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CONVERGENCE OF SEQUENCES OF SEMIGROUPS OF NONLINEAR OPERATORS WITH AN APPLICATION TO GAS KINETICS[1]

BY

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ABSTRACT. Let $A_1, A_2, \cdots$ be dissipative sets that generate semigroups of nonlinear contractions $T_1(t), T_2(t) \cdots$. Conditions are given on $\{A_n\}$ which imply the existence of a limiting semigroup $T(t)$. The results include types of convergence besides strong convergence.

As an application, it is shown that solutions of the pair of equations

$$u_t = -\alpha u_x + \alpha^2(v^2 - u^2)$$

and

$$v_t = \alpha v_x + \alpha^2(u^2 - v^2),$$

α a constant, approximate the solutions of

$$u_t = \tfrac{1}{4}(d^2/dx^2) \log u$$

as α goes to infinity.

1. **Introduction.** A general theorem concerning the convergence of sequences of semigroups of linear operators was given in [5]. The basis of the proof was the following corollary to the Hille-Yosida theorem.

Proposition (1.1). *Let $T(t)$ be a strongly continuous semigroup of linear operators on a Banach space L with infinitesimal operator A. Let M be a closed subspace of L. If $(\lambda - A)^{-1}: M \longrightarrow M$ for all λ sufficiently large, then $T(t): M \longrightarrow M$.*

Crandall and Liggett [3] have developed a theory for semigroups of nonlinear operators generated by accretive sets that implies essentially the same result. Consequently, many of the results in [5] can now be carried over to nonlinear semigroups of this type.

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In §2, we state the relevant results of Crandall and Liggett and develop the necessary background for the convergence theorem in §3. In §4 we consider the corresponding discrete parameter convergence theorem. In §5, we consider an example arising in gas kinetics and obtain a limit theorem analogous to a result of Pinsky [11] for a related linear problem.

2. **Background.** We state the results of Crandall and Liggett [3] in terms of dissipative rather than accretive sets so they will look more like the corresponding results in the linear case.

Let L be a Banach space. If $A \subset L \times L$ define the domain of A by

$$\mathfrak{D}(A) = \{x: (x, y) \in A \text{ for some } y\};$$

the range of A by

$$\mathfrak{R}(A) = \{y: (x, y) \in A \text{ for some } x\};$$

the inverse of A by

$$A^{-1} = \{(y, x): (x, y) \in A\};$$

and scalar multiplication by

$$\alpha A = \{(x, \alpha y) : (x, y) \in A\}.$$

If $A \subset L \times L$ and $B \subset L \times L$ then define

$$A + B = \{(x, y_1 + y_2) : (x, y_1) \in A (x, y_2) \in B\}.$$

Note that $\mathfrak{D}(A + B) = \mathfrak{D}(A) \cap \mathfrak{D}(B)$. For fixed x, Ax denotes $\{y: (x, y) \in A\}$.

If $A_1, A_2 \cdots$ is a sequence of subsets of $L \times L$ then

$$\lim_{n\to\infty} A_n \equiv \bigcap_{\epsilon} \bigcup_{m} \bigcap_{n>m} A_{n,\epsilon}$$

where

$$A_{n,\epsilon} \equiv \{(\hat{x}, \hat{y}) : \exists (x, y) \in A_n \|x - \hat{x}\|, \|y - \hat{y}\| \leq \epsilon\}.$$

This definition of the limit of a sequence of sets is equivalent to saying $(x, y) \in \lim_{n\to\infty} A_n$ if and only if there are $(x_n, y_n) \in A_n$ such that $\lim_{n\to\infty}(\|x - x_n\| + \|y - y_n\|) = 0$. A set $A \subset L \times L$ is called dissipative if $(x_1, y_1), (x_2, y_2) \in A$ implies

$$\|(x_1 - \alpha y_1) - (x_2 - \alpha y_2)\| \geq \|x_1 - x_2\| \tag{2.1}$$

for all $\alpha > 0$.

It follows from (2.1) that $I - \alpha A$ has a single valued inverse satisfying

$$\|(I - \alpha A)^{-1} x - (I - \alpha A)^{-1} y\| \leq \|x - y\| \tag{2.2}$$

for $x, y \in \mathfrak{R}(I - \alpha A)$, $\alpha > 0$.

Theorem (2.3) (Crandall and Liggett). *Let $A \subset L \times L$ and w be a real number such that $A - wI$ is dissipative. If $\mathfrak{R}(I - \alpha A) \supset \overline{\mathfrak{D}(A)}$ for all sufficiently small positive α, then for all $x \in \overline{\mathfrak{D}(A)}$*

$$S(t)x \equiv \lim_{m\to\infty} \left(I - \frac{t}{m}A\right)^{-m} x \tag{2.4}$$

exists; $S(t)x$ is a strongly continuous function of t; $S(t+s)x = S(t)S(s)x$; and

$$\|S(t)x - S(t)y\| \le e^{wt}\|x - y\|$$

all $x, y \in \overline{\mathfrak{D}(A)}$.

Corollary (2.5). *If C is a closed subset of $\overline{\mathfrak{D}(A)}$ and $(I - \alpha A)^{-1}: C \to C$ for all sufficiently small positive α, then*

$$S(t): C \to C, \quad \text{all } t > 0.$$

A corollary to the main limit theorem (Theorem (3.2)) we will prove is the following:

Theorem (2.6). *Let $\{A_n\}$ be a sequence of subsets of $L \times L$ satisfying the conditions of Theorem (2.3) with a common w, and let $\{T_n(t)\}$ be the corresponding sequence of semigroups. Define $A = \lim_{n\to\infty} A_n$. Then $A - wI$ is dissipative and if $\mathfrak{R}(I - \alpha A) \supset \overline{\mathfrak{D}(A)}$ for all sufficiently small positive α, then (by Theorem (2.3)) A generates a semigroup $T(t)$ and*

$$T(t)x = \lim_{n\to\infty} T_n(t)x_n$$

for all $x \in \overline{\mathfrak{D}(A)}$ and all sequences $x_n \in \overline{\mathfrak{D}(A_n)}$ with $\lim_{n\to\infty} x_n = x$.

We actually will prove a more general abstract theorem which will apply to notions of convergence other than strong convergence, as well as to "convergence" of semigroups defined on different Banach spaces in the manner introduced by Trotter [12]. To motivate the abstract formulation consider the following:

Let $\mathfrak{L}$ be the Banach space of bounded sequences $\{x_n\} \subset L$ with $\|\{x_n\}\| \equiv \sup_n \|x_n\|$. For $A_1, A_2 \cdots$ satisfying the conditions of Theorem (2.3) with common w let

$$\mathfrak{A} = \{(\{x_n\}, \{y_n\}) : (x_n, y_n) \in A_n;\ \{x_n\}, \{y_n\} \in \mathfrak{L}\}.$$

Then $\mathfrak{A} \subset \mathfrak{L} \times \mathfrak{L}$ satisfies the conditions of Theorem (2.3). Consequently,

$$\mathfrak{T}(t)\{x_n\} \equiv \lim_{m\to\infty} \left(I - \frac{t}{m}\mathfrak{A}\right)^{-m} \{x_n\}$$

exists for all $\{x_n\} \in \overline{\mathfrak{D}(\mathfrak{A})}$ and $\mathfrak{T}(t)\{x_n\} = \{T_n(t)x_n\}$, where $T_n(t)$ is the semigroup corresponding to A_n. Let $\mathfrak{M} \subset \mathfrak{L}$ be the subspace of strongly convergent sequences. If

$$\mathcal{T}(t):\mathfrak{M}\cap\overline{\mathfrak{D}(\mathfrak{A})}\to\mathfrak{M}\cap\overline{\mathfrak{D}(\mathfrak{A})}$$

then defining

$$T(t)x \equiv \lim_{n\to\infty} T_n(t)x_n, \tag{2.7}$$

whenever $\{x_n\}\in\mathfrak{M}\cap\overline{\mathfrak{D}(\mathfrak{A})}$ and $x=\lim_{n\to\infty}x_n$, we obtain a strongly continuous semigroup on $\{x: x=\lim_{n\to\infty}x_n, \{x_n\}\in\mathfrak{M}\cap\overline{\mathfrak{D}(\mathfrak{A})}\}$ satisfying $\|T(t)x-T(t)y\|\le e^{wt}\|x-y\|$. This expains our interest in Corollary (2.5) and motivates the following abstract formulation of the problem which will be considered in §3.

Let $\mathfrak{L}$ and L be Banach spaces, and let P be a bounded linear mapping defined on a closed subspace $\mathfrak{D}(P)$ of $\mathfrak{L}$,

$$P:\mathfrak{D}(P)\to L.$$

Let $\mathfrak{A}\subset\mathfrak{L}\times\mathfrak{L}$ satisfy the conditions of Theorem (2.3) and let $\mathcal{T}(t)$ be the corresponding semigroup. Under what conditions does

$$T(t)Px \equiv P\mathcal{T}(t)x, \qquad x\in\overline{\mathfrak{D}(\mathfrak{A})}\cap\mathfrak{D}(P),$$

determine a semigroup on a subset of L? In the case of primary interest $\mathfrak{L}$ is the sequence space and $P\{x_n\}=\lim_{n\to\infty}x_n$, the limit being the strong limit. However, note that other notions of convergence (e.g. weak convergence) determine bounded linear operators on subspaces of the sequence space in exactly the same way.

Similarly in Trotter's setting we have a Banach space L, a sequence of Banach spaces L_n, and linear maps $P_n: L\to L_n$ satisfying $\lim_{n\to\infty}\|P_nx\|=\|x\|$. Let $\mathfrak{L}=\{\{x_n\}: x_n\in L_n \|\{x_n\}\|\equiv\sup_n\|x_n\|<\infty\}$. Define $P\{x_n\}=x$ if $\lim_{n\to\infty}\|P_nx-x_n\|=0$. This defines a bounded linear operator on a subspace of $\mathfrak{L}$.

When this paper was in its final draft, the author learned that Professor Jerome A. Goldstein [14] had proved a theorem virtually identical to Theorem (2.6) using a somewhat different sequence space approach. For most purposes Theorem (2.6) is also equivalent to the results of Brezis and Pazy [1] which in turn generalize results in [7], [8], [9], [10]. Most of this work assumes convergence of $(I-\alpha A_n)^{-1}$ in some sense rather than convergence of A_n. Conditions of one type can of course be translated into conditions of the other type.

3. Transformation of semigroups.

Lemma (3.1). *Let $\mathfrak{L}$ and L be Banach spaces. Let P be a continuous linear mapping of a closed subspace $\mathfrak{D}(P)\subset\mathfrak{L}$ into L and let $\mathfrak{B}$ be a Lipschitz continuous (but not necessarily linear) mapping of a closed subset $\mathfrak{D}(\mathfrak{B})\subset\mathfrak{L}$ into $\mathfrak{L}$.*

Let $D=\mathfrak{D}(\mathfrak{B})\cap\mathfrak{D}(P)$.

For $y_1, y_2\in P(D)$ define $\rho(y_1,y_2)=\inf\{\|x_1-x_2\|: x_1, x_2\in D, y_1=Px_1, y_2=Px_2\}$ and suppose there is a constant c such that $\rho(y_1,y_2)\le c\|y_1-y_2\|$ all $y_1, y_2\in P(D)$.

Suppose $x, y \in D$, *and* $Px = Py$ *implies* $\mathfrak{B}x - \mathfrak{B}y \in \mathfrak{D}(P)$ *and* $P(\mathfrak{B}x - \mathfrak{B}y) = 0$. *Let* $\mathcal{C} \equiv \{x \in D\colon \mathfrak{B}x \in \mathfrak{D}(P)\}$ *and let* C *be the closure of* $P(\mathcal{C})$. *Then* $\mathcal{C} = \{x \in D\colon Px \in C\}$, *and* $BPx \equiv P\mathfrak{B}x$, $x \in \mathcal{C}$ *determines a Lipschitz continuous mapping* B *of* C *into* L.

Proof. If $x \in D$ and $Px \in C$, then there is a sequence $\{z_n\} \subset \mathcal{C}$ with $\lim_{n\to\infty} Pz_n = Px$. Using the assumptions on $\rho(y_1, y_2)$ we can find sequences $\{y_n\}$ and $\{x_n\}$ in D such that $Py_n = Pz_n$, $Px_n = Px$ and $\lim_{n\to\infty}\|y_n - x_n\| = 0$. The fact that $\mathfrak{B}z_n \in \mathfrak{D}(P)$ and the assumptions on P and $\mathfrak{B}$ imply $\mathfrak{B}y_n$ and $\mathfrak{B}x_n - \mathfrak{B}x$ are in $\mathfrak{D}(P)$.

The Lipschitz continuity of $\mathfrak{B}$ implies $\lim_{n\to\infty}\mathfrak{B}y_n - \mathfrak{B}x_n = 0$, and hence

$$\lim_{n\to\infty} (\mathfrak{B}y_n - \mathfrak{B}x_n + \mathfrak{B}x) = \lim_{n\to\infty} (\mathfrak{B}y_n - (\mathfrak{B}x_n - \mathfrak{B}x)) = \mathfrak{B}x.$$

Since $\mathfrak{B}y_n - (\mathfrak{B}x_n - \mathfrak{B}x) \in \mathfrak{D}(P)$ we must have $\mathfrak{B}x \in \mathfrak{D}(P)$, and hence $x \in \mathcal{C}$. (Note we have *not* shown that $P(\mathcal{C})$ is closed.)

The Lipschitz continuity of B follows from the Lipschitz continuity of $\mathfrak{B}$, the fact that B is well defined, and the assumptions on $\rho(y_1, y_2)$.

Theorem (3.2). *Let* $\mathfrak{L}$ *and* L *be Banach spaces, and let* P *be a continuous linear mapping of a closed subspace* $\mathfrak{D}(P) \subset \mathfrak{L}$ *into* L.

Let $\mathfrak{A} \subset \mathfrak{L} \times \mathfrak{L}$ *satisfy the conditions of Theorem* (2.3) *and denote the corresponding semigroup by* $\mathfrak{T}(t)$. *Suppose the conditions of Lemma* (3.1) *are satisfied taking* $\mathfrak{B} = \mathfrak{T}(t)$ *or* $\mathfrak{B} = (I - \alpha\mathfrak{A})^{-1}$. (*This is always true under the conditions of Theorem* (2.6).) *Define*

$$A = \{(Px, Py) : (x, y) \in \mathfrak{A} \cap \mathfrak{D}(P) \times \mathfrak{D}(P)\}.$$

If $\overline{\mathfrak{R}(I - \alpha A)} \supset \overline{\mathfrak{D}(A)}$ *for all sufficiently small positive* α, *then* $(I - \alpha A)^{-1}$ *can be extended to a Lipschitz continuous map* $J(\alpha)$ *on* $\overline{\mathfrak{R}(I - \alpha A)}$;

$$(I - \alpha A)^{-1} y = P(I - \alpha\mathfrak{A})^{-1} x \tag{3.3}$$

if $y = Px \in \mathfrak{R}(I - \alpha A)$;

$$T(t)y = P\mathfrak{T}(t)x \tag{3.4}$$

for $y = Px \in \overline{\mathfrak{D}(A)}$ *determines a strongly continuous semigroup of Lipschitz continuous mappings on* $\overline{\mathfrak{D}(A)}$; *and*

$$T(t)x = \lim_{n\to\infty} J\left(\frac{t}{n}\right)^n x \tag{3.5}$$

for $x \in \overline{\mathfrak{D}(A)}$.

Proof. Let $D_\alpha = \mathfrak{R}(I - \alpha\mathfrak{A}) \cap \mathfrak{D}(P)$ and $D = \overline{\mathfrak{D}(\mathfrak{A})} \cap \mathfrak{D}(P)$. By Lemma (3.1)

$$\mathcal{C}_\alpha = \{x \in D_\alpha : (I - \alpha\mathfrak{A})^{-1}x \in \mathfrak{D}(P)\} = \{x \in D_\alpha : Px \in C_\alpha\},$$

where C_α is the closure of $P(\mathcal{C}_\alpha)$. Observe that

$$C_\alpha \supset \overline{\mathfrak{R}(I - \alpha A)} \supset \overline{\mathfrak{D}(A)},$$

and

$$(I - \alpha\mathfrak{A})^{-1} : \mathcal{C}_\alpha \to \{x \in D : Px \in \overline{\mathfrak{D}(\mathfrak{A})}\} \equiv \mathcal{C} \subset \mathcal{C}_\alpha.$$

It follows that

$$(I - \alpha A)^{-1}Px = P(I - \alpha\mathfrak{A})^{-1}x$$

for $x \in \mathcal{C}_\alpha$ and $(I - \alpha A)^{-1}$ can be extended to a Lipschitz continuous map on $\overline{\mathfrak{R}(I - \alpha A)}$. Since $(I - \alpha\mathfrak{A})^{-1}: \mathcal{C} \to \mathcal{C}$, Corollary (2.5) implies

$$\mathfrak{T}(t): \mathcal{C} \to \mathcal{C}. \tag{3.6}$$

Again applying Lemma (3.1),

$$\{x \in D : \mathfrak{T}(t)x \in \mathfrak{D}(P)\} = \{x \in D : Px \in C_t\}$$

where C_t is a closed subset of L. By (3.6) $C_t \supset \mathfrak{D}(A)$ and (3.4) follows. The limit in (3.5) follows from (3.3) and the corresponding limit for $\mathfrak{T}(t)$.

In the linear case the converse of Theorem (3.2) holds as a result of the fact that the converse of Corollary (2.5) holds. Unfortunately, due to the nonuniqueness of the dissipative set corresponding to a nonlinear semigroup (see [3]), counterexamples to the converse exist. On the positive side, the following can also be found in [3].

Proposition (3.7). *Let $\mathfrak{A}$ satisfy the conditions of Theorem (2.3), and $\overline{\mathfrak{D}(\mathfrak{A})}$ be convex. If*

$$\lim_{t \to 0+} \left(I - \alpha\left(\frac{\mathfrak{T}(t) - I}{t}\right)\right)^{-1} x = (I - \alpha\mathfrak{A})^{-1}x \tag{3.8}$$

for all $x \in \overline{\mathfrak{D}(\mathfrak{A})}$, then $\mathfrak{T}(t): \mathcal{C} \to \mathcal{C}$, $\mathcal{C}$ a closed convex subset of $\overline{\mathfrak{D}(\mathfrak{A})}$, implies $(I - \alpha\mathfrak{A})^{-1}: \mathcal{C} \to \mathcal{C}$.

We note that (3.8) is equivalent to

$$\lim_{t \to 0+} \mathfrak{A}_t = \mathfrak{A} \cap \bigcup_\alpha \{((I - \alpha\mathfrak{A})^{-1}x, x) : x \in \overline{\mathfrak{D}(\mathfrak{A})}\} \tag{3.9}$$

where

$$\mathfrak{A}_t = \{(x, (\mathfrak{T}(t) - I)x/t) : x \in \overline{\mathfrak{D}(\mathfrak{A})}\},$$

and the limit is defined in the same way as the limit of a sequence of sets was in §2.

4. **Discrete parameter limit theorem.** In the linear case [4], [5], [12], discrete parameter limit theorems are obtained by comparing the sequence of discrete parameter semigroups $\{T_n(k)\}$ with a corresponding sequence of continuous parameter semigroups generated by

$$A_n = \{(x, (T_n(1) - I)x/b_n) : x \in \mathfrak{D}(T_n(1))\}$$

where $\{b_n\}$ is an appropriately chosen sequence of positive numbers with $\lim_{n\to\infty} b_n = 0$. The following, an immediate consequence of results due to Miyadera and Oharu [10], gives the nonlinear analogue.

Theorem (4.1). *Let $\{T_n\}$ be a sequence of contractions with $\mathfrak{R}(T_n) \subset \mathfrak{D}(T_n)$. Let $T_n(k)$ denote the kth power of T_n and define*

$$A_n = \{(x, b_n^{-1}(T_n - I)x) : x \in \mathfrak{D}(T_n)\},$$

where $\lim_{n\to\infty} b_n = 0$. Let $S_n(t)$ denote the contraction semigroup generated by A_n. If $x_n \in \mathfrak{D}(T_n) \equiv \mathfrak{D}(A_n)$ satisfies $\sup_n \|A_n x_n\| < \infty$, then

$$\|S_n(t)x_n - T_n([t/b_n])x_n\| \le b_n(b_n + \sqrt{t})\|A_n x_n\|$$

and hence

$$\lim_{n\to\infty} \|S_n(t)x_n - T_n([t/b_n])x_n\| = 0.$$

As a consequence of the above we state the following analogue of Theorem (2.6).

Theorem (4.2). *Let $\{T_n\}$ and $\{A_n\}$ be as in Theorem (4.1) and define $A = \lim_{n\to\infty} A_n$. Then A is dissipative and if $\overline{\mathfrak{R}(I - \alpha A)} \supset \overline{\mathfrak{D}(A)}$ for all sufficiently small positive α, then A generates a semigroup $T(t)$ and*

$$T(t)x = \lim_{n\to\infty} T_n([t/b_n])x_n$$

for all $x \in \overline{\mathfrak{D}(A)}$ and all sequences $x_n \in \mathfrak{D}(T_n)$ with $\lim_{n\to\infty} x_n = x$.

5. **An example from gas kinetics.** In [2] H. Conner considers a number of discrete velocity models related to the Boltzmann equation. One of these, due to Carleman,

$$u_t = -u_x + v^2 - u^2, \qquad v_t = v_x + u^2 - v^2$$

corresponds to a semigroup generated by a dissipative set. This can be considered as a model for a system with two types of particles, Type I particles move to the right on the line with speed one, Type II to the left with speed one. The local density of Type I particles is given by u and that of Type II particles by v. When two particles interact one Type I particle is produced and one Type II particle. Consequently an interaction between a Type I particle and a Type II particle has no effect on the system.

The results of this paper allow us to give a rigorous derivation of the associated fluid equation. The fluid equation corresponds roughly to the behavior of the total density $\rho = u + v$ when the local velocity distribution is in equilibrium (in this case $u = v$). This can only be approximately true (unless $u \equiv v \equiv$ constant) but the approximation may be good if the local approach to velocity equilibrium is sufficiently rapid.

To clarify what we mean by this, define

$$A(u, v) = (-u_x, v_x) \quad \text{and} \quad B(u, v) = (v^2 - u^2, u^2 - v^2).$$

Let $L = L_1(\mathrm{R}) \times L_1(\mathrm{R})$ with $\|(u, v)\| = \|u\|_{L_1} + \|v\|_{L_1}$, and $L^+ = \{(u, v) \in L: u \geq 0, v \geq 0\}$. Then A and B are defined and dissipative on a dense subset of L^+ as is $A + \alpha B$ for $\alpha > 0$, and the closure of $A + \alpha B$ generates a semigroup $T_\alpha(t)$ in the sense of Theorem (2.3). (See Appendix.)

If α is large then the approach to equilibrium by the local velocity distribution is rapid. Consequently we are interested in the behavior of $T_\alpha(t)$ as α goes to infinity. We will actually consider $T_\alpha(\alpha t)$ which is the semigroup corresponding to $\alpha A + \alpha^2 B$.

Theorem (5.1). *Let $T_\alpha(t)$ be as defined above. Let $u \geq 0$ and $u \in L_1(\mathrm{R})$. Then*

$$\lim_{\alpha \to \infty} T_\alpha(\alpha t)(u, u) = (T(t)u, T(t)u)$$

where $T(t)$ is the semigroup on $\{u \in L_1(\mathrm{R}): u \geq 0\}$ generated by the closure in $L_1(\mathrm{R})$ of

$$Cu = \frac{1}{4} \frac{d}{dx}\left(\frac{u_x}{u}\right) = \frac{1}{4} \frac{d^2}{dx^2} \log u, \tag{5.2}$$

with $\mathfrak{D}(C) = \{u \in L_1(\mathrm{R}): u > 0$, u and u_x absolutely continuous, $(d^2/dx^2)\log u \in L_1(\mathrm{R})$ and $\lim_{|x| \to \infty} u_x/u = 0\}$.

Proof. Let $\mathfrak{L}$ be the space of mappings of $[0, \infty)$ into L with norm

$$\|\{(u_\alpha, v_\alpha)\}\| = \sup_\alpha \|(u_\alpha, v_\alpha)\|.$$

Define a bounded linear operator from a subspace of $\mathfrak{L}$ into $L_1(\mathrm{R})$ by

$$P\{(u_\alpha, v_\alpha)\} = u \quad \text{if} \quad \lim_{\alpha \to \infty} u_\alpha = u \quad \text{and} \quad \lim_{\alpha \to \infty} v_\alpha = u.$$

To apply Theorem (3.2) we must find $\{(u_\alpha, v_\alpha)\} \in \mathfrak{L}$ such that

$$P\{(u_\alpha, v_\alpha)\} = u \quad \text{and} \quad P\{(\alpha A + \alpha^2 B)(u_\alpha, v_\alpha)\} = Cu$$

for sufficiently many u. Toward that end we prove the following lemma:

Lemma (5.3). *Let u be twice continuously differentiable, $(d^2/dx^2)\log u \in L_1(\mathbf{R})$, $u > 0$, $c_1/x^2 \le u \le c_2/x^2$ for some c_1 and c_2 and x sufficiently large, and $|u_x| \le |M/x^3|$ for some M.*

Let $x_\alpha = \inf\{|x|: |u_x(x)| \ge 4\alpha|u(x)|^2\}$ and define g_α so that

$$g_\alpha(x) = \begin{cases} 1, & |x| \le x_\alpha - 1, \\ 0, & |x| > x_\alpha, \end{cases}$$

$|g_\alpha'(x)| \le 2$ and $|g_\alpha''(x)| \le 4$. Define

$$u_\alpha = u - g_\alpha u_x/(4\alpha u) \quad \textit{and} \quad v_\alpha = u + g_\alpha u_x/(4\alpha u).$$

Then

$$P\{(u_\alpha, v_\alpha)\} = u \tag{5.4}$$

and

$$P\{(\alpha A + \alpha^2 B)(u_\alpha, v_\alpha)\} = Cu. \tag{5.5}$$

Proof. The functions g_α were selected so that u_α and v_α are nonnegative, and so that the dominated convergence theorem would imply (5.4).

Consider

$$(\alpha A + \alpha^2 B)(u_\alpha, v_\alpha)$$
$$= \left(\alpha(g_\alpha - 1)u_x + \frac{1}{4}g_\alpha' \frac{u_x}{u} + \frac{1}{4}g_\alpha \frac{d}{dx}\left(\frac{u_x}{u}\right), -\alpha(g_\alpha - 1)u_x + \frac{1}{4}g_\alpha' \frac{u_x}{u} + \frac{1}{4}g_\alpha \frac{d}{dx}\left(\frac{u_x}{u}\right)\right).$$

Since $|u_x|$ is bounded by $|M/x^3|$ for some M, we can bound the L_1 norm of the first term of each component by

$$2\alpha \int_{x_\alpha - 1}^{\infty} \frac{M}{x^3}\, dx = \frac{\alpha M}{(x_\alpha - 1)^2}.$$

The conditions on u imply $x_\alpha \ge c\alpha$ for some $c > 0$ and α sufficiently large, and hence this goes to zero as α goes to infinity. The limits of the other terms are straightforward and we have (5.5).

We must now check that Lemma (5.3) gives convergence for "sufficiently many" u. That is we must show that

$$\{u - \beta \tfrac{1}{4}(d^2/dx^2)\log u: u \text{ satisfies the conditions of Lemma (5.3)}\}$$

is dense in $\{u \in L_1(\mathbf{R}): u \ge 0\}$ for all sufficiently small β. If $u - (d^2/dx^2)\log u = f$,

$v(x) = u(ax)$ and $g(x) = f(ax)$ then $v - (1/a^2)(d^2/dx^2)\log v = g$. Consequently it is sufficient to consider the case $\beta = 4$.

Lemma (5.6). *Let f be nonnegative and continuous with compact support on* $\mathbf{R}$, *and not identically zero. Then there exists a solution of*

$$u - (d^2/dx^2)\log u = f \tag{5.7}$$

satisfying the conditions of Lemma (5.3).

Proof. Since the form of the equation is translation invariant we may assume without loss of generality that $f(0) > 0$. Let

$$d = \int_0^\infty f(y)\,dy, \qquad c = \int_{-\infty}^0 f(y)\,dy.$$

Select $a < -2/c$ and $b > 2/d$ so that the support of f is contained in $[a, b]$. Let g be a positive, continuous function satisfying

$$\int_0^x g(y) = d - \frac{2}{x} \quad \text{for } x > b$$

and

$$\int_x^0 g(y) = c + \frac{2}{x} \quad \text{for } x < a.$$

Define

$$\phi(x) = \exp\left\{-\int_0^x \int_0^y (f(z) - g(z))dzdy\right\}.$$

For $x > b$

$$\phi(x) = \exp\left\{-\int_0^b \int_0^y (f(z) - g(z))dzdy\right\} \exp\left\{-\int_b^x \frac{2}{y}\,dy\right\} = \frac{\text{const}}{x^2}.$$

Similarly $\phi(x) = \text{const}/x^2$ for $x < a$. We are interested in finding a solution of (5.7) of the form $u(x) = ke^{w(x)}\phi(x)$ where k is a constant which will be determined later. Substituting this form into the equation we obtain

$$ke^w\phi - (d^2/dx^2)w - (d^2/dx^2)\log\phi = f. \tag{5.8}$$

Considering the definition of ϕ, this becomes

$$ke^w\phi - (d^2/dx^2)w = g. \tag{5.9}$$

Since $g > 0$ and $g(x) = 2/x^2$ for $x > b$ and $x < a$ we note that g/ϕ is bounded above and bounded away from zero.

Let $M_1 = \inf_x g(x)/\phi(x)$ and $M_2 = \sup_x g(x)/\phi(x)$.

Select $0 < \epsilon < 1$ and $k > 0$ such that

$$M_2/ke > 1, \qquad \epsilon e^{-\epsilon} = (M_2/ke)\exp\{-M_2/ke\}, \quad \text{and} \quad M_1/k \geq e^{\epsilon}.$$

This is possible since making k small forces ϵ to be small and M_1/k large.

Suppose w is continuous and satisfies $\epsilon \leq w \leq M_2/ke$. The equation

$$(5.10) \qquad (ke^w\phi/w)v - (d^2/dx^2)v = g,$$

has a unique bounded solution v (see Appendix, Lemma (6.4)). Define $F(w)$ by $F(w) = v$. The solution v is twice continuously differentiable and satisfies

$$v \leq \sup_x w(x)e^{-w(x)} \frac{g(x)}{k\phi(x)} \leq \frac{M_2}{k} \sup_z ze^{-z} = \frac{M_2}{ke}$$

and

$$v \geq \inf_x w(x)e^{-w(x)} \frac{g(x)}{k\phi(x)} \geq \epsilon e^{-\epsilon} \frac{M_1}{k} \geq \epsilon.$$

Let $\Gamma = \{w\colon w$ is continuous and $\epsilon \leq w \leq M_2/ke\}$. Then $F\colon \Gamma \longrightarrow \Gamma$ and is continuous in the topology of uniform convergence on compact sets. Furthermore

$$(5.11) \qquad |v_x(z) - v_x(y)| = \left| \int_y^z \frac{d^2}{dx^2} v\, dx \right| \leq k \frac{e^\epsilon}{\epsilon} \frac{M_2}{ke} \int_y^z \phi dx + \int_y^z g\, dx.$$

Since ϕ and g are in $L_1(\mathrm{R})$ it follows that F maps Γ into a compact subset of Γ and hence has a fixed point w. But a fixed point of the mapping F is a solution of (5.9).

We now must verify that $u = ke^w\phi$ satisfies the conditions of Lemma (5.3). Since $\epsilon \leq w \leq M_2/ke$ the fact that $c_1/x^2 \leq u \leq c_2/x^2$ for some c_1 and c_2 and $|x|$ sufficiently large follows from the definition of ϕ.

Consider $u_x = ke^w\phi_x + ke^w\phi w_x$. Again, from the definition of ϕ, in order to verify $|u_x| \leq |M/x^3|$ for some M it is sufficient to show that xw_x is bounded.

From (5.11) it is clear that $\lim_{x\to\infty} w_x(x)$ exists and the boundedness of w implies the limit is zero. Therefore

$$w_x(x) = \int_x^\infty (g - ke^w\phi)\, dx.$$

But $g - ke^w\phi = O(1/x^2)$, and hence $w_x(x) = O(1/x)$ for $x \longrightarrow \infty$. Similarly $w_x(x) = O(1/x)$ for $x \longrightarrow -\infty$.

Remark. General results of this type are considered in the linear case in [6]. General material on discrete velocity models for the Boltzmann equation as well as some results related to the above for a different model can be found in [13].

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6. **Appendix.** We verify that the operators considered in §5 are dissipative.

Lemma (6.1). *Let* $L = L_1(\mathrm{R}) \times L_1(\mathrm{R})$ *with* $\|(u, v)\| = \|u\|_{L_1} + \|v\|_{L_1}$, *and let* $L^+ = \{(u, v) \in L\colon u \geq 0,\ v \geq 0\}$. *Let* $B(u, v) = (v^2 - u^2, u^2 - v^2)$ *with* $\mathfrak{D}(B) = \{(u, v) \in L^+\colon u^2, v^2 \in L_1(\mathrm{R})\}$. *Then* B *is dissipative.*

Proof.

$$\|(u_1, v_1) - (u_2, v_2) - \alpha(B(u_1, v_1) - B(u_2, v_2))\|$$
$$= \int |u_1 - u_2 - \alpha(v_1^2 - v_2^2 - (u_1^2 - u_2^2))|dx$$
$$+ \int |v_1 - v_2 - \alpha(u_1^2 - u_2^2 - (v_1^2 - v_2^2))|dx$$
$$\geq \int [(1 + \alpha(u_1 + u_2))|u_1 - u_2| - \alpha(v_1 + v_2)|v_1 - v_2|]dx$$
$$+ \int [(1 + \alpha(v_1 + v_2))|v_1 - v_2| - \alpha(u_1 + u_2)|u_1 - u_2|]dx$$
$$= \int |u_1 - u_2|dx + \int |v_1 - v_2|dx = \|(u_1, v_1) - (u_2, v_2)\|.$$

Since A is dissipative and generates a linear semigroup, it follows that $A + \alpha B$ is dissipative.

The fact that C is dissipative for functions satisfying the conditions of Lemma (5.3) follows from the fact that $A + \alpha B$ is dissipative.

However, we will verify that C is dissipative on all of $\mathfrak{D}(C)$ as defined in Theorem (5.1).

Lemma (6.2). *Let C be as defined in Theorem (5.1). Then C is dissipative.*

Proof. Let $u, v \in \mathfrak{D}(C)$ and let $\mathcal{S}_1$ be the collection of intervals $I = (a, b)$ such that $u - v > 0$ on I and either $a = -\infty$ or $u(a) = v(a)$ and either $b = \infty$ or $u(b) = v(b)$. Let $\mathcal{S}_2$ be the similar collection of intervals on which $u - v < 0$. For $I \in \mathcal{S}_1$, $u(a) = v(a)$ implies $u_x(a) - v_x(a) \geq 0$ and $u(b) = v(b)$ implies $u_x(b) - v_x(b) \leq 0$.

We then have

$$\int \left| u - v - \alpha\left(\frac{d^2}{dx^2}\log u - \frac{d^2}{dx^2}\log v\right)\right| dx$$
$$\geq \int |u - v|dx - \alpha \int \operatorname{sign}(u - v)\left(\frac{d^2}{dx^2}\log u - \frac{d^2}{dx^2}\log v\right)dx$$

(6.3)
$$= \int |u - v|dx + \alpha \sum_{I \in \mathcal{S}_1} - \int_I \left(\frac{d^2}{dx^2}\log u - \frac{d^2}{dx^2}\log v\right)dx$$
$$+ \alpha \sum_{I \in \mathcal{S}_2} \int_I \left(\frac{d^2}{dx^2}\log u - \frac{d^2}{dx^2}\log v\right)dx.$$

Suppose, for example, $I = (a, b) \in \mathcal{S}_1$, $a \neq -\infty$ and $b = \infty$. The fact that $\lim_{x\to\infty} u_x/u = \lim_{x\to\infty} v_x/v = 0$ implies

$$-\int_a^b \left(\frac{d^2}{dx^2}\log u - \frac{d^2}{dx^2}\log v\right) dx = \left(\frac{u_x(a)}{u(a)} - \frac{v_x(a)}{v(a)}\right) \tag{6.4}$$
$$= \frac{u_x(a) - v_x(a)}{u(a)} \geq 0.$$

It follows in a similar manner that all terms in the summations on the right-hand side of (6.3) are nonnegative, and hence it is bounded below by $\int |u - v|\, dx$.

Lemma (6.4). *Suppose $\gamma(x)$ and $g(x)$ are nonnegative, bounded and continuous and that there exists $\epsilon > 0$ such that $\gamma(x) \geq \epsilon$ for all x. Then*

$$\gamma v - \tfrac{1}{2} v'' = g \tag{6.5}$$

has a unique, bounded solution and

$$\inf_x \frac{g(x)}{\gamma(x)} \leq v \leq \sup_x \frac{g(x)}{\gamma(x)}. \tag{6.6}$$

Proof. The uniqueness follows from the fact that the only bounded solution of $v'' = \gamma v$ is $v \equiv 0$.

Since the author is a probabilist, to obtain existence we appeal to results concerning Brownian motion. (See Dynkin [15, p. 46].) In particular if $X(t)$ is standard Brownian motion, $(X(0) = 0,\ \mathrm{Var}(X(t)) = t)$ then the solution of (6.5) is

$$v(x) = E\left(\int_0^\infty \exp\{-\int_0^t \gamma(x + X(s))\}g(x + X(t))\, dt\right). \tag{6.7}$$

The inequalities in (6.6) follow immediately by making the change of variable $\tau = \int_0^t \gamma(x + X(s))$ in (6.7).

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