) \cup \mathfrak{U}_m$ is contracting and each set in $\mathfrak{U}_m$ is a nonlimit set of D_m.

Let C_1 be the family consisting of the singleton set $\{x\}$ and let $\mathfrak{U}_1 = \{G\}$. It is easy to see that conditions (a)–(f) are satisfied for $m = 1$.

Next, suppose $C_1, C_2, \cdots, C_m$ and $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_m$ have been selected and $m \leq n$. Let A be a set in C_m. We may suppose $A = \{a_1, a_2, \cdots, a_z\}$ where each a_i is in $S^{(n-m+1)}$ where $z = 1$ if $m = 1$ and $z = k$ if $m > 1$. There exists a neighborhood U_A of A in $\mathfrak{U}_m$ which does not intersect any other set in C_m. There is a family $\{U_i\}_{i=1}^{z}$ of closed disjoint sets such that for each i, U_i is a neighborhood of a_i and is a subset of U_A. Let $\{V_{ij}\}_{j=1}^{\infty}$ be a closed monotone neighborhood base for a_i with $V_{ij} \subset U_i$ for each j. In fact, we can suppose $\{V_{ij}\}_{j=1}^{\infty}$ is selected so that, for each i and j, $V_{ij} \sim V_{i(j+1)}$ is a neighborhood of a point a_{ij} in $S^{(n-m)}$. Then there is a family $\{W_{ij}\}_{j=1}^{\infty}$ of disjoint closed sets such that W_{ij} is a neighborhood of a_{ij} included in $V_{ij} \sim V_{i(j+1)}$. Then, we define

$$A_{ij} = \{a_{i(jk+r)} | 1 \leq r \leq k\} \text{ for } 1 \leq i \leq z,$$

$$U(A_{ij}) = \bigcup_{r=1}^{k} W_{i(jk+r)} \text{ for } 1 \leq i \leq z \text{ and } j = 0, 1, 2, \cdots,$$

$$C_{m+1} = \{A_{ij} |\ A \in C_m, 1 \leq i \leq z, \text{ and } j = 0, 1, 2, \cdots\},$$

$$\mathfrak{U}_{m+1} = \{U(A_{ij}) |\ A \in C_m, 1 \leq i \leq z, \text{ and } j = 0, 1, 2, \cdots\}.$$

Using these definitions, it is easy to see that hypotheses (a)–(e) are satisfied by C_{m+1} and $\mathfrak{U}_{m+1}$. The proof that (f) is satisfied is given in [4, paragraphs 2 and 3, p. 96]. This completes the inductive selection of $C_1, C_2, \cdots, C_{n+1}$ and $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_{n+1}$.

Let D be the decomposition of X consisting of the plural sets in $\bigcup_{j=1}^{n+1} C_{.j}$. It follows from Lemma 2.2 in [4] that since D_{n+1} is contracting, D is also contracting. (Let $M = D_{n+1}$ in Lemma 2.2.) This selection of D is easily seen to satisfy conclusions (1) and (3). Let q denote the quotient map of D. Since D is u.s.c. and G is a neighborhood of each plural set in D, D_G (the restriction of D to G) is u.s.c. and the identity map of G/D_G onto $q(Z)$ is a homeomorphism [10, I, §5.2, Proposition 4, p. 53]. Since G is normal, G/D_G is normal and $q(G)$ is a normal subspace of X/D. A similar argument with G replaced by Z shows $q(Z)$ is a normal subspace of X/D. But $q(G) \subset q(\text{Int } Z) = \text{Int } q(Z)$, and q is a homeomorphism on $X \sim G$. Since $X \sim G$ is Hausdorff, it follows that X/D is Hausdorff. If X is normal or compact, then X/D has the corresponding property [15, pp. 85 and 104]. If X is both compact and metrizable, it follows by a theorem of K. Morita and S. Hanai [19, Theorem 1] and also by A. H. Stone [26, Theorem 1] that X/D is also compact and metrizable. If X is first countable, X/D is also first countable since each decomposition set contains at most k elements. Thus D satisfies conclusion (2) also.

The following generalization of (5) is established next:

(5′) If M is an u.s.c. decomposition of X such that M contains each plural set of D and M is contracting at each plural set in D, then each projection P of $C(X)$ onto $C(X/M)$ has $\|P\| \geq 2n - 1 - (2n - 2)/k$.

To establish (5′), we let $S_i = D^{(i)} \sim D^{(i+1)}$ for $i = 1, 2, \cdots, n - 1$. Since each set in S_i is a plural set in D and M is contracting at each of these sets, M satisfies property $L_{n-1}(k, k, \cdots, k)$. By Proposition 3.2, each projection

P of $C(X)$ onto $C(X/M)$ has $\|P\| \geq 2n - 1 - (2n - 2)/k$. This proves (5′). Letting $D = M$, we obtain (5).

To establish (4), observe that it follows by induction that the decomposition $D^{(m)}$ satisfies the following properties provided $0 \leq m \leq n$:

(i) $\bigcup_{j=2}^{n-m+1} C_j$ is the family of plural sets in $D^{(m)}$.

(ii) The family of plural sets in C_{n-m+1} is the set of nonlimit plural sets in $D^{(m)}$.

By (i), $D^{(m)}$ contains a plural set if and only if $n - m + 1 \geq 2$ or $m \leq n - 1$. This proves (4).

In case the collection $S^{(n)}$ in Lemma 3.3 contains an isolated point with respect to its subset topology, the decomposition D of Lemma 3.3 can be selected so that two additional properties are satisfied.

Lemma 3.4. *If* $S^{(n)} \sim S^{(n+1)}$ *is nonempty in Lemma* 3.3, *then the decomposition* D *can be also selected so that*

(1′) *Each plural set of* $D^{(j)} \sim D^{(j+1)}$ *consists of* k *elements of* $S^{(j)} \sim S^{(j+1)}$.

(6) *For each ordinal number* α, $t \in S^{(\alpha)}$ *if and only if* $q(t) \in q(S)^{(\alpha)}$.

Proof. If $S^{(n)} \sim S^{(n+1)}$ is nonempty, then "$S^{(t)}$" can be replaced with "$S^{(t)} \sim S^{(t+1)}$" for each t in the proof of Lemma 1. [In this case, let $x \in (S^{(n)} \sim S^{(n+1)})$.] By the revised form of inductive hypothesis (a), it follows that each plural set in C_{n-m+1} consists of k-points from $S^{(m)} \sim S^{(m+1)}$. This establishes (1′).

Next, we establish (6) by transfinite induction. It is obvious if $\alpha = 0$. Suppose (6) is valid for all $\alpha < \gamma$ where $1 < \gamma$. Let $x \in S^{(\gamma)}$. Then for each $\alpha < \gamma$, there exists a sequence $\{x_n\}$ of distinct points in $S^{(\alpha)}$ such that $x_n \to x$. By inductive hypothesis, $\{q(x_n)\} \subset q(S)^{(\alpha)}$ and since $q(x_n) \to q(x)$, $q(x) \in q(S)^{(\gamma)}$.

Conversely, let $q(x) \in q(S)^{(\gamma)}$. Then, for each $\alpha < \gamma$, there exists a sequence $\{y_n\}$ of distinct points in $q(S)^{(\alpha)}$ such that $y_n \to q(x)$. Choose $x_n \in S$ with $q(x_n) = y_n$. By inductive hypothesis, $x_n \in S^{(\alpha)}$. If $q(x)$ is a singleton set, then $x_n \to x$ and it follows that $x \in S^{(\gamma)}$. If x is a plural set, then $\gamma < \omega$ by (1′) and $x = \{z_1, z_2, \cdots, z_k\}$ for some choice of z_i in S. Let $U_1, U_2, \cdots, U_k$ be disjoint neighborhoods of $z_1, z_2, \cdots, z_k$, respectively. Since D is contracting, we may assume each U_i is $(D \sim \{q(x)\})$-saturated. As before, there is a sequence $\{x_n\}$ of distinct points in $S^{(\gamma-1)}$ such that $q(x_n) \to q(x)$. But an infinite subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is included in some U_j. Then $x_{n_i} \to z_j$, so $z_j \in S^{(\gamma)}$. But by (1′), $z_j \in S^{(\gamma)}$ implies $\{z_1, z_2, \cdots, z_k\} \subset S^{(\gamma)}$. Since $x \in q(x)$, $x \in S^{(\gamma)}$.

Lemmas 3.5 and 3.6 are, respectively, the uncomplemented analogues of Lemmas 3.3 and 3.4.

Lemma 3.5. (Construction of uncomplemented subspaces of $C(X)$). *Suppose Z is a normal subspace of a topological space X and S is a subset of* Int (Z) *with $S^{(\omega)} \neq \emptyset$ such that each point in $S^{(1)}$ has a countable neighborhood base. Then for each positive integer $k > 1$ there exists an u.s.c. decomposition such that conclusions* (1)–(3) *of Lemma* 3.3 *are satisfied. Moreover,*

(4) *The decomposition $D^{(j)}$ of X contains a plural set if and only if $j < \omega$.*

(5) *The subspace $C(X/D)$ of $C(X)$ is uncomplemented in $C(X)$.*

Proof. Suppose $x \in S^{(\omega)}$. Let $\{O_n\}_{n=1}^{\infty}$ be an open monotone neighborhood base for x with $O_1 \subset \text{Int}\,(Z)$. We may assume this neighborhood base is selected so that $O_n \sim O_{n+1}$ is a neighborhood of a point $x_n \in S^{(n)}$. Let E_n be a closed neighborhood of x_n included in $O_n \sim O_{n+1}$. If R is the decomposition of X consisting of the plural sets $\{E_n\}_{n=1}^{\infty}$, then R is clearly contracting. Let $S_n = S \cap \text{Int}\,(E_n)$. Since $x_n \in S_n$, it follows by (5′) of Lemma 3.3 that there is a contracting decomposition M_n of X with each plural set in M_n a subset of S_n such that if M is a contracting decomposition containing each plural set in M_n and P is a projection of $C(X)$ onto $C(X/M)$, then $\|P\| \geq n$. Moreover, each M_n can be selected so that each plural set contains exactly k points of S_n. Let D be the decomposition consisting of the plural sets in M_n. By Lemma 2.2 in [4], D is contracting. Thus, there does not exist a projection P of $C(X)$ onto $C(X/D)$, since $\|P\| \geq n$ for each positive integer n is impossible. This proves (5). Parts (1) and (3) are trivial, and the proof of (2) is the same as in Lemma 3.3. Part (4) follows, since the decomposition M_n has the property that $M_n^{(j)}$ contains a plural set if and only if $j < n$.

Lemma 3.6. *If $S^{(\omega)} \sim S^{(\omega+1)}$ is nonempty in Lemma* 3.5, *then the decomposition D can also be selected so that*

(1′) *Each plural set of $D^{(j)} \sim D^{(j+1)}$ consists of k elements of $S^{(j)} \sim S^{(j+1)}$.*

(6) *For each ordinal number α, $t \in S^{(\alpha)}$ if and only if $q(t) \in q(S)^{(\alpha)}$.*

Proof. If $S^{(\omega)} \sim S^{(\omega+1)}$ is nonempty, then the point x in the proof of Lemma 3.5 can be selected from $S^{(\omega)} \sim S^{(\omega+1)}$. Also, the points x_n in that proof can be selected from $S^{(n)} \sim S^{(n+1)}$. Then by Lemma 3.4, each of the decompositions M_n in the proof of Lemma 3.5 can be selected so that M_n satisfies (1′). Since each plural set of D is contained in some M_n, (1′) is established.

The proof of (6) is identical to the proof of (6) in Lemma 3.4.

4. **Existence of uncomplemented $C(X)$-subalgebras of $C(X)$.** In this section, we apply the lemmas of the last section to certain topological spaces X to construct a subalgebra of $C(X)$ isometrically and algebraically isomorphic to $C(X)$ that either has a large lower bound for the norms of projections from $C(X)$ or is uncomplemented in $C(X)$.

A topological space X is *dispersed* (*scattered*) if it does not contain a perfect subset. If X is dispersed, there is a least ordinal α such that $X^{(\alpha)}$ is either finite or empty. The *characteristic* (characteristic system) of X is the ordered pair (α, n) where n is the cardinality of X. Note that $n \geq 1$ if X is compact. If ξ is an ordinal, $\Gamma(\xi)$ denotes the space of ordinals not exceeding ξ with the interval topology and $\Gamma_0(\xi)$ denotes the subspace of $\Gamma(\xi)$ consisting of the ordinals strictly less that ξ.

Theorem 4.1. *Let n be a positive integer. Suppose a topological space X includes a compact first-countable set K such that* Int $(K)^{(n)}$ *contains infinitely many isolated points. Then for each $\epsilon > 0$, there is a map ϕ of X onto itself such that if P is a projection of $C(X)$ onto $\phi^0[C(X)]$, then $\|P\| \geq 2n + 1 - \epsilon$.*

Proof. Let x be an accumulation point of the set of isolated points in Int $(K)^{(n)}$. Suppose $\{U_j\}_{j=1}^{\infty}$ is a neighborhood base for x. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of distinct isolated points of Int $(K)^{(n)}$ with x_j in U_j. By induction, there is a sequence $\{V_j\}_{j=1}^{\infty}$ of disjoint closed sets such that V_j is a neighborhood of x_j included in Int $(K) \cap U_j$. Since $X^{(n+1)}$ is closed and $X^{(n)} \sim X^{(n+1)}$ is discrete in its subset topology, we may assume $V_j \cap X^{(n)} = \{x_j\}$. Each V_i is dispersed and compact; hence, it is 0-dimensional [20]. Thus, there exists an open-and-closed (in V_j) neighborhood W_j of x_j included in Int (V_j). Since $W_j \subset$ Int (V_j), W_j is open-and-closed in X. Let $W = \bigcup_{i=1}^{\infty} W_i$ and $S = W \cup \{x\}$. The set S is first-countable, compact and has characteristic $(n + 1, 1)$, so it follows by a theorem due to Z. Semadeni [24] (see [5, Corollary 2]) that S is homeomorphic to $\Gamma(\omega^{n+1})$.

Let $\epsilon > 0$ and choose a positive integer k sufficiently large so that $(2n)/k < \epsilon$. Since $S^{(n+1)} \neq \emptyset$, it follows by Lemma 3.3 that there is an u.s.c. decomposition D of X such that each projection P of $C(X)$ onto $C(X/D)$ has $\|P\| \geq 2n + 1 - (2n)/k > 2n + 1 - \epsilon$. Since $S^{(n+1)} \sim S^{(n+2)} = \{x\}$, it follows by Lemma 3.4 that D can also be selected so that if q denotes the quotient map, then $x \in S^{(\alpha)}$ if and only if $q(x) \in q(S)^{(\alpha)}$ for each ordinal α. Thus, $q(S)$ also has characteristic $(n + 1, 1)$. The subset $q(S)$ is compact because S is compact. But $q(S)$ is first-countable as each plural set of D is finite, so, by [5, Corollary 2], $q(S)$ is also homeomorphic to $\Gamma(\omega^{n+1})$.

Let μ be a homeomorphism of S onto $q(S)$. As topological derivatives are preserved by homeomorphisms [23, Lemma 3.1 (e)], $\{\mu(x)\} = \mu(S^{(n+1)}) = \mu(S)^{(n+1)} = q(S)^{(n+1)} = \{q(x)\}$ and $\mu(x) = q(x)$. Since x is the only possible accumulation point of $X \sim S$ contained in S, we can extend μ to a map of X into $q(X)$ by letting $\mu(x) = q(x)$ for x in $X \sim S$. Then μ is one-to-one and onto X/D. Since $q(x)$ is a singleton set and q^{-1} is continuous on $q(X) \sim q(S)$, μ is a homeomorphism of X onto X/D.

Thus, $\phi = \mu^{-1} \circ q$ is a map of X onto X and $\phi^0[C(X)] = q^0[C(X/D)]$. Consequently, each projection P of $C(X)$ onto $\phi^0[C(X)]$ has $\|P\| \geq 2n + 1 - \epsilon$.

Our first "uncomplemented $C(X)$-algebra of $C(X)$" result is contained in the next theorem. This theorem is essentially the uncomplemented version of Theorem 4.1.

Theorem 4.2. *Suppose a topological space X includes a first-countable compact subset K such that* Int $(K)^{(n)}$ *contains an isolated point for each positive integer n. Then there is a map ϕ of X onto itself such that $\phi^0[C(X)]$ is an uncomplemented subalgebra of $C(X)$.*

Proof. Let t_n be an isolated point in Int $(K)^{(n)}$ for each positive integer n. Let x be an accumulation point of $\{t_n\}_{n=1}^{\infty}$. Suppose $\{U_n\}$ is a neighborhood base for x. For each n, let x_n be an isolated point of Int $(K)^{(n)}$ contained in U_n. By induction, there exists a sequence $\{V_n\}_{n=1}^{\infty}$ of disjoint sets such that V_n is a neighborhood of x_n included in Int $(K) \cap U_n$. Since $X^{(n+1)}$ is closed in its subset topology, we may assume $V_n \cap X^{(n)} = \{x_n\}$ for each n.

The remainder of the proof follows the proof of Theorem 4.1, except that $n + 1$ is replaced with ω and Lemmas 3.5 and 3.6 are used in place of Lemmas 3.3 and 3.4 respectively. □

Next, for each denumerable ordinal number α, we construct a compact subspace $\mathcal{C}_\alpha$ of the unit interval satisfying the two following properties: (1) $(\mathcal{C}_\alpha)^{(\alpha)} = \mathcal{C}$, and (2) each point in the subset $\mathcal{C}$ of $\mathcal{C}_\alpha$ is the limit of a well-ordered sequence in $\mathcal{C}_\alpha \sim \mathcal{C}$ homeomorphic to $\Gamma_0(\omega^\alpha)$. Let $I_{n,k}$ denote the kth open interval removed (counting from left to right) in the nth step of the construction of the Cantor set $\mathcal{C}$ [17, p. 70]. Let $a_{n,k}$ and $b_{n,k}$ denote the left and right endpoint, respectively, of $I_{n,k}$ and let $m_{n,k}$ be the midpoint of $I_{n,k}$. In each interval $[a_{n,k}, m_{n,k})$ select a well-ordered sequence $A_{n,k} = \{a_\mu\}_{\mu<\omega^\alpha}$ homeomorphic to $\Gamma_0(\omega^\alpha)$ with $\lim_{\mu<\omega^\alpha} a_\mu = a_{n,k}$. Similarly, select a well-ordered sequence $B_{n,k} = \{b_\mu\}_{\mu<\omega^\alpha}$ in $(m_{n,k}, b_{n,k}]$ homeomorphic to $\Gamma_0(\omega^\alpha)$ with $\lim_{\mu<\omega^\alpha} b_\mu = b_{n,k}$. Then let $\mathcal{C}_\alpha$ be the subspace of $[0, 1]$ defined by

$$\mathcal{C}_\alpha = \mathcal{C} \cup \left[\bigcup_{n,k=1}^{\infty} (A_{n,k} \cup B_{n,k}) \right].$$

Clearly, the choice of each $A_{n,k}$ and $B_{n,k}$ is possible but not unique. The fact that $\mathcal{C}_\alpha$ is independent up to homeomorphism of the choices of $A_{n,k}$ and $B_{n,k}$ is established in Proposition 4.4. It follows from Lemma 1 in [5] and the construction of $\mathcal{C}_\alpha$ that $\mathcal{C}_\alpha$ satisfies properties (1) and (2) of the preceding paragraph. Compact metric spaces which satisfy (1) and (2) will be called $\mathcal{C}_\alpha$-spaces. More specifically,

Definition. Let α be an ordinal number. A topological space X is called a $\mathcal{C}_\alpha$-*space* if and only if X is an uncountable 0-dimensional compact metric space, $X^{(\alpha)}$ is perfect, and each point in $X^{(\alpha)}$ is the limit of a well-ordered sequence in $X \sim X^{(\alpha)}$ homeomorphic to $\Gamma_0(\omega^\alpha)$.

If X is a topological space, there is a least ordinal λ such that $X^{(\lambda)} = X^{(\lambda+1)}$. Let $\operatorname{Ker}(X) = X^{(\lambda)}$ and observe that $\operatorname{Ker}(X)$ is the largest perfect subset of X.

An alternate characterization of $\mathcal{C}_\alpha$-spaces is given by the following lemma.

Lemma 4.3. *Let X be a compact 0-dimensional metric space. If α is an ordinal number, then X is a $\mathcal{C}_\alpha$-space if and only if*

$$\operatorname{Ker}(X) \subset \operatorname{Cl}[X^{(\gamma)} \sim \operatorname{Ker}(X)]$$

for each $\gamma < \alpha$, but $X^{(\alpha)} = \operatorname{Ker}(X) \neq \varnothing$.

Proof. Suppose X is a $\mathcal{C}_\alpha$-space. Since $X^{(\alpha)}$ is perfect, $X^{(\alpha)} = \operatorname{Ker}(X)$. By Lemma 1 in [5] it follows that $\operatorname{Ker}(X) \subset \operatorname{Cl}[X^{(\gamma)} \sim \operatorname{Ker}(X)]$ for each $\gamma < \alpha$.

Conversely, suppose $\operatorname{Ker}(X) \subset \operatorname{Cl}[X^{(\gamma)} \sim \operatorname{Ker}(X)]$ for each $\gamma < \alpha$, but $X^{(\alpha)} = \operatorname{Ker}(X) \neq \varnothing$. Thus, $X^{(\alpha)}$ is perfect. The proof that each point in $X^{(\alpha)}$ is the limit of a well-ordered sequence of points in $X \sim X^{(\alpha)}$ homeomorphic to $\Gamma_0(\omega^\alpha)$ is similar to the proof of Theorem 2 in [7] and is omitted.

Each uncountable, 0-dimensional, perfect, compact metric space is homeomorphic to $\mathcal{C}$. The following proposition establishes a similar homeomorphic characterization for each $\mathcal{C}_\alpha$.

Proposition 4.4. *All $\mathcal{C}_\alpha$-spaces are homeomorphic.*

Proof. Suppose X and Y are $\mathcal{C}_\alpha$-spaces. Since $\operatorname{Ker}(X) = X^{(\alpha)}$ and $\operatorname{Ker}(Y) = Y^{(\alpha)}$ are nonempty, 0-dimensional, perfect metric spaces, they are both homeomorphic to the Cantor set. Thus, there is a homeomorphism ϕ of $\operatorname{Ker}(X)$ onto $\operatorname{Ker}(Y)$. Let $G_1 = X \sim \operatorname{Ker}(X)$ and $G_2 = Y \sim \operatorname{Ker}(Y)$. Since $G_1^{(\gamma)}$ and $G_2^{(\gamma)}$ are infinite for each $\gamma < \alpha$ and $G_1^{(\alpha)} = G_2^{(\alpha)} = \varnothing$, it follows by Theorem 3 in [5] that both G_1 and G_2 are homeomorphic to $\Gamma_0(\omega^\alpha)$. By Lemma 4.3, $\phi[\operatorname{Ker}(X) \cap \operatorname{Cl}[X^{(\gamma)} \sim \operatorname{Ker}(X)]] = \phi(\operatorname{Ker} X) = \operatorname{Ker}(Y) = \operatorname{Ker}(Y) \cap \operatorname{Cl}[Y^{(\gamma)} \sim \operatorname{Ker}(Y)]$ for each $\gamma < \alpha$. Since $\operatorname{Cl}[X^{(\gamma)} \sim \operatorname{Ker}(X)] = \operatorname{Cl}[Y^{(\gamma)} \sim \operatorname{Ker}(X)] = \varnothing$ for all $\gamma \geq \alpha$, it follows that $\phi(\operatorname{Ker}(X) \cap \operatorname{Cl}[X^{(\gamma)} \sim \operatorname{Ker}(X)]) = \operatorname{Ker}(Y) \cap \operatorname{Cl}[Y^{(\gamma)} \sim \operatorname{Ker}(Y)]$ for each ordinal number γ. By Theorem 1.1 in [23], ϕ can be extended to a homeomorphism of X onto Y.

It is well known that a nondispersed compact metric space X contains a subset K homeomorphic to the Cantor set. However, consideration of the case $X = \mathcal{C}_1$ indicates that even when X is 0-dimensional, it is sometimes impossible to select K to be open in X. The next lemma establishes that if X is 0-dimensional, then one can find a compact open subset K of X homeomorphic to $\mathcal{C}_\alpha$ for some α.

Lemma 4.5. *Let X be an uncountable 0-dimensional compact metric space. Then X contains an open-and-closed subset homeomorphic to $\mathcal{C}_\alpha$ for some countable ordinal number α.*

Proof. Let α be the least ordinal such that there exists an open-and-closed, nondispersed subset Y of X with $Y^{(\alpha)}$ perfect [25, Theorem 4.7]. Let Y be such a subspace of X. Then for each y in Ker (Y) and each open-and-closed neighborhood U of y included in Y, it follows by the minimality of α that $U^{(\gamma)}$ is perfect if and only if $\gamma \geq \alpha$. Thus, $[Y^{(\gamma)} \sim \text{Ker}\,(Y)] \cap U$ is nonempty and $y \in \text{Cl}[Y^{(\gamma)} \sim \text{Ker}\,(Y)]$ for each $\gamma < \alpha$. Since this is true for each $y \in \text{Ker}\,(Y)$,

$$\text{Ker}(Y) \subset \text{Cl}\,[Y^{(\gamma)} \sim \text{Ker}(Y)]$$

for each $\gamma < \alpha$. Therefore, by Lemma 4.3, Y is a $\mathcal{C}_\alpha$-space.

The second "uncomplemented $C(X)$-subalgebra of $C(X)$" result is stated in the next theorem. In contrast to Theorem 4.2, this theorem is applicable to 0-dimensional compact metric spaces with a finite nonempty perfect derivative such as $\mathcal{C}$, the free union $\mathcal{C}_1 + \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ [13, p. 127], and $\Gamma(\Omega) \times \mathcal{C}$. However, Theorem 4.6 does not include Theorem 4.2, since it is easy to construct spaces which satisfy the hypothesis of Theorem 4.2 but not of Theorem 4.6 (e.g., the subset $(\mathcal{C}_{\omega\omega} \times \{0\}) \cup (\mathcal{C} \times [0, 1])$ of the unit square).

Theorem 4.6. *If a topological space X contains an open, 0-dimensional compact metric subspace K with $K^{(\omega)} \neq \emptyset$, then there is a map ϕ of X onto itself such that $\phi^0[C(X)]$ is uncomplemented in $C(X)$.*

Proof. If $K^{(n)}$ contains an isolated point for each positive integer n, this result follows by Theorem 4.2. Therefore, we may suppose $K^{(n)}$ is nonempty and perfect for some n By Lemma 4.5, there is an open-and-closed subset Y of K homeomorphic to $\mathcal{C}_t$ for some integer t. Let $S = \text{Ker}\,(Y)$. By Lemma 3.5 there is a decomposition D of X with each plural set of D a subset of S such that $C(X/D)$ is uncomplemented in $C(X)$.

Let q be the quotient map of D. We show $q(Y)$ is 0-dimensional by examining the construction of D in the proofs of Lemma 3.3 and Lemma 3.5. The notation used in these constructions will be preserved. Let $y \in Y$ and suppose V is a neighborhood of $q(y)$ in $q(Y)$. If $y = x$ where x is the point selected in the proof of Lemma 3.5, then there exists n such that O_n is included in the neighborhood $q^{-1}(V)$ of x. Then $q(O_n)$ is an open-and-closed neighborhood of $q(y)$ included in V.

Next, suppose y belongs to the complement of the closed set $F = (\bigcup_{n=1}^{\infty} E_n) \cup \{x\}$ where E_n and x are defined in the proof of Lemma 3.5. Then there is an open-and-closed neighborhood W of y contained in $q^{-1}(V)$ which does not intersect F. In this case, $q(W)$ is an open-and-closed subset of V in $q(Y)$ containing $q(y)$.

Finally, we suppose y belongs to E_n for some n and restrict our attention to the selection of M_n made in the proof of Lemma 3.3. This M_n was constructed so that each plural set of M_n would be a subset of $S_n = S \cap \operatorname{Int}(E_n)$. Suppose $q(y)$ belongs to one of the disjoint families C_m selected in the proof of Lemma 3.3. Then $q(y) = \{a_1, a_2, \cdots, a_z\}$ where $z = 1$ if $m = 1$ and $z = k$ if $m > 1$. We may assume that each set in the monotone neighborhood basis $\{V_{ij}\}$ of each a_i selected in that proof is open-and-closed. Recall the sets W_{ij} were selected so that $W_{ij} \subset V_{ij} \sim V_{i(j+1)}$. Since $q^{-1}(V)$ is a neighborhood of $q(y)$, there is a positive integer p such that $V_{ip} \subset q^{-1}(V)$ for $i = 1, 2, \cdots, z$. If $W = \bigcup_{i=1}^{z} V_{i(pk+1)}$, it is obvious from the construction of M_n in Lemma 3.3 that $q(W)$ is an open-and-closed neighborhood of $q(y)$ contained in V.

On the other hand, suppose $q(y)$ is not a set in any C_i. Then, by the construction of M_n, $q(y)$ is a nonlimit singleton set in D and there is an open-and-closed neighborhood W of y included in $q^{-1}(V)$ which does not intersect any plural set of D. In this case, $q(W)$ is an open-and-closed neighborhood of $q(y)$ included in V. This completes the proof that $q(Y)$ is 0-dimensional.

Since Y is compact and q is continuous, $q(Y)$ is compact. By a theorem of K. Morita and S. Hanai [19, Theorem 1] and also of A. H. Stone [26, Theorem 1], $q(Y)$ is metrizable. To establish $q(Y)$ is a $\mathcal{C}_t$-space, it remains to be shown that $q(Y)^{(t)}$ is a nonempty perfect set and each point in $q(Y)^{(t)}$ is the limit of a well-ordered sequence of points in $q(Y) \sim q(Y)^{(t)}$ homeomorphic to $\Gamma_0(\omega^t)$. Recall that each plural set in D is a subset of Ker (Y). Consequently,

$$\operatorname{Ker} q(Y) = q(\operatorname{Ker} Y) \tag{1}$$

and

$$q(Y) \sim \operatorname{Ker} q(Y) = q(Y \sim \operatorname{Ker} Y). \tag{2}$$

Since $\operatorname{Ker} q(Y)$ is perfect and the restriction of q to $Y \sim \operatorname{Ker}(Y)$ is a homeomorphism, it follows from the fact that topological derivatives are preserved by homeomorphisms [23, Lemma 2.1 (e)] and equalities (1) and (2) above that

$$q(Y)^{(t)} = [q(Y) \sim \operatorname{Ker} q(Y)]^{(t)} \cup [\operatorname{Ker} q(Y)]$$

$$= q([Y \sim \operatorname{Ker} q(Y)]^{(t)}) \cup [\operatorname{Ker} q(Y)] = \operatorname{Ker} q(Y).$$

Thus, $q(Y)^{(t)}$ is perfect.

Next, suppose z is in $q(Y)^{(t)}$. By equality (1) above, there exists $y \in \operatorname{Ker}(Y)$ with $q(y) = z$. Let $\{x_\mu\}_{\mu<\omega^t}$ be a well-ordered sequence in $Y \sim \operatorname{Ker}(Y)$ homeomorphic to $\Gamma_0(\omega^t)$ which converges to z. Since the restriction of q to $Y \sim \operatorname{Ker}(Y)$ is a homeomorphism, it follows from line (2) that $\{q(y_\mu)\}_{\mu<\omega^t}$ is a well-ordered sequence in $q(Y) \sim \operatorname{Ker} q(Y)$ homeomorphic to $\Gamma_0(\omega^t)$ which converges to z. This completes the proof that $q(Y)$ is homeomorphic to $q(Y)$ (see Proposition 4.4). Since Y is an open-and-closed set and q is a homeomorphism of $X \sim Y$ onto $q(X) \sim q(Y)$, X is homeomorphic to $q(X)$.

Corollary 4.7. *If X is a 0-dimensional compact metric space with $X^{(\omega)} \neq \emptyset$, then there exists a map ϕ onto itself such that $\phi^0[C(X)]$ is an uncomplemented subalgebra of $C(X)$.*

5. **Applications of uncomplemented $C(X)$-subspaces of $C(X)$.** The next two theorems are the uncomplemented analogues to Theorem 1 and Theorem 1a, respectively, in [22] as they essentially replace "complemented" in the two theorems by A. Pełczyński with "uncomplemented".

Theorem 5.1. *Let S be a compact metric space with $S^{(\omega)} \neq \emptyset$. If a Banach space X contains a subspace Y isomorphic to $C(S)$, then there is a subspace Z of Y such that Z is isomorphic to $C(S)$ and Z is not complemented in X.*

Proof. Let μ be an isomorphism of $C(S)$ onto Y. First, suppose S is countable. Then S is dispersed, and, by Theorem 4.2, there is a map ϕ of X onto itself such that $\phi^0[C(S)]$ is uncomplemented in $C(S)$. In this case, $\mu\phi^0[C(S)]$ is a subset of Y isomorphic to $C(S)$ which is not complemented in X.

Next suppose S is uncountable. By Milutin's Theorem (see [21, Theorem 8.5] or [11]) there is an isomorphism ν of $C(\mathcal{C})$ onto $C(S)$. By Corollary 4.7 there is a map ϕ of $\mathcal{C}$ onto itself such that $\phi^0[C(\mathcal{C})]$ is uncomplemented in $C(\mathcal{C})$. Then $\mu\nu\phi^0[C(\mathcal{C})]$ is an uncomplemented subspace of X included in Y.

Theorem 5.2. *Let S be a 0-dimensional compact metric space with $S^{(\omega)} \neq \emptyset$. If a Banach space X contains a subspace Y (algebraically) isometrically isomorphic to $C(S)$, then there is a subspace Z of Y such that Z is (algebraically) isometrically isomorphic to $C(S)$ and Z is not complemented in $C(S)$.*

Proof. Let μ be an (algebraic) isometric isomorphism of $C(S)$ onto Y. By Corollary 4.7 there is a map ϕ of S onto itself such that $\phi^0[C(S)]$ is uncomplemented in $C(Y)$. Then $\mu\phi^0[C(S)]$ is an uncomplemented subspace of $C(X)$ contained in Y which is (algebraically) isometrically isomorphic to $C(Y)$.

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