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THE MODULE DECOMPOSITION OF $I(\bar{A}/A)$

BY

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ABSTRACT. Let A and B be scalar rings with B an A-algebra. The B-algebra $D^n(B/A) = I(B/A)/I^n(B/A)$ is universal for n-truncated A-Taylor series on B. In this paper, we consider the $\bar{A}$ module decomposition of $D^n(\bar{A}/A)$ with a view to classifying the singularity A which is assumed to be the complete local ring of a point on an algebraic curve at a one-branch singularity. We assume that $A/M = k < A$ and that k is algebraically closed with no assumption on the characteristic.

We show that $D^n(\bar{A}/A) = I(\bar{A}/A)$ for n large and that the decomposition of $I(\bar{A}/A)$ as a module over the P.I.D. $\bar{A}$ is completely determined by the multiplicity sequence of A. The decomposition is displayed and a length formula for $I(\bar{A}/A)$ developed. If B is another such ring, where $\bar{B} = \bar{A} = k[[t]]$, we show that $I(\bar{A}/A) \cong I(\bar{B}/B)$ as $k[[t]]$ modules if and only if the multiplicity sequence of A is equal to the multiplicity sequence of B. If $A < B < \bar{A}$, then $I(\bar{A}/A) \cong I(\bar{B}/B)$ as $\bar{A} = \bar{B}$ modules if and only if the Arf closure of A and B coincide. This is equivalent to the existence of an algebra isomorphism between $I(\bar{A}/A)$ and $I(\bar{B}/B)$.

1. Preliminaries. Suppose B and A are commutative rings with identity elements. Such rings are known as scalar rings. Suppose also that B is an A algebra.

Definition. An A linear map T from B to a B algebra S is called an S-valued A-Taylor series if, for each $x, y \in B$, $T(xy) = xT(y) + yT(x) + T(x)T(y)$ and if $T(1) = 0$.

Denote by π the map from $B \otimes_A B$ to B given by multiplication: i.e. $\pi(a \otimes_A b) = a \cdot b$. If $I(B/A)$ is defined by the exact sequence

$$0 \to I(B/A) \to B \otimes_A B \xrightarrow{\pi} B \to 0,$$

then the ideal $I(B/A)$ has the structure of a left B-algebra and a left B module (for $b \in B$ and $c \otimes_A d \in B \otimes_A B$, $b(c \otimes_A d) = (b \otimes_A 1)(c \otimes_A d)$). The B

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module $I(B/A)$ is generated by the set of elements $(1 \otimes_A x - x \otimes_A 1)$ for $x \in B$. If z_i $(i \in H)$ is a set of A algebra generators for B, then the elements $(1 \otimes_A z_i - z_i \otimes_A 1)$ are B algebra generators for $I(B/A)$ [6, p. 4].

Let δ_A be the map from B to $I(B/A)$ given by $\delta_A(x) = 1 \otimes_A x - x \otimes_A 1$. This map is easily checked to be an $I(B/A)$-valued A-Taylor series and is called the canonical A-Taylor series on B. It is known that the pair $(\delta_A, I(B/A))$ is universal for S-valued A-Taylor series [6, p. 5]. Hence, if $r: B \to S$ is an S-valued A-Taylor series, then there exists a unique B-algebra homomorphism r^* from $I(B/A)$ to S so that $r^* \delta_A = r$.

Definition. Suppose A and B are scalar rings with B an A-algebra. If S is a B-algebra then S is said to be n-truncated if for each sequence of $n+1$ elements $s_0, \cdots, s_n$ in S, $s_0 \cdot s_1 \cdots s_n = 0$.

Definition. An S-valued A-Taylor series r is said to be n-truncated if S is n-truncated.

Denote by θ the natural map of $I(B/A)$ to $I(B/A)/I(B/A)^{n+1}$. If δ_A is the canonical A-Taylor series on B, then $\delta_A^n = \theta\delta_A$ is an n-truncated A-Taylor series and the pair $(\delta_A^n, I(B/A)/[I(B/A)]^{n+1})$ is universal for n-truncated S-valued A-Taylor series [6, p. 17].

The universal object $I(B/A)/[I(B/A)]^{n+1}$ will be denoted by $D^n(B/A)$ and has again the structure of a left B-algebra and hence B-module.

Lemma 1.1. *Suppose A and B are scalar rings and B is a finitely generated A-module. Then $I(B/A)$ is finitely generated as a B-module.*

Proof. Let $I(B/A)$ be defined by the sequence

$$0 \longrightarrow I(B/A) \longrightarrow B \otimes_A B \xrightarrow{\pi} B \longrightarrow 0$$

where $\pi(a \otimes_A b) = ab$. If $\Sigma_{i=1}^n (a_i \otimes b_i) \in I(B/A)$, then $\Sigma_{i=1}^n a_i b_i = 0$ and consequently

$$\sum_{i=1}^n (a_i \otimes b_i) = \sum_{i=1}^n (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1).$$

Hence, $I(B/A)$ is generated as a B-module by elements of the form $(1 \otimes_A b - b \otimes_A 1) = \delta_A(b)$ where $b \in B$. But B finitely generated as an A-module implies $b = \Sigma_{i=1}^s x_i y_i$ where $x_i \in A$ and y_i are the generators. Since δ_A is A-linear, $\delta_A(b) = \Sigma_{i=1}^s x_i \delta(y_i)$ and hence $I(B/A)$ is finitely generated as a B-module. Q.E.D.

Let A be a Noetherian local domain and F its field of quotients. If $\bar{A}$ is the integral closure of A in F, assume that $\bar{A}$ is finitely generated as an A-module. Lemma 1.1 then implies that $I(\bar{A}/A)$ is finitely generated as an $\bar{A}$-module and, consequently, also $D^n(\bar{A}/A)$.

Theorem 1.2. *Assume that A is a local Noetherian domain of Krull dimension 1. If M is the maximal ideal of A, assume $A/M = k$ is algebraically closed and*

$k < A$. Let $\bar{A}$ be the integral closure of A inside its field of quotients and suppose $\bar{A}$ is finitely generated as an A-module. If $\bar{A}$ is local then the ideal $I(\bar{A}/A)$ is nilpotent.

Proof. Since $\bar{A}$ is finitely generated over A, the conductor $C = \mathrm{Ann}_A(\bar{A}/A) \neq 0$. (Take the product of the denominators of the generators of A.) We may assume that $C \neq (1)$, since otherwise $A = \bar{A}$.

Let $x_1, \cdots, x_s$ be the generators for $\bar{A}$ over A. Then $\delta_A(x_1), \cdots, \delta_A(x_s)$ are $\bar{A}$ algebra generators for $I(\bar{A}/A)$. For the maximal ideal M' of $\bar{A}$, $\bar{A}/M' \cong k$ since k is algebraically closed. If $\alpha + y \in \bar{A}$ where $\alpha \in k$, then $\delta_A(\alpha + y) = \delta_A(y)$. Hence, we may assume that the x_i's lie in M'. Because A is one dimensional, we may assume that $x_i^d \in C < A$ for some d and all i. But

$$\delta_A(x_i^d) = \binom{d}{1} x_i^{d-1}\delta_A(x_i) + \binom{d}{2} x_i^{d-2}[\delta_A(x_i)]^2 + \cdots + [\delta_A(x_i)]^d$$

[6, p. 16]. Since $x_i^d \in A$, $\delta_A(x_i^d) = 0$ and hence

$$[\delta_A(x_i)]^d = (-x_i)\left[\binom{d}{1} x_i^{d-2}\delta_A(x_i) + \cdots + \binom{d}{d-1}[\delta_A(x_i)]^{d-1}\right].$$

Since $(x_i)^d I(\bar{A}/A) = 0$, we conclude that $\delta_A(x_i)$ is nilpotent for $i = 1, \cdots, s$. But these are algebra generators for $I(\bar{A}/A)$ over $\bar{A}$. Therefore, $I(\bar{A}/A)$ is nilpotent. Q.E.D.

The theorem states that with the assumptions on A, $D^n(\bar{A}/A) \cong I(\bar{A}/A)$ as $\bar{A}$-algebras for $n >> 0$. Specifically, if A is the complete local ring of a point on algebraic curve at a "one-branch singularity", then $D^n(\bar{A}/A) \cong I(\bar{A}/A)$ for n large and it is this observation which leads to the study of $I(\bar{A}/A)$ in the following sections.

Definition. Suppose A and B are scalar rings with B an A-algebra. Suppose M is a B-module. An A-linear map L from B to M is called a qth order derivation of B/A into M if it satisfies the following conditions:

(1)
$$L(x_0 x_1 \cdots x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} x_{i_1} \cdots x_{i_s} L(x_0 \cdots \hat{x}_{i_1} \cdots \hat{x}_{i_s} \cdots x_q)$$

for any set $x_0, \cdots, x_q$ of $(q+1)$ elements in B.

(2)
$$L(1) = 0.$$

Note that a derivation of order 1 is a standard derivation from B to M over A.

The map δ_A^q is itself a qth order derivation of B/A into $D^q(B/A)$. If L is a qth order derivation of B/A into M, then there exists a unique B-homomorphism h from $D^q(B/A)$ to M so that $h \cdot \delta_A^q = L$ [6, p. 35]. Therefore, when considered as a B-module, $D^q(B/A)$ together with δ_A^q is universal for qth order derivations of B/A into M (cf. [8]).

2. The blow-up and strict closure of A. Throughout this section we will assume that A is the complete local ring of a point on an algebraic curve at a "one-branch singularity" whose residue field k is algebraically closed and contained in A. Hence, A is a complete local Noetherian domain whose integral closure $\overline{A}$ inside its quotient field F is a power series ring in one variable with coefficients in k. If $k[[t]]$ is the power series ring where t is a uniformizing parameter, then $A \subset k[[t]]$ and the inclusion is proper since we assume that A is the local ring of a singular point.

Since A is a complete local integral domain, $\overline{A}$ is a finitely generated A module [7, p. 112]. Hence, $I(\overline{A}/A)$ is nilpotent according to Theorem 1.1.

The natural valuation on F will be denoted by v and for any $x \in A$, $v(x)$ is called the order of x. The valuation ring of v is, of course, $k[[t]]$.

Let E be a set contained in A. Then E is said to be open in the M-adic topology of A if $M^n < E$ for some n. Note that for any $x \in A$, x is not a zero divisor. Hence, since A is one-dimensional, xA is an open ideal. Let E be any A module and denote $\lambda_A(E)$ as the length of this module. For any open ideal $J < A$, $\lambda_A(A/J^n)$ becomes a polynomial of degree 1 in n for n large [11, Volume 1, p. 284]. This polynomial, say $en + b$, is known as the characteristic polynomial of J and e is called the multiplicity of the ideal J. We shall denote the multiplicity of an ideal $J < A$ as $e(J)$. The multiplicity of the maximal ideal M is by definition the multiplicity of A. We shall write $e(A)$ for $e(M)$. It is easy to see that $\lambda_A(M^n/M^{n+1}) = e(A)$ for n large. If $v(M)$ is the least integer in the set $\{v(x) : x \in M\}$, where v is the valuation, then $v(M) = e(M)$.

Definition. Let M be the maximal ideal of A. Then $x \in M$ is said to be transversal to M if $v(x) = e(M)$.

Let $A < \overline{A} = k[[t]]$ and let $e(A) = e > 0$. For any $x \in A$ where $v(x) = e$, $x = t^e u$ where u is a unit in $k[[t]]$. Hensel's lemma [11, Volume 2, p. 279] assures that there exists a unit $u' \in k[[t]]$ so that $(u')^e = u$. Setting $t' = tu'$, t' is necessarily of order one and $(t')^e = x$. Since $v(t') = 1$, $k[[t]] = k[[t']] = \overline{A}$ and consequently, $F = k((t'))$. Hence, we may always assume that if x is transversal to $M < A$, then $x = t^e$ where $\overline{A} = k[[t]]$ and $e = e(A)$.

Let n be any positive integer. The power series ring $k[[t^n]]$ is contained in $k[[t]]$ and $k[[t]]$ is freely generated over $k[[t^n]]$ by the elements $1, t, \cdots, t^{n-1}$. Denote by R the power series ring $k[[t^n]]$ and let $\overline{A} = k[[t]]$.

Lemma 2.1. *The $\overline{A}$ module $I(\overline{A}/R)$ is a free $\overline{A}$-module on the generators $\delta_R(t), \cdots, \delta_R(t^{n-1})$.*

Proof. The sequence

$$0 \longrightarrow I(\overline{A}/R) \longrightarrow \overline{A} \otimes_R \overline{A} \xrightarrow{\pi} \overline{A} \longrightarrow 0$$

is split exact and hence $\overline{A} \otimes_R \overline{A} \cong \overline{A} \otimes I(\overline{A}/R)$. But $\overline{A} \otimes_R \overline{A}$ is a free $\overline{A}$-module of rank n and hence, $I(\bar{A}/R)$ is a free $\bar{A}$-module of rank $n-1$. Since by Lemma 1.1, $\delta_R(t), \cdots, \delta_R(t^{n-1})$ generate $I(\bar{A}/R)$, they must form a free basis. Q.E.D.

Definition. Let x be transversal to M, the maximal ideal of A and let $\{x_1, \cdots, x_n\}$ generate M. The ring $A^M = A[x_1/x, \cdots, x_n/x]$ is called the blow-up of A along M [5, p. 651], (cf. [9]).

Note that A^M is finitely generated over A and that $\overline{A^M} = k[[t]]$. Hence, $\overline{A^M}$ is itself the complete local ring of an algebraic curve at a one-branch singularity. Since M is contained in the maximal ideal of A^M, $e(A^M) \leq e(A)$.

Set $A^M = A_1$ and let M_1 be the maximal ideal of this ring. We may form the blow up of A_1 along the maximal ideal M_1 and set $A_1^{M_1} = A_2$ with maximal ideal M_2. Continuing in this way, we have a sequence of increasing rings

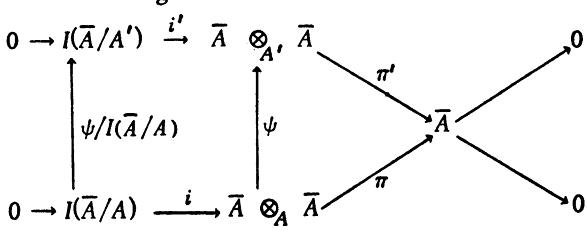
$$A < A_1 < \cdots < A_N < \cdots < \bar{A}.$$

Since A is Noetherian and for each i, A_i is finitely generated, $A_N = A_{N+1}$ for some N. Hence, by [9, p. 372], A_N is a regular local ring and therefore, $A_N = \bar{A} = k[[t]]$. That is, the singularity can be resolved after applying a finite number of blow-ups. The sequence $A = A_0 < A_1 < \cdots < A_N \cdots < \bar{A}$ is known as the branch sequence of A along the maximal ideal $\bar{M}$ of $\bar{A}$ and $\{e(A), e(A_1), \cdots\}$ is called the multiplicity sequence of A.

Definition. Suppose $\bar{A}$ is the integral closure of A inside its quotient field. Then the ring $A' = \{x \in \bar{A}: 1 \otimes_A x - x \otimes_A 1 = 0\}$ is called the strict closure of A inside $\bar{A}$. Clearly, $A < A'$ and $\bar{A}' = \bar{A}$.

Lemma 2.2. *Let $A < \bar{A}$ and let A' be the strict closure of A in $\bar{A}$. Then $\bar{A} \otimes_A \bar{A} \cong \bar{A} \otimes_{A'} \bar{A}$, $I(\bar{A}/A) \cong I(\bar{A}/A')$ and $D^n(\bar{A}/A) \cong D^n(\bar{A}/A')$.*

Proof. Consider the diagram



where ψ is the canonical map.

Note that for any three scalar rings $A < B < C$, the kernel of the canonical map from $C \otimes_A C$ to $C \otimes_B C$ is generated as an ideal by the elements $\{1 \otimes_A x - x \otimes_A 1: x \in B\}$ [6, p. 12]. Hence, if K is the kernel of ψ, then K is generated as an ideal by the elements $1 \otimes_A x - x \otimes_A 1$ where $x \in A'$. Therefore $K = 0$. The map ψ is clearly onto and the restriction of ψ to $I(B/A)$ gives the required B-algebra isomorphism. Hence, $I(B/A) \cong I(B/A')$ and $D^n(B/A) \cong D^n(B/A')$. Q.E.D.

Remark 2.3. The proof of the lemma shows that $(A')' = A'$. If $A = A'$, then A is said to be strictly closed in $\bar{A}$. Under our standing assumption that $k < A$ (k

a field), [5, p. 677] shows that A' coincides with the Arf closure of A. In particular, $e(A) = e(A')$ [5, p. 668] and if M' is the maximal ideal of A', then $(A')^{M'} = (A^M)'$ [5, p. 668]. The geometric properties of the strict closure of A (= Arf closure of A) will be used in §4.

Remark 2.4. The blow-up of A' along its maximal ideal has an easier form. In fact, if A is strictly closed (i.e. $A = A'$) with maximal ideal M and if x is transversal to M, then $A^M = \{m/x\colon m \in M\}$. To show this we need only prove that Mx^{-1} forms a ring. But for any $y/x,\ z/x \in Mx^{-1}$,

$$y \cdot z/x \otimes_A 1 = y/x \otimes_A z = y/x \otimes_A zx/x = y \otimes_A z/x = 1 \otimes_A yz/x.$$

Hence, $y \cdot z/x \in A$ since A is strictly closed. Equivalently, $y \cdot z = x \cdot a$, $a \in M < A$. Therefore, $y/x \cdot z/x = a/x \in Mx^{-1}$.

3. The relation of $I(\overline{A}/A)$ to $I(\overline{A}/A^M)$. Suppose A is as in §2 with maximal ideal M. Choose a transversal element x so that $x = t^e$ where $e = e(A)$ and $\overline{A} = k[[t]]$. Hence, A is properly contained in $\overline{A} = k[[t]]$ and if A^M is the blow-up of A along M, then also $\overline{A^M} = k[[t]]$. Let θ be the canonical homomorphism from $I(\overline{A}/A)$ to $I(\overline{A}/A^M)$ given by

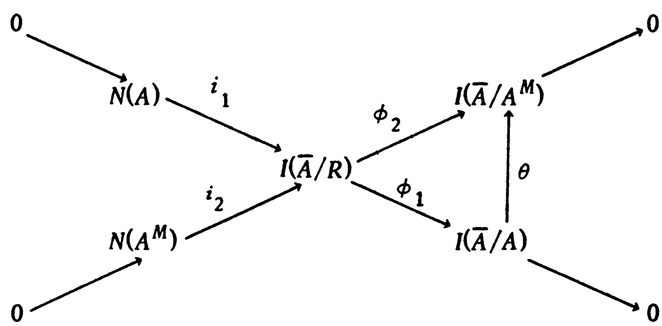
$$\theta\left(\sum_{i=1}^{n} (a_i \otimes_A b_i)\right) = \sum_{i=1}^{n} (a_i \otimes_{A^M} b_i).$$

Note that θ is onto.

Let $R = k[[t^e]]$; it is clear that R is a complete subring of both A and A^M.

Lemma 2.1 asserts that $I(\overline{A}/R)$ is a free $\overline{A} = k[[t]]$ module with $\delta_R(t), \cdots, \delta_R(t^{e-1})$ as generators. Let ϕ_1 and ϕ_2 be the canonical maps from $I(\overline{A}/R)$ to $I(\overline{A}/A)$ and $I(\overline{A}/A^M)$ respectively. Since $I(\overline{A}/A)$ is generated by $\delta_A(t), \cdots, \delta_A(t^{e-1})$ (Lemma 1.1) and since $e(A^M) \leq e(A)$, it follows that ϕ_1 and ϕ_2 are both onto maps. Let $N(A)$ and $N(A^M)$ be the kernel of ϕ_1 and ϕ_2 respectively. We have then the diagram:

(*)



where clearly, $\phi_1 \theta = \phi_2$.

Both $N(A)$ and $N(A^M)$ are submodules of the free $\overline{A}$ module $I(\overline{A}/R)$ and are finitely generated.

Lemma 3.1. *With the situation as above, suppose* $\eta \in N(A^M)$, *then* $t^e\eta \in N(A)$.

Proof. If A' is the strict closure of A in $\bar{A}$, then $I(\bar{A}/A) \cong I(\bar{A}/A')$ where the $\bar{A}$-algebra isomorphism is given by the canonical morphism ψ_1 from $\bar{A} \otimes_A \bar{A}$ to $\bar{A} \otimes_{A'} \bar{A}$.

Let M' be the maximal ideal of A'. Since the blow-up of A commutes with taking the strict closure (Remark 2.3), we have $(A')^{M'} = (A^M)'$ and hence the commutative diagram:

$$\begin{array}{ccc} I(\bar{A}/A^M) & \xrightarrow[\sim]{\psi_2} & I(\bar{A}/(A^M)') \\ \uparrow \theta & & \uparrow \theta' \\ I(\bar{A}/A) & \xrightarrow[\sim]{\psi_1} & I(\bar{A}/A') \end{array}$$

where ψ_2 and θ' are the obvious canonical maps.

If $\eta = \sum_{j=1}^{e-1} \alpha_j \delta_R(t^j) \in N(A^M)$, then

$$\psi_2\phi_2(\eta) = \sum_{j=1}^{e-1} \alpha_j \delta_{(A^M)'}(t^j) = 0.$$

Since $e(A) = e(A')$ (Remark 2.3), t^e is also transversal to M' and since $(A')^{M'} = (A^M)'$, $A'^{M'}$ is strictly closed and $(A')^{M'} = \{m'/t^e : m' \in M'\}$ (Remark 2.4). But

$$\sum_{j=1}^{e-1} \alpha_j \delta_{(A^M)'}(t^j) = \sum_{j=1}^{e-1} \alpha_j \delta_{(A')^{M'}}(t^j) = 0$$

and hence

$$\sum_{j=1}^{e-1} \alpha_j \delta_{A'}(t^j) = \sum_{i=1}^{n} c_i \otimes_{A'} d_i \left(1 \otimes_{A'} \frac{m_i'}{t^e} - \frac{m_i'}{t^e} \otimes_{A'} 1\right)$$

where $m_i' \in M'$ for all i and $c_i, d_i \in \bar{A}$. Then clearly

$$(t^e)\left(\sum_{j=1}^{e-1} \alpha_j \delta_{A'}(t^j)\right) = 0.$$

Since ψ_1 is an isomorphism it follows that $t^e \sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) = 0$ and consequently, $\phi_1[t^e(\sum_{j=1}^{e-1} \alpha_j \delta_R(t^j))] = 0$. Hence, $t^e\eta \in N(A)$. Q.E.D.

Let $\eta_1, \cdots, \eta_s \in N(A^M)$ be a set of generators for $N(A^M)$. (This generating set is finite since the rank of $I(\bar{A}/R)$ is finite.) The next theorem asserts that $t^e\eta_1, \cdots, t^e\eta_s$ is then a generating set for $N(A)$. In fact, let $\eta_1, \cdots, \eta_s$ form a basis for $N(A^M)$. That is, if $\sum_{i=1}^{s} \alpha_i \eta_i = 0$ where $\alpha_i \in \bar{A}$, then necessarily $\alpha_i \eta_i = 0$ for all i. We may then claim that $t^e\eta_1, \cdots, t^e\eta_s$ is a basis for $N(A^M)$.

Theorem 3.2. *Suppose* $\eta_1, \cdots, \eta_s \in N(A^M)$ *form a basis for* $N(A^M)$. *Then the* *elements* $t^e\eta_1, \cdots, t^e\eta_s$ *form a basis for* $N(A)$.

Proof. Lemma 3.1 assures that $t^e\eta_1, \cdots, t^e\eta_s \in N(A)$. In order to show that these elements generate $N(A)$, we refer to diagram (*).

If $\xi = \Sigma_{j=1}^{e-1} \alpha_j \delta_R(t^j) \in N(A)$, then

$$\phi_1(\xi) = \sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) = 0 \quad \text{in } k[[t]] \otimes_A k[[t]].$$

Hence,

$$\xi = \sum_{i=1}^{n} (a_i \otimes_R b_i)(1 \otimes_R x_i - x_i \otimes_R 1)$$

where $a_i, b_i \in \bar{A}$ and $x_i \in M < A$. Since $t^e \in R = k[[t^e]]$,

$$\xi = t^e \left(\sum_{i=1}^{n} (a_i \otimes_R b_i) \left(1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1\right)\right).$$

Now,

$$\phi_2 \left(\sum_{i=1}^{n} (a_i \otimes_R b_i) \left(1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1\right)\right)$$

$$= \theta \cdot \phi_1 \left(\sum_{i=1}^{n} (a_i \otimes_R b_i) \left(1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1\right)\right) = 0$$

since $x_i/t^e \in A^M$. Therefore,

$$\sum_{j=1}^{n} (a_i \otimes_R b_i) \left(1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1\right) = \sum_{i=1}^{s} \beta_i \eta_i$$

where $\beta_i \in k[[t]]$ for $i = 1, \cdots, s$. Hence,

$$\xi = t^e \left(\sum_{i=1}^{s} \beta_i \eta_i \right) = \sum_{i=1}^{s} \beta_i t^e \eta_i$$

and, consequently, $t^e\eta_1, \cdots, t^e\eta_s$ generate $N(A)$.

To show that $t^e\eta_1, \cdots, t^e\eta_s$ form a basis we need only point out that the $\bar{A}$-module homomorphism p from the free $\bar{A}$ module $I(\bar{A}/R)$ to $I(\bar{A}/R)$ given by

$$p \left(\sum_{j=1}^{e-1} \alpha_j \delta_R(t^j) \right) = \sum_{j=1}^{e-1} t^e \alpha_j \delta_R(t^j)$$

is injective. Q.E.D.

It should be noted that in the case where A^M is the full power series ring $k[[t]]$, then a set of generators for $N(A)$ is $\{t^e\delta_A(t), \cdots, t^e\delta_A(t^{e-1})\}$ where $e =$

$e(A) > 1$. When A is itself equal to $k[[t]]$, then $I(\bar{A}/A) = I(\bar{A}/A^M) = 0$ and the theorem is trivial.

Corollary 3.3. *If* $z = \sum_{j=1}^{e-1} a_j \delta_A(t^j) = 0$ *in* $\bar{A} \otimes_A \bar{A}$ *where* $e = e(A) > 1$, *then* $v(a_j) \geq e$ *for all* j. *Consequently,* $a_j/t^e = \alpha_j \in k[[t]]$. *Furthermore,*

$$\sum_{j=1}^{e-1} \alpha_j \delta_{A^M}(t^j) = 0 \quad \text{in } \bar{A} \otimes_{A^M} \bar{A}.$$

Proof. Let $\sum_{j=1}^{e-1} a_j \delta_R(t^j)$ be the preimage of z under ϕ_1. The theorem asserts that

$$\sum_{j=1}^{e-1} a_j \delta_R(t^j) = t^e \left(\sum_{j=1}^{e-1} \alpha_j \delta_R(t^j) \right)$$

for some $\alpha_j \in k[[t]]$. But $I(\bar{A}/R)$ is a free module and hence, $a_j = t^e \alpha_j$ for all j. Consequently, $v(a_j) = e + v(\alpha_j) \geq e$ since $\alpha_j \in k[[t]]$.

Since $z = 0$ in $\bar{A} \otimes_A \bar{A}$ and because $t^e \in R = k[[t^e]]$ we have

$$\sum_{j=1}^{e-1} a_j \delta_R(t^j) = \sum_{i=1}^{n} (c_i \otimes_R d_i)(1 \otimes_R y_i - y_i \otimes_R 1)$$

$$= t^e \left[\sum_{i=1}^{n} (c_i \otimes_R d_i) \left(1 \otimes_R \frac{y_i}{t^e} - \frac{y_i}{t^e} \otimes_R 1\right)\right],$$

where $c_i, d_i \in \bar{A}$ and $y_i \in A$.

But $a_j = t^e \alpha_j$, and hence

$$(t^e)\left[\sum_{j=1}^{e-1} \alpha_j \delta_R(t^j)\right] = (t^e)\left[\sum_{i=1}^{n} (c_i \otimes_R d_i) \left(1 \otimes_R \frac{y_i}{t^e} - \frac{y_i}{t^e} \otimes 1\right)\right].$$

But $I(\bar{A}/R)$ is a free module (Lemma 2.1) and therefore,

$$\sum_{j=1}^{e-1} \alpha_j \delta_R(t^j) = \sum_{i=1}^{n} (c_i \otimes_R d_i) \left(1 \otimes_R \frac{y_i}{t^e} - \frac{y_i}{t^e} \otimes_R 1\right).$$

Under the mapping ϕ_2, the right side of this equation goes to zero since $y_i/t^e \in A^M$ and hence,

$$\sum_{j=1}^{e-1} \alpha_j \delta_{A^M}(t^j) = 0 \quad \text{in } \bar{A} \otimes_{A^M} \bar{A}. \quad \text{Q.E.D.}$$

We refer once more to diagram (*). Since $I(\bar{A}/R)$ is a free module over the principal ideal domain $k[[t]] = \bar{A}$, $N(A) = \ker \phi_1$ is itself free and finitely generated.

Let $\eta_i \in N(A)$, $i = 1, \cdots, s$ be a basis for $N(A)$. If $\eta_i = \sum_{j=1}^{e-1} \alpha_{ij} \delta_R(t^j)$ for each i, let (α_{ij}) be the matrix of the coefficients. It is known that a set of invariant factors of (α_{ij}) are found by considering the highest common factor of all $(k \times k)$ subdeterminants of (α_{ij}) [4, p. 92]. These invariant factors completely determine the structure of the $k[[t]]$ module $I(\bar{A}/A)$ and are unique up to units from $k[[t]]$.

Likewise, let (β_{ij}) be the matrix of the coefficients of a basis for $N(A^M) = \ker \phi_2$. We shall relate the invariant factors of (α_{ij}) to those of (β_{ij}).

Lemma 3.4. *Suppose e is the multiplicity of A and A^M is the blow-up of A. Let $\{E_1, \cdots, E_{e-1}\}$, $E_i \in k[[t]]$ be a set of invariant factors of (β_{ij}). Then $\{t^e E_1, \cdots, t^e E_{e-1}\}$ constitutes a set of invariant factors of (α_{ij}).*

Conversely, suppose $\{F_1, \cdots, F_{e-1}\}$ is a set of invariant factors of (α_{ij}) then $F_i/t^e \in k[[t]]$ for all $i = 1, \cdots, e-1$ and $\{F_1/t^e, \cdots, F_{e-1}/t^e\}$ is a set of invariant factors of (β_{ij}).

Proof. Let (β_{ij}) be the matrix of relations for $N(A^M)$. Theorem 3.2 implies that $(t^e \beta_{ij})$ is the matrix of the relations of $N(A)$. Let σ_k be the highest common factor of the $(k \times k)$ subdeterminants of (β_{ij}). We have [4, p. 92]

$$\sigma_1 = E_1, \quad \sigma_2/\sigma_1 = E_2, \quad \cdots, \quad \sigma_{e-1}/\sigma_{e-2} = E_{e-1}.$$

Hence,

$$t^e\sigma_1 = t^e E_1, \quad \frac{(t^e)^2\sigma_2}{(t^e)\sigma_1} = t^e E_2, \quad \cdots, \quad \frac{(t^e)^{e-1}\sigma_{e-1}}{(t^e)^{e-2}\sigma_{e-2}} = t^e E_{e-1}$$

is a set of invariant factors of (α_{ij}).

Conversely, Theorem 3.2 asserts that $(t^e \beta_{ij})$ is the matrix of relations of $N(A)$. Hence, $t^e E_i$ and F_i are associates for each i. The conclusion follows easily. Q.E.D.

Suppose $A = A_0 < A_1 < \cdots < A_N < \cdots < \bar{A}$ is the branch sequence of A along $\bar{M} < \bar{A}$, the maximal idea of $\bar{A}$. Let $e_i = e(A_i)$ for each i. Assume that $A_{N+1} = k[[t]]$ and $e(A_N) = e_N > 1$. Hence, the multiplicity sequence of A has the form $\{e_0, e_1, \cdots, e_N, 1, \cdots\}$.

Theorem 3.5. *The decomposition of $I(\bar{A}/A)$ as a module over the P.I.D. $\bar{A}$ depends only on the multiplicity sequence of A.*

Proof. If (α_i) is the matrix of relations for $N(A_i)$ and $\{F_1, \cdots, F_{e_i-1}\}$ constitutes a set of invariant factors of (α_i) then $I(\bar{A}/A_i) \cong \bar{A}/F_1 \oplus \cdots \oplus \bar{A}/F_{e_i-1}$ [4, p. 86], as $\bar{A}$ modules. We shall write $I(\bar{A}/A_i) \sim \{F_1, \cdots, F_{e_i-1}\}$ to mean that $\{F_1, \cdots, F_{e_i-1}\}$ is a set of invariant factors of (α_i). Since $A_{N+1} = k[[t]]$, we may proceed backwards to A by using Lemma 3.4. In fact, if $\{e_0, \cdots, e_N, 1, \cdots\}$ is the multiplicity sequence of A, write $E_i = t^{e_i}$ for $i = 0, \cdots, N$. Then using Lemma 3.4 repeatedly:

$$I(\bar{A}/A_N) \sim \{\underbrace{E_N, \cdots, E_N}_{e_N - 1}\}$$

$$I(\bar{A}/A_{N-1}) \sim \{\underbrace{E_{N-1}, \cdots, E_{N-1}}_{e_{N-1} - e_N \geq 0}, \underbrace{E_N \cdot E_{N-1}, \cdots, E_N \cdot E_{N-1}}_{e_N - 1}\}$$

$$I(\bar{A}/A_{N-2}) \sim \{\underbrace{E_{N-2}, \cdots, E_{N-2}}_{e_{N-2} - e_{N-1}}, \underbrace{E_{N-1} \cdot E_{N-2}, \cdots, E_{N-1} \cdot E_{N-2}}_{e_{N-1} - e_N},$$

$$\underbrace{E_N \cdot E_{N-1} \cdot E_{N-2}, \cdots, E_N E_{N-1} E_{N-2}}_{e_N - 1}\}$$

$$\vdots \qquad\qquad \vdots$$

$$I(\bar{A}/A) = I(\bar{A}/A_0) \sim \{\underbrace{E_0, \cdots, E_0}_{e_0 - e_1}, \underbrace{E_1 E_0, \cdots, E_1 E_0}_{e_1 - e_2},$$

$$\cdots \underbrace{E_N \cdot E_{N-1} \cdots E_0, \cdots, E_N E_{N-1} \cdots E_0}_{e_N - 1}\}$$

Note that it may happen that $e_{i+1} - e_i = 0$. Q.E.D.

Theorem 3.6. *Let* $\{e_0, e_1, \cdots, e_N, 1, \cdots\}$ *be the multiplicity sequence of* A. *Then*

$$\lambda_{\bar{A}}(I(\bar{A}/A)) = \sum_{i=0}^{\infty} e_i(e_i - 1).$$

Proof. If $I(\bar{A}/A) \sim \{F_1, \cdots, F_r\}$, then it is clear that $\lambda_{\bar{A}}(I(\bar{A}/A)) = v(F_1) + \cdots + v(F_r)$ where v is the valuation. Hence, by Theorem 3.5

$$\begin{aligned}\lambda_{\bar{A}}(I(\bar{A}/A)) &= (e_0 - e_1)v(E_0) + (e_1 - e_2)[v(E_0) + v(E_1)] \\ &\quad + \cdots + (e_N - 1)[v(E_0) + \cdots + v(E_N)] \\ &= (e_0 - 1)v(E_0) + (e_1 - 1)v(E_1) + \cdots + (e_N - 1)v(E_N)] \\ &= \sum_{i=0}^{N} e_i(e_i - 1) \quad \text{since } v(E_i) = e_i \text{ for all } i.\end{aligned}$$

Since $e_{N+k} = 1$ for $k \geq 1$, the formula holds when taking the infinite sum. Q.E.D.

We mention that if A is the complete local ring of a plane curve with only one characteristic pair, i.e. if $A = k[[t^p, t^q + a_1 t^{q+1} + \cdots]]$ where $(p, q) = 1$, then the length formula reduces to the following: $\lambda_{\overline{A}}(I(\overline{A}/A)) = (p-1)(q-1)$.

4. Comparison of $I(\overline{A}/A)$ to $I(\overline{B}/B)$. The purpose of this section is to study the relationship of the $k[[t]]$ module $I(\overline{A}/A)$ to that of $I(\overline{B}/B)$ where A and B are two arbitrary rings which satisfy the previous assumption. Namely, A and B are the complete local rings of an algebraic curve at a "one-branch singularity." We shall assume that $\overline{A} = \overline{B} = k[[t]]$ for some uniformizing parameter t (k algebraically closed). If C is any ring satisfying the above, we write I_C to mean $I(\overline{C}/C)$ since $\overline{C} = k[[t]]$ for all of these rings. We shall, for the most part, be interested in the structure of I_A as a module over $k[[t]]$ even though I_A is also an algebra over $k[[t]]$. We will mention explicitly which structure is intended.

Recall that every k automorphism σ on $k[[t]]$ is of the form $\sigma: t \longrightarrow ut$ where u is a unit in $k[[t]]$. Conversely, every mapping of the form σ is a k-automorphism on $k[[t]]$.

Definition. If $A, B \leq k[[t]]$, both complete, A and B are said to be analytically equivalent if there exists a k-automorphism σ on $k[[t]]$ so that $\sigma(A) = B$.

Let

$$A = A_0 < A_1 < A_2 < \cdots < k[[t]] \quad \text{and} \quad B = B_0 < B_1 < B_2 < \cdots < k[[t]]$$

be the branch sequence of A and B respectively. Let

$$\{e(A_0) = e(A), e(A_1), \cdots\} \quad \text{and} \quad \{e(B_0) = e(B), e(B_1), \cdots\}$$

be the multiplicity sequence of A and B respectively.

Definition. A and B are said to have the same multiplicity sequence if $e(A_i) = e(B_i)$ for every $i = 0, 1, \cdots$.

Lemma 4.1. *If A and B are analytically equivalent, then $I_A \cong I_B$ as $k[[t]]$ modules.*

Proof. If M and M' are the maximal ideals of A and B respectively, then the k-automorphism σ of $k[[t]]$ between A and B extends to one between A^M and $B^{M'}$. Hence the multiplicity sequences of A and B are the same, since $\sigma(A) = B$ implies $e(A) = e(B)$. By Theorem 3.5, the decomposition of I_A as $k[[t]]$ modules is dependent only on the multiplicity sequence of A. Hence, the decomposition of I_A and I_B are equivalent. Q.E.D.

Lemma 4.2. *Let $\overline{M}$ be the maximal ideal of $\overline{A}$, then*

$$\dim_{\overline{A}/\overline{M}}(I_A/\overline{M}I_A) = \dim_k(I_A/\overline{M}I_A) = e(A) - 1.$$

Proof. We may assume that a transversal element has been so chosen that

$x = t^e$ where $e = e(A)$ and $\overline{A} = k[[t]]$. Hence, $\overline{M} = (t)$ and we need to show

$$\dim_k(I_A/(t)I_A) = e(A) - 1.$$

Lemma 2.1 says that $\delta_A(t), \cdots, \delta_A(t^{e-1})$ generate I_A as a $k[[t]]$ module. Hence, $\overline{\delta_A(t)}, \cdots, \overline{\delta_A(t^j)}$ span the k vector space $I_A/(t)I_A$. We need to show these are independent. But if $\Sigma_{j=1}^{e-1} \overline{a}_j \overline{\delta_A(t^j)} = 0$ in $I_A/(t)I_A$ then $\Sigma_{j=1}^{e-1} a_j \delta_A(t^j) \in (t)I_A$ or, equivalently, $\Sigma_{j=1}^{e-1} \alpha_j \delta_A(t^j) \in (t)I_A$ where $\alpha_j \in k$ is the constant term of the power series a_j. Hence $\Sigma_{j=1}^{e-1} (\alpha_j - b_j)\delta_A(t^j) = 0$ where $b_j \in (t)$. But Corollary 2.3 implies $v(\alpha_j - b_j) \geq e(A)$ for all j. Hence, $\alpha_j = 0$ and $\overline{a}_j = 0$ for all j. Q.E.D.

Lemma 4.3. *If M and M^* are the maximal ideals of A and B respectively, then $I_A \cong I_B$ implies $I_{A^M} \cong I_{B^{M^*}}$. (Both isomorphisms as $k[[t]]$ isomorphisms.)*

Proof. By Lemma 4.2, $I_A \cong I_B$ implies $e(A) - 1 = e(B) - 1$ and hence $e(A) = e(B) = e$. Therefore, if t^e and $(t')^e$ are transversal parameters to M and M^* respectively, then $t^e = u(t')^e$ where u is a unit in $k[[t]]$.

Let $E_1, \cdots, E_{e-1}$ and $F_1, \cdots, F_{e-1}$ be a set of invariant factors of I_A and I_B respectively (cf. Lemma 3.4). We may assume E_i and F_i are associates and $E_i \neq 1$ for each i.

Lemma 3.4 implies that $E_1/t^e, \cdots, E_{e-1}/t^e$ and $F_1/(t')^e, \cdots, F_{e-1}/(t')^e$ are then a set of invariant factors of I_{A^M} and $I_{B^{M^*}}$ respectively. Since $t^e = u(t')^e$, E_i/t^e and $F_i/(t')^e$ are associates for each i and the conclusion follows. Q.E.D.

The converse of this is clearly false. Let $A = k[[t^2, t^3]]$ and $B = k[[t^3, t^4]]$ with maximal ideal $M = (t^2, t^3)$ and $M^* = (t^3, t^4)$ respectively. Then $A^M = B^{M^*} = k[[t]]$ and hence, $I_{A^M} = I_{B^{M^*}} = 0$. But Theorem 3.5 gives

$$I_A \cong \overline{A}/(t^2) \text{ and } I_B \cong \overline{A}/t^3 \oplus \overline{A}/t^3$$

as $k[[t]]$ modules.

Theorem 4.4. *$I_A \cong I_B$ as $k[[t]]$ modules if and only if A and B have the same multiplicity sequence.*

Proof. Let $A = A_0 < A_1 < \cdots < A_N = k[[t]]$ and $B = B_0 < B_1 < \cdots < B_{N'} = k[[t]]$ be the branch sequence of A and B respectively. Lemma 4.2 implies that if $I_A \cong I_B$, then $e(A) = e(B)$. Lemma 4.3 asserts that $I_{A_1} \cong I_{B_1}$ and hence $e(A_1) = e(B_1)$. Continuing, the result follows.

Theorem 3.5 asserts the converse. Q.E.D.

Before continuing, we need to indicate some of the geometric properties of the strict closure A' of A in $\overline{A}$. Let C be a ring so that $A < C < \overline{A} = k[[t]]$. (Our assumptions on A imply that C is necessarily local and complete.) Let $C = C_0$

$< C_1 < \cdots < C_N < \bar{A}$ be the branch sequence of C and let $e(C_i) = e_i$. The ring C is said to be an Arf ring (cf. [1]) if it satisfies any one of the following conditions.

(1) The embedding dimension of C_i is equal to the multiplicity of C_i for every i.

(2) $\lambda_C(\bar{C}/C) = \Sigma_{i=0}^{\infty} (e_i - 1)$. (Since $e_n = 1$ for n large, the formula makes sense.)

(3) The semigroup $G(C) = \{v(x)\colon x \in C\}$ has the form $G(C) = \{0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \cdots\}$.

J. Lipman in [5] shows the equivalence of the above conditions. In the same paper, he shows that if A is any ring among the collection of all Arf rings between A and $\bar{A}$, there exists one, say A^*, contained in all the others [5, p. 666]. The ring A^* is called the Arf closure of A and coincides with the strict closure A' since we assume that A contains a field k [5, p. 677]. Hence, we shall continue to denote the Arf closure A^* of A as A'.

Remark 4.5. Note that if $A < C < \bar{A}$, then $A' < C'$. Using (2), the ring A' (= strict closure of A = Arf closure of A) can be characterized as the largest ring between A and $\bar{A}$ whose multiplicity sequence is equal to A [5, p. 671]. This implies that if $A < C < \bar{A}$, then $A' = C'$ if and only if the multiplicity sequence of A is equal to the multiplicity sequence of C. Similarly, one shows by using (3) that if $A < C < \bar{A}$, then $A' = C'$ if and only if $G(C') = G(A')$.

Definition. Let $d \in G(A)$ be the least integer in $G(A)$ so that $d + j \in G(A)$ for any integer $j \geq 0$. Then d is called the degree of the conductor of A.

Theorem 4.6. *The annihilator ideal of I_A in $\bar{A} = k[[t]]$ is (t^d) where d is the degree of the conductor of A'.*

Proof. By induction on the number of blow-ups needed to "resolve the singularity". Note that if $A = k[[t]]$, then $A' = k[[t]]$ and since $d = 0$ and $I_A = 0$, the theorem holds true in this case.

Next note that for an ideal $Q \leq k[[t]]$, $QI_A = 0$ if and only if $Q\delta_A(t) = 0$. For if $x = \Sigma_{i=0}^{\infty} a_i t^i \in k[[t]]$, $\delta_A(x) = \Sigma_{i=1}^{\infty} a_i \delta(t^i)$. But

$$\delta_A(t^n) = \binom{n}{1} t^{n-1}\delta_A(t) + \binom{n}{2} t^{n-2}\delta_A^2(t) + \cdots + \delta^n(t)$$

and hence, $QI_A = 0$ if $Q\delta_A(t) = 0$. The converse is clear. Therefore, the theorem asserts that the order ideal of $\delta_A(t)$ is (t^d) where d is the degree of the conductor of A'.

Let A have the multiplicity sequence $\{e(A), e(A_1), e(A_2), \cdots, e(A_N), 1, \cdots\}$ where N is the largest integer so that $e(A_N) > 1$. By Remark 4.5, $G(A') = \{0, e_0, e_0 + e_1, \cdots\}$ where $e_i = e(A_i)$, and hence, the degree of the conductor of A' is $d = e_0 + e_1 + \cdots + e_N$. Note that A_1 has the multiplicity sequence $\{e_1,$

$e_2, \cdots, e_N, 1, \cdots\}$ and also by Remark 4.5, $G(A_1') = \{0, e_1, e_1 + e_2, \cdots\}$ so that the degree of the conductor of A_1' is $d' = e_1 + e_2 + \cdots + e_N$. By the inductive hypothesis we may assume that the order ideal of $\delta_{A_1}(t)$ is $(t^{d'})$. Now the proof of Lemma 3.1 implies that $(t^d)\delta_A(t) = 0$. If $(t^a)\delta_A(t) = 0$ for $a < d$, then Corollary 3.3 implies $a \geq e_0$ and $(t^{a-e_0})\delta_{A'}(t) = 0$ which contradicts the assumption on A_1 since $a - e_0 < d'$. Q.E.D.

We come to the main theorem of this section.

Theorem 4.7. *Let A and B be complete local rings of points on an algebraic curve at one-branch singularities and assume that $A < B < \bar{A}$. Then the following are equivalent.*

1. $I_A \cong I_B$ *as* $k[[t]] = \bar{A} = \bar{B}$ *modules.*
2. *The multiplicity sequence of A is eaual to the multiplicity sequence of* B.
3. $A' = B'$.
4. $I_A \cong I_B$ *as $\bar{A}$-algebras.*
5. $D^n(\bar{A}/A) \cong D^n(\bar{B}/B)$ *as $\bar{A}$-algebras for all n.*
6. $G(A') = G(B')$.

Proof. (1) implies (2) by Theorem 4.4, (2) implies (3) by Remark 4.5. To show that (3) implies (4), we consider the canonical isomorphism ψ from $\bar{A} \otimes_A \bar{A}$ to $\bar{A} \otimes_B \bar{A}$. (By assumption, $\bar{A} \otimes_{A'} \bar{A} = \bar{A} \otimes_{B'} \bar{A}$.) From the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_B & \longrightarrow & \bar{A} \otimes_B \bar{A} & \longrightarrow & \bar{A} & \longrightarrow & 0 \\
 & & \uparrow \psi/I_A & & \uparrow \psi & & \uparrow & & \\
0 & \longrightarrow & I_A & \longrightarrow & \bar{A} \otimes_A \bar{A} & \longrightarrow & \bar{A} & \longrightarrow & 0
\end{array}$$

it is clear that the restriction of ψ gives the desired algebra isomorphism.

Clearly, (4) implies (1).

Since by Theorem 1.1, $D^n(\bar{A}/A) = I(\bar{A}/A)$ and $D^n(\bar{B}/B) = I(\bar{B}/B)$ for $n >> 0$, (4) is equivalent to (5).

(6) is equivalent to (3) by Remark 4.5. Q.E.D.

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