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A KUROSH SUBGROUP THEOREM FOR FREE PRO-$\mathcal{C}$-PRODUCTS OF PRO-$\mathcal{C}$-GROUPS

BY

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Dedicated to the memory of A. G. Kurosh

ABSTRACT. Let $\mathcal{C}$ be a class of finite groups, closed under finite products, subgroups and homomorphic images. In this paper we define and study free pro-$\mathcal{C}$-products of pro-$\mathcal{C}$-groups indexed by a pointed topological space. Our main result is a structure theorem for open subgroups of such free products along the lines of the Kurosh subgroup theorem for abstract groups. As a consequence we obtain that open subgroups of free pro-$\mathcal{C}$-groups on a pointed topological space, are free pro-$\mathcal{C}$-groups on (compact, totally disconnected) pointed topological spaces.

0. Introduction. Let $\mathcal{C}$ be a class of finite groups, closed under the formation of subgroups, finite direct products and homomorphic images. It is natural (and useful) to look at free pro-$\mathcal{C}$-groups generated by pointed topological spaces (see Gildenhuys and Lim [3]). On the other hand, the concept of a free pro-$\mathcal{C}$-group generated by an infinite set has not proved to be a very useful one. For one thing, such a free pro-$\mathcal{C}$-group has very large cardinality (loc. cit). The categorical concept of a coproduct of an infinite family of pro-$\mathcal{C}$-groups has not proved to be very useful either. Neukirch [11] has defined "corestringierte" free pro-$\mathcal{C}$-products of families of pro-$\mathcal{C}$-groups, and has applied his concept to algebraic number theory. In this paper we extend his notion, by defining the free pro-$\mathcal{C}$-product of a family $\{A_x | x \in X\}$ of pro-$\mathcal{C}$-groups, indexed by a pointed topological space $(X,*)$, where it is assumed that the map $x \mapsto A_x$ is locally constant outside $\{*\}$ and $A_* = (1)$ (Definition 1.2, Proposition 1.5).

Our main result is a "Kurosh subgroup theorem" for open subgroups of free pro-$\mathcal{C}$-products over pointed compact Hausdorff totally disconnected index spaces X, where it is assumed that $\mathcal{C}$ is also closed under group extension (Theorem 4.1). Our proof is based upon Mac Lane's proof of the Kurosh subgroup theorem for discrete groups (Kurosh [7, p. 147]). When applied to the special case where X is the one-point compactification of a discrete space, our theorem gives somewhat

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more information than a similar result(1) by Binz, Neukirch and Wenzel [1] (Theorem 5.3). Like Binz, Neukirch and Wenzel, we also look at the case where X is discrete, in which case our free pro-$\mathcal{C}$-product coincides with the coproduct in the category of pro-$\mathcal{C}$-groups. As a consequence of our main result (Theorem 4.1), we prove that every open subgroup of a free pro-$\mathcal{C}$-group on a pointed topological space X is freely generated by a compact Hausdorff totally disconnected pointed space Y (Theorem 5.1). If X is a one-point compactification of a discrete space, then so is Y (Corollary 5.4; see also Binz, Neukirch and Wenzel [1]).

1. **Definitions and terminology.** Let $(X, *)$ be a pointed Hausdorff topological space. Let $\{A_x | x \in X\}$ be a set of pro-$\mathcal{C}$-groups indexed by X, such that A_* is the trivial group (1), and the mapping $X\backslash\{*\} \to \{A_x | x \in X\}$ given by $x \mapsto A_x$ is locally constant, i.e. for all $x \in X\backslash\{*\}$ there exists a so-called *constant open* neighborhood U of x in $X\backslash\{*\}$, with $A_y = A_x$ for all $y \in U$. Our aim in this section is to define the "free product" of the pro-$\mathcal{C}$-groups A_x, $x \in X$, and describe its essential properties. First, we define a pointed topological space $E = \bigvee_{x \in X} A_x$, called the *étale space*. As a pointed set, E is the coproduct of the pointed sets $(A_x, 1)$, $x \in X$, i.e. E is the set obtained by forming the disjoint union of the sets A_x, and then identifying the identity elements of the groups A_x. If N is a constant open subset of X ($A_x = A_y$ for all $x, y \in N$) and $x \in N$, we define

$$p_N: N \times A_x \to E, \qquad (n, t) \mapsto t \in A_n, \qquad (n, t) \in N \times A_x,$$

and we endow E with the topology defined by letting $W \subset E$ be open in E if the following conditions are satisfied: (i) for every constant open subset N of X, the set $p_N^{-1}(W)$ is open with respect to the product topology on $N \times A_x$ ($x \in N$); (ii) if W contains the distinguished point 1 of E, there is a neighborhood V of $*$ in X such that $A_y \subset W$ whenever $y \in V$. Note that if $1 \neq \alpha \in E$, the sets $p_{V^x}(V^x \times U)$ form a neighborhood basis of α in E, where V^x runs through the constant open neighborhoods of x in X, missing $*$, and U runs through the open neighborhoods of α in A_x, missing 1. A neighborhood basis for $1 \in E$ is given by the sets

$$\left(\bigcup_{x \in X\backslash\{*\}} p_{V^x}(V^x \times U^x)\right) \cup \left(\bigcup_{x \in V^*} A_x\right),$$

where V^* runs through the open neighborhoods of $*$ in X, V^x runs through the constant open neighborhoods of x in X missing $*$, and U^x runs through the open neighborhoods of 1 in A_x.

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(1) In fact in a subsequent revision of their paper, they obtain precisely this result (our Theorem 5,3) completely. Cf. E. Binz. J. Neukirch and G. H. Wenzel, J. Algebra 19 (1971), 104–109.

Consider the map

$$\pi\colon E = \bigvee_{x\in X} A_x \to X$$

of pointed spaces, given by $\pi(1) = *$, and $\pi(e) = x$ if $e \in E$ and $e \in A_x \backslash \{1\}$. It is easily verified that π is continuous and open. The pro-$\mathcal{C}$-group $A_x = \pi^{-1}(\{x, *\})$ is called the *fiber above* x.

Lemma 1.1. *If* X *is compact, so is* E.

Proof. Assume X is compact. Let $\{O_\gamma\}_{\gamma\in\Gamma}$ be an open cover of E. Suppose $1 \in O_{\gamma_0}$. For each $x \in X$, $x \neq *$ let $\Gamma_x = \{\gamma \in \Gamma | O_\gamma \cap (A_x \backslash \{1\}) \neq \emptyset\}$. For each $\gamma \in \Gamma_x$ and each $a_\gamma \in O_\gamma \cap (A_x \backslash \{1\})$ choose an open neighborhood U_γ of a_γ in A_x and a constant open neighborhood W_γ of x in X such that $p_{W_\gamma}(W_\gamma \times U_\gamma) \subset O_\gamma$. The family consisting of $O_{\gamma_0} \cap A_x$ and U_γ $(\gamma \in \Gamma_x)$ forms an open cover of A_x. Let $O_{\gamma_0} \cap A_x$, $U_{\gamma_1}, \cdots, U_{\gamma_{n_x}}$ be a finite subcover. Put $V_x = \bigcap_{i=1}^{n_x} W_{\gamma_i}$ and $S_x = \{\gamma_1, \cdots, \gamma_{n_x}\}$. Suppose $\bigcup_{x\in N} A_x \subset O_{\gamma_0}$, where N is an open neighborhood of $*$. The family consisting of N and the open sets V_x $(* \neq x \in X)$ forms an open cover for X. Let $N, V_{x_1}, \cdots, V_{x_m}$ be a finite subcover. Then $\{O_{\gamma_0}\} \cup \{O_\gamma | \gamma \in \bigcup_{i=1}^m S_{x_i}\}$ is easily seen to be a cover of E. □

Definition 1.2. Let X be any pointed Hausdorff topological space, and let $\{A_x | x \in X\}$ and $E = \bigvee_{x\in X} A_x$ be as before. A pair (A, ϕ) consisting of a pro-$\mathcal{C}$-group $A = \coprod_{x\in X} A_x$, and a continuous mapping $\phi\colon E \to A$ such that $\phi | A_x$ is a monomorphism for every $x \in X$, is said to be a *free pro-$\mathcal{C}$-product of the groups* A_x *over the index space* X, if whenever B is a pro-$\mathcal{C}$-group and $\psi\colon E \to B$ is continuous with each $\psi | A_x$ a homomorphism, there is a unique continuous homomorphism $\bar{\psi}\colon A \to B$, such that $\bar{\psi}\phi = \psi$.

Proposition 1.3. *The free pro-$\mathcal{C}$-product* (A, ϕ) *of* $\{A_x | x \in X\}$ *always exists and is unique in the obvious sense. The subgroup of* A, *generated algebraically by the set* $\phi(E)$, *is dense in* A.

Proof. Let $A^d = \coprod_{x\in X}^d A_x$ be the coproduct (or free product) of the underlying discrete groups A_x $(x \in X)$ in the category of discrete groups. Let $f\colon E = \bigvee_{x\in X} A_x \to A^d$ be such that $f | A_x\colon A_x \to B$ is the natural inclusion.

Consider the topological group structure on A^d given by the following fundamental system of open neighborhoods of 1:

$$\mathfrak{N} = \{N \lhd A^d \mid A^d/N \in \mathcal{C},\ f^{-1}(bN) \text{ is open in } E, \forall b \in A^d\}$$

Let $A = \varprojlim_{N\in\mathfrak{N}} A^d/N$, and let $\bar{f}\colon A^d \to A$ be the natural continuous homomorphism. Put $\phi = \bar{f} \circ f$. Clearly $\bar{f}(A^d)$ is dense in A. One easily verifies that the pair (A, ϕ) is a free pro-$\mathcal{C}$-product of the A_x's. The uniqueness is obvious. □

To justify our definition of free product, we shall prove that it includes, as special cases, the concepts of free pro-$\mathcal{C}$-group and free pro-C-product as defined previously in the literature (see Binz, Neukirch and Wenzel [1], Gildenhuys and Lim [3], Gruenberg [5], Iwasawa [6] and Neukirch [11]).

First, we note that if $(X, *)$ is a discrete pointed space, then the free product $\coprod_{x\in X} A_x$, as we have defined it, coincides with the coproduct of the A_x's in the category of pro-$\mathcal{C}$-groups (where we can let x range over the set X or the set $X\setminus\{*\}$).

Let $\tilde{F}(X, *)$ be the free pro-$\mathcal{C}$-group on a pointed topological Hausdorff space $(X, *)$ (cf. Gildenhuys and Lim [3]), and let $\hat{\underset{\sim}{Z}}_{\mathcal{C}}$ denote the free pro-$\mathcal{C}$-group on one generator. Then one would expect that $\tilde{F}(X, *)$ is the free pro-$\mathcal{C}$-product of the groups A_x, $x \in X$, where $A_* = 1$ and $A_x = \hat{Z}_{\mathcal{C}}$ if $x \in X\setminus\{*\}$. In fact this is a clear consequence of the following result.

Proposition 1.4. *Let $(X, *)$ and $(Y, *)$ be pointed Hausdorff topological spaces, with Y compact (by abuse of notation, $*$ denotes the distinguished points of both X and Y). Let $X \circ Y$ be the quotient $X \times Y/\sim$ of the product of X and Y, where $(x, y) \sim (x', y')$ if $x = *$ and $y' = *$, or $x' = *$ and $y = *$. The equivalence class $[(*, *)]$ of $(*, *)$ is taken as the distinguished point of $X \circ Y$, and is also denoted by $*$. Suppose now that $A_x = \tilde{F}(Y, *)$ for all $x \in X\setminus\{*\}$ and $A_* = (1)$. Then $\coprod_{x\in X} A_x = \tilde{F}(X \circ Y, *)$.*

Proof. Let $\mu_x: (Y, *) \to A_x = \tilde{F}(Y, *)$ be the canonical maps, $x \in X\setminus\{*\}$, with $\mu_*: (Y, *) \to \tilde{F}(Y, *)$ the trivial map. Let μ be the composite map:

$$(X \circ Y, *) \xrightarrow{\nu} E = \bigvee_{x\in X} A_x \xrightarrow{\phi} \coprod_{(X,*)} A_x$$

where ϕ is canonical and $\nu[(x, y)] = \mu_x(y)$. We refer to the description of a neighborhood basis for a point in E and, using the notation introduced there, we note that $\nu^{-1}(p_{V^x}(V^x \times U))$ is the image in $X \circ Y$ of the open subset $V^x \times (\mu_x^{-1}(U))$ of $X \times Y$ $(* \neq x \in X)$ and $\nu^{-1}(\bigcup_{x\in V^*} A_x)$ is the image in $X \circ Y$ of the open subset $V^* \times Y$ of $X \times Y$. It follows that ν and μ are continuous. Let σ: $X \circ Y \to G$ be a continuous mapping of $X \circ Y$ into a pro-$\mathcal{C}$-group G, with $\sigma(*) = 1$. It suffices to show that there exists a unique continuous homomorphism $\bar{\psi}$: $\coprod_{x\in X} A_x \to G$, such that $\bar{\psi}\mu = \sigma$ (Gildenhuys and Lim [3]). For each $x \in X\setminus\{*\}$, let $\sigma_x: (Y, *) \to G$ be given by $\sigma_x(y) = \sigma[(x, y)]$. Then σ_x is continuous, with $\sigma_x(*) = 1$, and so there is a unique morphism $\psi_x: F(Y, *) = A_x \to G$ such that $\psi_x\mu_x = \sigma_x$. We claim that the map $\psi: E = \bigvee_{x\in X} A_x \to G$ defined by $\psi \mid A_x = \psi_x$ is continuous. For this, it suffices to show that for every open normal subgroup W of G, the composite $\pi_W\psi$ is continuous, where $\pi_W: G \to G/W$ is canonical. Let $\alpha \in A_{x_0} \subset E$. For each $y \in Y$, let V_y be an open neighborhood of y in Y, and let $U^y_{x_0}$ be a constant open neighborhood of x_0 in X, missing $*$ if $x_0 \neq *$, and such that

$$\pi_W \sigma[(x, t)] = \pi_W \sigma[(x_0, y)]$$

for every $(x, t) \in U^y_{x_0} \times V_y$. Let $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$ cover Y, and put $U_{x_0} = \bigcap_{i=1}^n U^{y_i}_{x_0}$. Then

$$\pi_W \sigma[(x, y)] = \pi_W \sigma[(x_0, y)]$$

for all $(x, y) \in U_{x_0} \times Y$. Thus,

$$\pi_W \psi_x = \pi_W \psi_{x'}: A_x = A_{x'} \longrightarrow G/W \text{ if } x, x' \in U_{x_0}.$$

Let $\bar{g} = \pi_W \psi_{x_0}(\alpha) \in G/W$. Then there is an open neighborhood T_{x_0} of α in A_{x_0} such that

$$\pi_W \psi(p_{U_{x_0}}(U_{x_0} \times T_{x_0})) = \{\bar{g}\},$$

where, as before, $p_{U_{x_0}}: U_{x_0} \times A_{x_0} \longrightarrow E$ is given by $p_{U_{x_0}}(x, t) = t \in A_x$. If $\alpha \neq 1$, then $p_{U_{x_0}}(U_{x_0} \times T_{x_0})$ is an open neighborhood of α in E, and the above shows that $\pi_W \psi$ is continuous at α. If $\alpha = 1$, choose U_{x_0} and T_{x_0} as above for every $x_0 \in X \setminus \{*\}$, and let U_* be an open neighborhood of $* \in X$ such that $\pi_W \sigma[(x, y)] = 1$, whenever $(x, y) \in U_* \times Y$. Then $\pi_W \psi(A_x) = \{1\}$ if $x \in U_*$. Thus, the open neighborhood.

$$S = \left(\bigcup_{x_0 \in X \setminus \{*\}} p_{U_{x_0}}(U_{x_0} \times T_{x_0}) \right) \cup \left(\bigcup_{x \in U_*} A_x \right)$$

of 1 in E is mapped by $\pi_W \psi$ onto the identity element of G/W. So $\pi_W \psi$ is also continuous at $1 \in E$. Since ψ is continuous and each ψ_x is a homomorphism, there is a unique continuous homomorphism $\bar{\psi}: \coprod_{x \in X} A_x \longrightarrow G$ with $\bar{\psi} \phi = \psi$. Then it is plain that $\bar{\psi} \mu = \sigma$ and $\bar{\psi}$ is unique with this property. □

Next, we consider free products in the sense of Neukirch [11] and Binz, Neukirch and Wenzel [1]. First we recall their definition. If A_x, $x \in X$ are pro-$\mathcal{C}$-groups indexed by a set X, one says that a system (A, ϕ_x) consisting of a pro-$\mathcal{C}$-group A and of maps $\phi_x: A_x \longrightarrow A$ $(x \in X)$ converging to 1 (i.e., every open subgroup of A contains all but a finite number of the groups $\phi_x(A_x)$) is a *restricted free pro-$\mathcal{C}$-product* of the A_x's, if whenever G is a pro-$\mathcal{C}$-group and $\psi_x: A_x \longrightarrow G$ $(x \in X)$, is a family of continuous homomorphisms converging to 1, there is a unique continuous homomorphism $\bar{\psi}: A \longrightarrow G$ with $\bar{\psi} \phi_x = \psi_x$ for all $x \in X$.

A restricted free pro-$\mathcal{C}$-group F on a set X (cf. Serre [5] or Ribes [13] where it is called simply a free pro-$\mathcal{C}$-group) is easily seen to be the restricted free pro-$\mathcal{C}$-product of X copies of $\hat{Z}_{\mathcal{C}}$, the free pro-$\mathcal{C}$-group on one generator (cf. Neukirch [10]).

Proposition 1.5. *Let A_x, $x \in X$ be pro-$\mathcal{C}$-groups indexed by a set X. Let $\overline{X} = X \cup \{*\}$ be the one-point compactification of the discrete space X. Put $A_* = 1$. Then the free pro-$\mathcal{C}$-product $\coprod_{x \in \overline{X}} A_x$ coincides with the restricted free pro-$\mathcal{C}$-product of the groups A_x, $x \in X$.*

Proof. The result easily follows from the fact that a mapping ϕ: $E = \bigvee_{x \in \overline{X}} A_x \to G$, with each $\phi_x = \phi | A_x$ a continuous homomorphism, is itself continuous iff $\{\phi_x\}_{x \in X}$ converges to 1. □

To conclude this section, we wish to point out to those readers acquainted with the concept of a category object in a category with pullbacks, that our concept of free product can also be described formally as a "colimit" of a "functor" whose "domain" is a category object "without maps" in the category of pointed topological Hausdorff spaces, and which takes values in the category of pro-$\mathcal{C}$-groups.(2)

2. The structure of free products over compact Hausdorff totally disconnected index spaces. In this section we shall restrict ourselves to free pro-$\mathcal{C}$-products $A = \coprod_{x \in X} A_x$, where X is compact Hausdorff and totally disconnected. Our purpose is to express A as a projective limit of free pro-$\mathcal{C}$-products over finite discrete spaces. First, we introduce some notation. If R is an equivalence relation on a set T, then tR will denote the equivalence class of $t \in T$. Let $\{A_x | x \in X\}$ be pro-$\mathcal{C}$-groups indexed by X and satisfying the conditions of §1. Let R be an equivalence relation on X such that (i) X/R is a finite discrete space with the quotient topology, and (ii) $A_x = A_y$ whenever xRy and $x \notin *R$. For $x \in X$, put $A_{xR} = A_x$ if $x \notin *R$ and $A_{xR} = 1$ if $xR*$. Let $E = \bigvee_{x \in X} A_x$ be as in §1, and define a function

$$\rho_R: E \to \coprod_{xR \in X/R} A_{xR} = A_R$$

by sending A_x identically onto A_{xR} if $x \notin *R$ and onto 1 if $xR*$. Then ρ_R is continuous. To see this, let $\alpha \in A_x \subset E$, and let W be an open neighborhood of $\rho_R(\alpha)$. We choose a neighborhood V_α^x of α in A_x such that $\rho_R(V_\alpha^x) \subset A_{xR} \cap W$ and, in addition, $1 \notin V_\alpha^x$ if $\alpha \neq 1$. If $x \neq *$, we also choose a constant open neighborhood U_x of x in X such that $* \notin U_x$. If $\alpha \neq 1$, then $p_{U_x}(U_x \times V_\alpha^x)$ is an open neighborhood of α in E, and

$$\rho_R(p_{U_x}(U_x \times V_\alpha^x)) \subset W.$$

If $\alpha = 1$, we choose V_α^x and U_x as above for each $x \in X\backslash\{*\}$. Then ρ_R maps the open neighborhood

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(2) Cf. D. Gildenhuys and L. Ribes, *On the cohomology of certain topological colimits of pro-$\mathcal{C}$-groups*, J. Algebra (to appear).

$$\left(\bigcup_{x\in X\setminus\{*\}} p_{U_x}(U_x \times V_\alpha^x)\right) \cup \left(\bigcup_{x\in *R} A_x\right)$$

of $\alpha = 1$ into W. Thus, ρ_R is continuous at all $\alpha \in E$.

By the universal property of $A = \coprod_{x\in X} A_x$, there is then a unique continuous homomorphism

$$\pi_R : A \longrightarrow \coprod_{xR\in X/R} A_{xR} = A_R$$

with $\pi_R\phi = \rho_R$, where ϕ is as in Definition 1.2.

If R and R' are equivalence relations on X satisfying conditions (i) and (ii) above, and $R \subset R'$, then define

$$\pi_{RR'} : A_R = \coprod_{xR\in X/R} A_{xR} \longrightarrow A_{R'} = \coprod_{xR'\in X/R'} A_{xR'}$$

to be the unique continuous homomorphism induced by the maps

$$A_{xR} \xrightarrow{\eta_{xR}} A_{xR'} \xrightarrow{\phi_{xR'}} A_{R'},$$

where η_{xR} is the identity mapping if $x \notin *R'$ and the trivial mapping otherwise, and where the $\phi_{xR'}$'s are the canonical monomorphisms.

Proposition 2.1. *Let* $(X, *)$, $\{A_x \mid x \in X\}$ *and* $A = \coprod_{x\in X} A_x$ *be as above. Let* H *be an open subgroup of* A. *Consider the set* $\mathfrak{R}$ *of equivalence relations* R *on* X *satisfying conditions* (i) *and* (ii) *above, as well as the following*: (iii) *the canonical projection* $\pi_R : A \longrightarrow A_R = \coprod_{x\in X/R} A_{xR}$ *described above, preserves the index of* H, *i.e.*

$$(A : H) = (A_R : \pi_R(H)).$$

Then, $A = \varprojlim_{R\in\mathfrak{R}} A_R$.

Proof. Denote by H_A the open normal subgroup $\bigcap_{a\in A} H^a$ of A (here $H^a = a^{-1}Ha$). Let V be an open normal subgroup of A; put $W = W_V = V \cap H_A$, and let $q_W : A \longrightarrow A/W$ be the canonical homomorphism. Consider the equivalence relation R_V on X characterized by: if $x \in X$, then xR_V* when $q_W(A_x) = \{1\}$; if $x, y \in X$ and $x \neq * \neq y$, we have $xRVy$ iff $A_x = A_y$ and $q_W(a_x) = q_W(a_y)$ whenever $a_x \in A_x$, $a_y \in A_y$ and $a_x = a_y$ under the equality $A_x = A_y$. We proceed to show that R_V satisfies conditions (i), (ii) and (iii). Since X is compact, (i) is satisfied if R_V is open, i.e., if every xR_Vy is open. Assume $x \neq *$; then for every $\alpha \in A/W$ consider the open and closed subset $O_\alpha = \phi^{-1}(q_W^{-1}(\alpha)) \cap E$, where ϕ: $E \longrightarrow A$ is as in Definition 1.2. Let $\pi: E \longrightarrow X$ be the projection (see §1). For every $a \in A_x \cap O_\alpha$ let $U_{x,\alpha}^a$ be a constant open neighborhood of x in X

missing $*$ and let $D^a_{x,\alpha}$ be an open neighborhood of a in A_x with $p_U(U^a_{x,\alpha} \times D^a_{x,\alpha}) \subset O_\alpha$; let $\{D^{a_i}_{x,\alpha}\}_{i=1}^{n_\alpha}$ be a finite cover of $A_x \cap O_\alpha$, and set $U_x = \bigcap_\alpha \bigcap_{i=1}^{n_\alpha} U^{a_i}_{x,\alpha}$, where α runs through those elements of A/W for which $A_x \cap O_\alpha \neq \emptyset$. Clearly U_x is a constant open neighborhood of x. Moreover if $y \in U_x$, $a_y \in A_y$, $a_x \in A_x$ and $a_y = a_x$ under the equality $A_y = A_x$, then a_x belongs to some $D^{a_i}_{x,\alpha}$ and

$$a_y \in p_{U^{a_i}_{x,\alpha}}(U^{a_i}_{x,\alpha} \times D^{a_i}_{x,\alpha}),$$

hence $q_W(a_y) = q_W(a_x) = \alpha$. Therefore, yRx, and every $x \neq *$ is an interior point of xR. Assume now that $x = *$. Since $q_W^{-1}(1)$ is an open neighborhood of 1 in E, there is an open neighborhood of $*$ in X, such that $x \in U_* \Rightarrow A_x \subset q_W^{-1}(1)$. Then $x \in U_* \Rightarrow xR*$, and hence $*$ is an interior point of $*R$. This establishes condition (i). Obviously R_V satisfies condition (ii). To verify condition (iii) consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad q_W \quad} & A/W \\ & {\scriptstyle \pi_{R_V}} \searrow \quad \nearrow {\scriptstyle \psi^V} & \\ & A_{R_V} = \coprod_{x \in X/R} A_{xR_V} & \end{array}$$

where ψ^V is the continuous homomorphism induced by the homomorphism ψ_x: $A_{xR_V} \to A/W$, given by $\psi_x(a) = q_W(a)$ $(a \in A_{xR_V} = A_x$ if $x \notin *R_V)$. Then $\ker(\pi_{R_V}) \subset \ker(q_W) = W \subset H$. Thus, $(A:H) = (A_{R_V}:\pi_{R_V}(H))$ and $R_V \in \mathfrak{R}$.

Now, the maps $\{\psi^V | V \lhd A,\ V \text{ open in } A\}$ are easily seen to induce a continuous homomorphism:

$$\varprojlim_V A_{R_V} \longrightarrow \varprojlim_V A/W_V = A$$

and hence a continuous homomorphism

$$\eta\colon \varprojlim_{R \in \mathfrak{R}} A_R \longrightarrow A.$$

Now it is easily checked that η and the homomorphism $A \to \varprojlim_{R \in \mathfrak{R}} A_R$, induced by the π_R's, $R \in \mathfrak{R}$, are inverse isomorphisms. □

Remark 2.2. Let $\mathfrak{R}$ be as in Proposition 2.1. Using the notation of the proof of Proposition 1.3, we define

$$\pi_R^d\colon \coprod_{x \in X}^{d} A_x \longrightarrow \coprod_{xR \in X/R}^{d} A_{xR}$$

to be the homomorphism that sends A_x identically onto A_{xR} if $* \notin xR$, and onto 1 otherwise. One then has a commutative diagram

$$\begin{array}{ccc} A^d = \coprod^d_{x \in X} A_x & \xrightarrow{\quad \overline{f} \quad} & A = \varprojlim_{R \in \mathfrak{R}} A_R \\ \Big\downarrow \pi^d_R & & \Big\downarrow \pi_R \\ A^d_R = \coprod^d_{xR \in X/R} A_{xR} & \xrightarrow{\overline{f}_R} & A_R = \coprod_{xR \in X/R} A_{xR} \end{array}$$

in which the maps $\overline{f}$ and $\overline{f}_R$ are defined as in the proof of Proposition 1.3.

The following corollary extends a result of Ribes [14].

Corollary 2.3. *Let $(X, *)$ be a pointed, compact, totally disconnected, Hausdorff topological space. Let C be closed under group extensions and let $\{A_x \mid x \in X\}$ be pro-$\mathcal{C}$-groups satisfying the conditions of §1. Then the canonical homomorphism*

$$\overline{f}: A^d = \coprod_{x \in X} A_x \longrightarrow A = \coprod_{x \in X} A_x$$

defined in Proposition 1.3 is a monomorphism.

Proof. Assume for a moment that the result holds true when the groups A_x are indexed by a finite discrete space. Let $1 \neq a \in \coprod^d_{x \in X} A_x$, and assume $x_1, x_2, \cdots, x_n \in X$, $x_i \neq *$, are the indexes involved in a when written as a reduced word. Choose $R \in \mathfrak{R}$ (where $\mathfrak{R}$ is as in Proposition 2.1) so that $x_i \notin x_j R$, if $i \neq j$, and $x_i \notin *R$, $i = 1, 2, \cdots, n$. Using the notation of the above remark we have $\pi^d_R(a) \neq 1$ and thus $\overline{f}(\pi^d_R(a)) \neq 1$ by our assumption. Hence $\overline{f}(a) \neq 1$.

Now we prove our result when $(X, *)$ is finite discrete. Since $\mathcal{C}$ is closed under formations of subgroups and extensions, if a prime p divides the order of some group in $\mathcal{C}$, then $\mathcal{C}$ contains all finite p-groups. Hence all free discrete groups are residually $\mathcal{C}$ (cf. Hall [12]). Then, using a slight variation of an argument of Gruenberg (cf. [4, Theorem 4.1]), one shows that $\bigcap_{N \in \mathfrak{N}} N = 1$, where $\mathfrak{N}$ is as in Proposition 1.3. Thus $\ker \overline{f} = 1$. □

Remark 2.4. The last part of the above argument in fact shows that if $(X, *)$ is discrete, not necessarily finite, then $\coprod^d_{x \in X} A_x$ is canonically embedded in $\coprod_{x \in X} A_x$.

3. **The subgroup theorem for free products over discrete spaces.** The following lemma plays an important role.

Lemma 3.1. *Let $\mathcal{C}$ be a class of finite groups, closed under the formation of subgroups and extensions. Let H be a subgroup of a group G, let $H_G = \bigcap_{g \in G} H^g$ denote the core of H in G and suppose that $G/H_G \in \mathcal{C}$. Then every normal sub-*

group N of H, with $H/N \in \mathcal{C}$, contains a normal subgroup M of G, such that $G/M \in \mathcal{C}$.

Proof.(3) Put $K = H_G \cap N$; then $H/K \subset (H/H_G) \times (H/N)$, and hence $H/K \in \mathcal{C}$. Choose $g_1, g_2, \cdots, g_r \in G$ so that $K_G = \bigcap_{i=1}^r K^{g_i}$. Then $K^{g_i} \lhd H_G$ and $H_G/K^{g_i} \in \mathcal{C}$. Therefore $H_G/K_G \subset (H_G/K^{g_i}) \times \cdots \times (H_G/K^{g_r})$; and hence $H_G/K_G \in \mathcal{C}$. Thus the extension G/K_G of H_G/K_G by G/H_G belongs to $\mathcal{C}$, and we can take $M = K_G$. □

Theorem 3.2. *Let $\mathcal{C}$ be a class of finite groups, closed under the formation of subgroups, homomorphic images and extensions. Let $\{A_x \mid x \in X\}$ be pro-$\mathcal{C}$-groups, indexed by a discrete space X, and let $A = \coprod_{x \in X} A_x$ be their coproduct in the category of pro-$\mathcal{C}$-groups (i.e. A is the free pro-$\mathcal{C}$-product $\coprod_{y \in Y} A_y$, where $Y = X \cup \{*\}$, and $A_* = 1$). Let H be an open subgroup of A. Then*

$$H = \left(\coprod_{x \in X,\, i \in I_x} (H \cap A_x^{s_{x,i}})\right) \coprod F$$

where F is a free pro-$\mathcal{C}$-group on a discrete space, and $\{s_{x,i}, i \in I_x\}$ runs for each $x \in X$, through a complete and irredundant system of double coset representatives of A with respect to A_x and H.

Proof. Let $A^d = \coprod_{x \in X}^d A_x$ be the discrete free product of the A_x's considered as discrete groups. We shall identify A^d with its canonical image in A (see Remark 2.4). Put $H^d = H \cap A^d$. By the Kurosh subgroup theorem (cf. Kurosh [7], [8]) we have

$$H^d = \left(\coprod_{x \in X,\, i \in I_x} (H^d \cap A_x^{s_{x,i}})\right) \coprod F^d$$

where F^d is a free discrete group, and $\{s_{x,i}, i \in I\}$ runs through a complete and irredundant set of double coset representatives of A^d with respect to A_x and H^d. Notice that for each $x \in X$, $\{s_{x,i} \mid i \in I_x\}$ forms also a complete and irredundant set of double coset representatives of A with respect to A_x and H; indeed $A = \bigcup_{i \in I_x} A_x s_{x,i} H$, and if $A_x s_{x,i} H = A_x s_{x,j} H$, then $s_{x,j} = a s_{x,i} b$ for some $a \in A_x$, $b \in H$, and it follows that $b \in A^d \cap H = H^d$ and $s_{x,i} = s_{x,j}$. (Note that I_x is finite for all $x \in X$.)

We shall consider two topological group structures on the group H^d. One, denoted by τ^i, is induced by the topology on A^d (cf. Proposition 1.3); and another, denoted τ^n, is determined by a fundamental system of neighborhoods of the iden-

(3) We thank J. Poland for this proof which is simpler than our original one.

tity, consisting of normal subgroups N of H^d, such that

(i) $H^d/N \in \mathcal{C}$;

(ii) $N \cap (H^d \cap A_x^{s_{x,i}}) = N \cap A_x^{s_{x,i}}$ is open in $H^d \cap A_x^{s_{x,i}} = H \cap A_x^{s_{x,i}}$ for all $i \in I_x$, $x \in X$, where the topology on $H \cap A_x^{s_{x,i}}$ is induced by H (and also by A_x);

(iii) $N \cap F^d$ is open with respect to the natural pro-$\mathcal{C}$-topology on F^d ($M \lhd F^d$ is open iff $F^d/M \in \mathcal{C}$).

We know that H is the completion of H^d endowed with the topology τ^i (cf. Bourbaki [2, p. 224]). Looking back at the proof of Proposition 1.3, we see that our result will follow if we can prove that $\tau^i = \tau^n$. Let M be an open normal subgroup of A^d (cf. the proof of Proposition 1.3), and $N = M \cap H^d$. Then H^d/N is isomorphic to a subgroup of A^d/M, hence is in $\mathcal{C}$. Also,

$$N \cap (H^d \cap A_x^{s_{x,i}}) = M \cap A_x^{s_{x,i}}$$

is open in $A_x^{s_{x,i}}$, and hence in $H^d \cap A_x^{s_{x,i}}$. Clearly $N \cap F^d = M \cap F^d$ is open in F^d. Thus, $\tau^i \subset \tau^n$. Conversely, suppose that N is a normal subgroup of H^d satisfying (i), (ii) and (iii) above. We will prove that N is open with respect to τ^i. Since $\mathcal{C}$ is closed under quotients, we need only show that $A^d/N_{A^d} \in \mathcal{C}$ and $N_{A^d} \cap A_x$ is open in A_x for all $x \in X$, where N_{A^d} denotes the core of N in A^d. Since H^d is open in A^d, we have $A^d/H^d_{A^d} \in \mathcal{C}$, and hence, by Lemma 3.1, N contains a normal subgroup M of A^d, such that $A^d/M \in \mathcal{C}$. It follows that the homomorphic image A^d/N_{A^d} of A^d/M belongs to $\mathcal{C}$. To show that $N_{A^d} \cap A_x$ is open in A_x it suffices to see that for each $a \in A^d$, $N^a \cap A_x$ is open (notice that N is of finite index in A^d). Now $N^a \cap A_x = (N \cap A_x^{a^{-1}})^a$. So it suffices to show that for each $a \in A^d$, $N \cap A_x^{a^{-1}}$ is open in $A_x^{a^{-1}}$. But $a^{-1} = a_x s_{x,i} b$ for some $i \in I_x$, where $a_x \in A_x$ and $b \in H^d$; therefore $N \cap A_x^{a^{-1}} = N \cap A_x^{s_{x,i}b} = (N \cap A_x^{s_{x,i}})^b$. Now, $N \cap A_x^{s_{x,i}}$ is open in $H^d \cap A_x^{s_{x,i}}$ by assumption, and hence is open in $A_x^{s_{x,i}}$; so $N \cap A_x^{a^{-1}}$ is open in $A_x^{s_{x,i}b} = A_x^{a^{-1}}$. □

Remark 3.3. From the above proof one easily deduces that in the statement of the theorem one can assume that F is the completion of F^d where F^d has any topology finer that the topology induced by A^d, and coarser than its natural pro-$\mathcal{C}$-topology.

4. The subgroup theorem for compact Hausdorff totally disconnected index spaces. We henceforth assume that $\mathcal{C}$ is a class of finite groups, closed under subgroups, homomorphic images and extensions. In this section we prove our main result:

Theorem 4.1. *Let H be an open subgroup of the free pro-$\mathcal{C}$-product $A = \coprod_{x \in X} A_x$ of pro-$\mathcal{C}$-groups A_x ($x \in X$), where X is a compact Hausdorff totally* *disconnected pointed space* $(X, *)$, *Then there are closed subgroups* B *and* F *of* H, *such that* $H = B \amalg F$ *(the coproduct in the category of pro-*$\mathcal{C}$*-groups), where* F *is a free pro-*$\mathcal{C}$*-group on a certain pointed compact totally disconnected Hausdorff space* Z, *and* B *is a free pro-*$\mathcal{C}$*-product* $B = \coprod_{y \in Y} B_y$ *over a pointed compact totally disconnected Hausdorff space* $(Y, *)$, *the* B_y*'s being conjugates of closed subgroups of the* A_x*'s.*

The proof of this theorem will require a series of preliminary lemmas. Before we launch into their statement and proof, we shall sketch the general outline of the proof of the theorem. By means of a result similar to Proposition 2.1 we express A as a projective limit of free pro-$\mathcal{C}$-products indexed by finite discrete spaces; we apply Theorem 3.3 to each of these free products to find an expression for the projection of H into each of them; then we take the projective limit of these expressions to obtain the desired result for H. The heart of the matter consists of finding a convenient set $\mathfrak{R}_1$ of equivalence relations on X that will allow us to take this last projective limit.

Throughout this section we keep the notation of Proposition 2.1 and Remark 2.4, and the hypothesis that X is compact Hausdorff totally disconnected. Since we are indulging in the dangerous practice of identifying equal pro-$\mathcal{C}$-groups A_x and A_y $(x \neq y)$, with their respective distinct images in the étale space E, it should be pointed out once and for all that if we write "$A_x \cap S = A_y \cap S$ in $A_x = A_y$" or "$a_x = a_y$ in $A_x = A_y$", where S is a subset of E and $a_x \in A_x$, $a_y \in A_y$, this does not imply equality of $A_x \cap S$ and $A_y \cap S$ in E, or of a_x and a_y in E.

Lemma 4.2. *Let* $\mathfrak{R}$ *be as in Proposition 2.1. There exists a cofinal subset* $\mathfrak{R}'$ *of* $\mathfrak{R}$ *with the property that if the following holds:* $R' \in \mathfrak{R}'$, C *is a left coset of* H^d *in* A^d, $x, y \in X$, $xR'y$ *and* $* \notin xR'$, *then*

(I) $C \cap A_x = C \cap A_y$, *in* $A_x = A_y$, *and*

(II) $a_x C = a_y C$ *in* A^d, *whenever* $a_x = a_y$ *in* $A_x = A_y$.

Proof. Let $R \in \mathfrak{R}$ be given. Let $V = *R$, $x \in X \setminus V$ and $a_x \in A_x$. For each left coset C of H^d in A^d, choose an open neighborhood $U = U^{x,C}_{a_x}$ of a_x in A_x, and a constant open and closed neighborhood $W = W^{x,C}_{a_x}$ of x in X, missing V, such that

(i) $p_W(W \times U)$ is contained either in C or in $E \setminus (E \cap C)$ (note that $E \cap C$ is open and closed in E), and

(ii) $t \in p_W(W \times U) \Longrightarrow tC = a_x C$.

Let $\{U^{x,C}_{a_{x,i}}\}_{n=1}^{n_C}$ be a finite cover of A_x; put

$$V_x = \bigcap_{C \in A^d/H^d} \bigcap_{i=1}^{n_C} W^{x,C}_{a_{x,i}}$$

and let $\{V_{x_j}\}_{j=1}^n$ be a finite cover of $X \setminus V$. Suppose that $x, y \in V_{x_j}$. We will show that for every coset $C \in A^d/H^d$, the equalities (I) and (II) above are satisfied. We may, without loss in generality, take $y = x_j$. If $e \in A_{x_j} \cap C$ then

$$(x, e) \in W^{x_j, C}_{a_{x_j, i}} \times U^{x_j, C}_{a_{x_j, i}}$$

for some $U' = U^{x_j,C}_{a_{x_j,i}}$ and $W' = W^{x_j,C}_{a_{x_j,i}}$. Now, $p_{W'}(W' \times U') \subset C$, since $p_{W'}(x_j, e) = e \in A_{x_j} \cap C$; so, when we consider e as an element of A_x, we have $e = p_{W'}(x, e) \in C$; i.e., $e \in A_x \cap C$. Conversely, suppose $c \in A_x \cap C$; then, viewing c as an element of A_{x_j}, we have $(x, c) \in W^{x_j,C}_{a_{x_j,k}} \times U^{x_j,C}_{a_{x_j,k}}$ for some $U'' = U^{x_j,C}_{a_{x_j,k}}$ and $W'' = W^{x_j,C}_{a_{x_j,k}}$. On the other hand, $p_{W''}(W'' \times U'')$ meets C, because $p_{W''}(x, c) \in A_x \cap C$; hence $p_{W''}(W'' \times U'')$ is contained in C, and $c = p_{W''}(x_j, c) \in A_{x_j} \cap C$. This proves equation (I). To prove (II), suppose $C \in A^d/H^d$, $x \in V_{x_j}$ and $a_x = a_{x_j}$ in $A_x = A_{x_j}$; then

$$(x, a_x) \in W^{x_j, C}_{a_{x_j, b}} \times U^{x_j, C}_{a_{x_j, b}}$$

for some $\overline{U} = U^{x_j,C}_{a_{x_j,b}}$, $\overline{W} = W^{x_j,C}_{a_{x_j,b}}$, so that $a_x = p_{\overline{W}}(x, a_x) \in p_{\overline{W}}(\overline{W} \times \overline{U})$, and hence $a_x C = a_{x_j} C$.

Define now an equivalence relation R' on X as follows: $*R' = *R$, and if $x, y \notin *R'$, then $xR'y$ iff xRy and either $x, y \in V_{x_j}$ or $x, y \notin V_{x_j}$ for all $j = 1, 2, \cdots, n$. Then $R' \subset R$, $R' \in \mathfrak{R}$ and R' satisfies the required property. □

In our next lemma we refine a result of Mac Lane (cf. Kurosh [7, Lemma 1, p. 147]) in order to make it suitable to our setting.

Lemma 4.3. *Let* $(X, *)$, $\{A_x | x \in X\}$, A^d, H^d *and* $\mathfrak{R}'$ *be as in Lemma* 4.2. *Then for each* $x \in X$ *there is a system* $r_x\colon A^d/H^d \to A^d$ *of representatives for the left cosets* C *of* H^d *in* A^d *satisfying the following conditions*:

(1) $r_x(H^d) = 1$.

(2) *If* $a_x \in A_x$, *then* $r_x(a_x C) = a'_x r_x(C)$, *where* $a'_x \in A_x$.

(3) *If* $r_x(C) = a_x s$, *where* $a_x \in A_x$ *and* $s \neq 1$, *and if the first index* y *of* s *is different from* x, *then* $r_x(sH^s) = r_y(sH^d) = s$.

(4) *If* $\lambda(A_x C)$ *denotes the minimal length of elements of the double coset* $A_x C$ *of* A^d *with respect to the subgroups* A_x *and* H^d, *we have* $\lambda(r_x(C)) \leq 1 + \lambda(A_x C)$.

(5) *Let* D *be a double coset of* A^d *with respect to the pair of subgroups* A_x *and* H^d *for some* $x \in X$. *Let* $s(x, D)$ *be the unique element of minimal length among the representatives* $r_x(C)$, *where* $A_x C = D$. *Then* $s(x, D)$ *is of*

minimal length in D, *and* $s(x, D) = s(y, D)$, *whenever* $A_x s(x, D)H^d = A_y s(y, D)H^d$.

(6) *Suppose* $R' \in \mathfrak{R}'$, $xR'y$, $x \notin *R'$, $a_x = a_y$ *in* $A_x = A_y$, $s = s(x, D)$ *and* $a_x(H^d)^{s-1} = a_y(H^d)^{s-1}$; *then* $\pi_R^d(r_x(C)) = \pi_R^d(r_y(C))$.

Proof. To define $r_x(C)$ we will proceed by induction on $\lambda(A_x C)$. In the course of this proof, the term "double coset of A^d" will mean a set of the form $A_x aH^d$ for some $x \in X$ and $a \in A^d$. Let D be a double coset of A^d, and C a left coset of H^d in A^d, with $A_x C = D$ for some $x \in X$; then we put $T_{C,D} = \{y \in X \mid A_y C = D\}$. Suppose $\lambda(D) = 0$ and $x \in T_{C,D}$. Then $D = A_x C = A_x H^d$ and $A_x \cap C \neq \emptyset$. Let S_x be the set of all those elements y of $T_{C,D}$ for which $A_x = A_y$, $C \cap A_x = C \cap A_y$ in $A_x = A_y$, and $a_x H^d = a_y H^d$ whenever $a_x \in C \cap A_x$, $a_y \in C \cap A_y$ and $a_x = a_y$ in $A_x = A_y$. We have $x \in S_x$, and we can write $T_{C,D}$ as a disjoint union of sets S_{x_i}, $x_i \in X$, $i \in I$. For each x_i, we choose some element $a_{x_i} \in A_{x_i} \cap C$, with $a_{x_i} = 1$ if $C = H^d$, and we put $r_{x_i}(C) = a_{x_i}$. If $x \in S_{x_i}$, we can find $a_x \in A_x \cap C$ with $a_x = a_{x_i}$ in $A_x = A_{x_i}$, and we put $r_x(C) = a_x$. If D and D' are distinct double cosets of A^d, then $T_{C,D} \cap T_{C,D'} = \emptyset$; so the element $r_x(C)$ is well defined for every $C \in A^d/H^d$ and every $x \in X$ with $\lambda(A_x C) = 0$. Conditions (1), (2), (4), (5), and (6) are easily seen to be satisfied; condition (3) is empty in this case.

Suppose now that $D = A_x C$ is such that $\lambda(D) = n > 0$, and assume that $r_z(C')$, $z \in X$, has been defined satisfying conditions (1)–(6) whenever $\lambda(A_z C') < n$ and $C' \in A^d/H^d$. Let $g \in D$ be an element of length n (for a given D the element g will be fixed throughout the discussion). Then the first index y of g is different from z whenever $A_z C = D$, for otherwise D would contain elements of length less than n. In particular $y \neq x$. Since $\lambda(A_y gH^d) < n$, the representative $r_y(gH^d) = s$ has already been chosen according to our induction hypothesis, and depends only on g. By (4), $\lambda(s) \leq n$. Since $D = A_x sH^d$, we have $n = \lambda(D) \leq \lambda(x)$; and consequently $\lambda(s) = n$, so that the first index t of s is different from all $z \in T_{C,D}$, and by condition (3) and the induction hypothesis, $r_t(sH^d) = s$. Now, if $C = sH^d$, we put $r_z(C) = s$ for all $z \in T_{C,D}$. If $C \neq sH^d$, for each $z \in T_{C,D}$ we form the set S_z of all $v \in T_{C,D}$ such that $A_v = A_z$ and $a_v(H^d)^{s-1} = a_x(H^d)^{s-1}$ whenever $a_v = a_z$ in $A_v = A_z$. Clearly $z \in S'_z$. Now, $T_{C,D}$ can be expressed as the disjoint union of sets S'_{x_j}, $x_j \in T_{C,D}$, $j \in J$. For each $j \in J$, we choose $a_{x_j} \in A_{x_j}$, $a_{x_j} \neq 1$, such that $C = a_{x_j} sH^d$. Define $r_{x_j}(C) = a_{x_j} s$. Then $a_{x_j} \in Cs^{-1} = a_{x_j}(H^d)^{s-1}$. For all $v \in S'_{x_j}$, the element $a_v \in A_v$ with $a_v = a_{x_j}$ under the equality $A_v = A_{x_j}$, is such that $a_v(H^d)^{s-1} = a_{x_j}(H^d)^{s-1}$, and we define $r_v(C) = a_v s$. Then the elements $r_v(C)$, are well defined for all $v \in T_{C,D}$. Moreover s is of minimal length among the representatives $r_x(C')$, where x is fixed and C' runs through those left cosets of H^d in A^d such that $D = A_x C'$, i.e., $s(x, D) = s$. Thus $s(x', D) = s(x, D)$ whenever $x, x' \in T_{C,D}$, and condition (5) is verified. If $R' \in \mathfrak{R}'$, $x, x' \in T_{C,D}$, $xR'x'$, $x \notin *R'$, and $a_x(H^d)^{s-1} = a_{x'}(H^d)^{s-1}$ whenever $a_x = a_{x'}$ in $A_x = A_{x'}$, we must have $x' \in S'_x$; hence by construction there are elements $\bar{a}_x \in A_x$ and $\bar{a}_{x'} \in A_{x'}$ with $\bar{a}_x = \bar{a}_{x'}$ in $A_x = A_{x'}$, such that $r_x(C) = \bar{a}_x s$ and $r_{x'}(C) = \bar{a}_{x'} s$; thus $\pi^d_{R'}(r_x(C)) = \pi^d_{R'}(\bar{a}_x s) = \pi^d_{R'}(\bar{a}_{x'} s) = \pi^d_{R'}(r_{x'}(C))$, i.e., condition (6) is verified. Condition (1) is not applicable, and conditions (2), (3), and (4) are easy to check. □

Corollary 4.4. *Let r_x and $s(x, D)$ be as above. Let S be the set consisting of all elements $s(x, D)$, where $x \in X$ and D runs through the double cosets $A_x \alpha H^d$, $\alpha \in A^d$. Then, S is finite.*

Proof. Since H^d is open and of finite index in A^d, the group $\bigcap_{\alpha \in A^d} (H^d)^\alpha$ is still open. Hence (cf. §1), there is an open neighborhood W of $*$ in X such that $x \in W \Rightarrow A_x \subset \bigcap_{\alpha \in A^d} (H^d)^\alpha$. Let $C \in A^d/H^d$, say $C = \alpha H^d$; then $x \in W \Rightarrow A_x C = A_x (H^d)^{\alpha^{-1}} \alpha = C$.

Thus it follows from the definition of $s(x, D)$, and from (5) of Lemma 4.3, that

$$x, y \in W \Rightarrow r_x(C) = r_y(C) = s(x, A_x C) = s(y, A_y C).$$

Since H^d is of finite index in A^d, the set

$$\{s(x, D) \in A^d \mid x \in W,\ D = A_x C,\ C \in A^d/H^d\}$$

is finite. Now, let $\mathfrak{R}'$ be as in Lemma 4.2, and choose $R' \in \mathfrak{R}'$ with $*R' \subset W$. Then by Lemma 4.2 (II) $A_x C = A_y C$ whenever $C \in A^d/H^d$, $xR'y$ and $x \notin *R'$. Consequently, by (5) of Lemma 4.3, the set

$$\{s(x, D) \in A^d \mid x \notin W,\ D = A_x C,\ C \in A^d/H^d\}$$

is also finite, and the result follows. □

Lemma 4.5. *Let $\mathfrak{R}'$ be as in Lemma 4.2, S as in Corollary 4.4, $r_x\colon A^d/H^d \to A^d$ as in Lemma 4.3 ($x \in X$), and T an open neighborhood of $*$ in X missing all the indexes appearing in the irreducible presentations of all $s \in S$. Let $\mathfrak{R}_1$ be the set of those $R \in \mathfrak{R}'$ such that*

(i) *$*R \subset T$;*

(ii) *if $s \in S$, xRy, $x \notin *R$, $a_x = a_y$ in $A_x = A_y$, then $a_x(H^d)^{s-1} = a_y(H^d)^{s-1}$;*

(iii) *R separates the indexes appearing in the elements of S (i.e. $xR \neq x'R$ whenever x and x' are distinct indexes appearing in some $s \in S$).*

Then $\mathfrak{R}_1$ is a cofinal subset of $\mathfrak{R}'$.

Proof. Clearly the subset $\mathfrak{R}''$ of $\mathfrak{R}'$ consisting of equivalence relations R'' that satisfy (i) and (iii) above, is cofinal in $\mathfrak{R}'$. So it suffices to prove that such an $\mathfrak{R}''$ contains an open equivalence relation R, satisfying (ii). For each $s \in S$,

$x \notin *R''$ and $a \in A_x$, choose a constant open and closed neighborhood $W = W_{a,s,x}$ of x in X, missing R'', and an open neighborhood $U = U_{a,s,x}$ of a on A_x, such that if $t \in p_W(W \times U)$, then $t(H^d)^{s-1} = a(H^d)^{s-1}$. Let $\{U_{a_i,s,x}\}_{i=1}^{n_s}$ be a finite cover of A_x and put $V_x = \bigcap_{s \in S} \bigcap_{i=1}^{n_s} W_{a_i,s,x}$. Let $\{V_{x_j}\}_{j=1}^{m}$ be a finite cover of $X \backslash *R''$. Define R by writing $*R = *R''$, and if $x, y \notin *R''$, then xRy iff $xR''y$ and for all $j = 1, 2, \cdots, m$, either $x, y \in V_{x_j}$ or $x, y \in X \backslash V_{x_j}$. Clearly R satisfies (iii) and $R \subset R''$. □

Then, as an immediate consequence of Lemmas 4.3 and 4.5, we get

Corollary 4.6. *Let r_x, $x \in X$ be as in Lemma 4.3, S as in Corollary 4.4 and $\mathfrak{R}_1$ as in Lemma 4.5. Suppose that $R \in \mathfrak{R}_1$, $C \in A^d/H^d$, xRy and $x \notin *R$. Then $\pi_R^d(r_x(C)) = \pi_R^d(r_y(C))$.* □

Now, let $\mathfrak{R}_1$ be as in Lemma 4.5 and r_x: $A^d/H^d \rightarrow A^d$ as in Lemma 4.3. For every $R \in \mathfrak{R}_1$ there is a bijective correspondence between the left cosets of H^d in A^d and of $H_R^d = \pi_R^d(H^d)$ in A_R^d, (cf. Proposition 2.1), given by $C \mapsto C_R = \pi_R^d(C)$. If $*R \neq xR \in X/R$ and $C \in A^d/H^d$, then Corollary 4.6 assures us that the representative $r_{xR}^R(C_R)$ of C_R in H_R^d is well defined by putting $r_{xR}^R(C_R) = \pi_R^d(r_x(C))$. We let $r_{*R}^R(C_R) = \pi_R^d(r_*(C))$.

Lemma 4.7. *For each $R \in \mathfrak{R}_1$, the systems of representatives of left cosets r_{xR}^R: $A_R^d/H_R^d \rightarrow A_R^d$ satisfy conditions (1), (2) and (3) of Lemma 4.3. Moreover, $r_{xR}^R(C_R) \in A_R^d$ has the same length as $r_x(C)$ for all $x \notin *R$.*

Proof. Conditions (1) and (2) are easily checked. To verify condition (3) suppose that $r_{xR}^R(C_R) = a_{xR}t, t \neq 1$, where $a_{xR} \in A_{xR}$ and the first index in t is distinct from xR. Let $\pi_R^d(C) = C_R$, where $C \in A^d/H^d$. We know (cf. Lemma 4.3, part (5)) that $r_x(C) = a'_x s(x, A_xC)$ where $a'_x \in A_x$ ($s(x, A_xC) \neq 1$ for otherwise $t = 1$). Now, if y is the first index of $s = s(x, A_xC)$ then $y \notin xR$; indeed, if yRx then $A_xC = A_yC$ by Lemma 4.2(II); so s is not of minimal length in A_yC, contradicting Lemma 4.3, part (5). Consequently, if $s = a_y a_{z_1} \cdots a_{z_n}$ is an irreducible presentation of s ($a_y \in A_y$, $a_{z_i} \in A_{z_i}$), then $a'_{xR} a_{yR} a_{z_1R} \cdots a_{z_nR}$ is an irreducible presentation for $\pi_R^d(r_x(C)) = a_{xR}t$. Hence $a_{xR} = a'_{xR}$ and $t = \pi_R^d(s)$. So, using Lemma 4.3, part (3), $r_{xR}^R(tH_R^d) = \pi_R^d r_x(sH^d) = \pi_R^d(s) = t$ and $r_{yR}^R(tH_R^d) = \pi_R^d(r_y(sH^d)) = \pi_R^d(s) = t$ as desired. At the same time, we have shown that $r_{xR}^R(C_R)$ has the same length as $r_x(C)$. □

Now for $R \in \mathfrak{R}_1$ (as in Lemma 4.5) put $s^R = s^R(xR, A_{xR}C_R) = \pi_R^s(s(x, A_xC))$, where $C_R = \pi_R^d(C)$. Then s^R is the element $r_{xR}^R(C'_R)$ of shortest length among those for which $C'_R \subset A_{xR}C_R$, $C'_R \in A_R^d/H_R^d$. Indeed, since $\ker \pi_R^d \subset H$, one has that if $\pi_R^d(C') = C'_R \subset A_{xR}C_R$, then $C' \subset A_x C$, and the length of $r_{xR}^R(C'_R)$ equals the length of $r_x(C')$.

For each $R \in \mathfrak{R}_1$ let $Y'_R = (X/R) \times (A_R^d/H_R^d)$, where A_R^d/H_R^d has the discrete topology. Define an equivalence relation $\sim$ on Y'_R as follows: $(xR, C_R) \sim (x'R, C'_R)$ iff $xR = x'R = *R$ or $xR = x'R$ and $A_{xR}C_R = A_{x'R}C'_R$. Denote by $[(xR, C_R)]$ the equivalence class of (xR, C_R), and write $* = [(*R, C_R)]$. Put $Y_R = (Y'_R/\sim, *)$. With this notation, the Kurosh subgroup theorem for discrete groups, as applied to the subgroup H_R^d of A_R^d, is expressed by:

$$H_R^d = \left(\coprod^d_{[(xR, C_R)] \in Y_R} H_R^d \cap A^{s^R(xR, A_{xR}C_R)}\right) \coprod^d F_R^d,$$

where F_R^d is a finitely generated free discrete group. In fact the set Z'_R of elements of the form $r_{xR}^R(C_R)^{-1} r_{*R}^R(C_R) \neq 1$ where $x \in X$ and $C_R \in A_R^d/H_R^d$, constitute a set of free generators of F_R^d (cf. Kurosh [7, p. 147, Lemmas 3,4, and 5]). Consider the pointed discrete topological space $Z_R = (Z'_R \cup \{1\}, 1)$. Then the free pro-$\mathcal{C}$-group F_R on the finite discrete pointed space Z_R is precisely the pro-$\mathcal{C}$-completion of F_R^d (cf. Gildenhuys and Lim [3, Proposition 1.3]), and by Remark 3.3 F_R is the closure in $H_R = \pi_R(H)$ of F_R^d. It is plain that if $R, R' \in \mathfrak{R}_1$ and $R \subset R'$ one has $\pi_R(Z_R) = Z_{R'}$. Put $F = \varprojlim_{R \in \mathfrak{R}_1} F_R$. Then F is the free pro-$\mathcal{C}$-group on the pointed topological space $Z = \varprojlim_{R \in \mathfrak{R}_1} Z_R$, with 1 as the distinguished point (cf. Gildenhuys and Lim [3, Proposition 1.7]).

Proof of Theorem 4.1. Let r_x and $s(x, D)$ be as in Lemma 4.3. By the Kurosh subgroup theorem for discrete groups (cf. Kurosh [7, p. 146]) we have

$$H^d = \left[\coprod^d (H^d \cap s(x, D_x)^{-1} A_x s(x, D_x))\right] \coprod^d F^d,$$

where D_x ranges over the double cosets $A_x \alpha H^d$ of H^d with respect to A_x and H^d, and where F^d is the free discrete group on the free generators $Z' = \{r_x(C)^{-1} r_*(C) \neq 1 \mid x \in X,\ C \in A^d/H^d\}$ (cf. Kurosh [7, p. 147, Lemmas 3, 4, and 5]). Put $Z'' = (Z' \cup \{1\}, 1)$. Then Z'' is a pointed topological space whose topology is the one induced by H. By Corollary 4.6, $\pi_R^d(Z'') = Z'_R$ for every $R \in \mathfrak{R}_1$. Thus Z, as defined above, is the closure of Z'' in H.

Now, consider the product space $Y' = X \times (A^d/H^d)$ where A^d/H^d has the discrete topology. Define an equivalence relation $\sim$ on Y' as follows: $(x, C) \sim (x', C')$ iff $x = x' = *$ or $x = x'$ and $A_x C = A_{x'} C'$. Denote by $[(x, C)]$ the equivalence class of (x, C) and put $* = [(*, C)]$. Clearly $Y = (Y'/\sim, *)$ is a compact, totally disconnected, Hausdorff pointed topological space. For each $y = [(x, C)] \in Y$ consider the well-defined group

$$\beta(y) = (s(x, A_xC)Hs(x, A_xC)^{-1}) \cap A_x = (s(x, A_xC)H^d s(x, A_xC)^{-1}) \cap A_x,$$

and write

$$B_y = s(x, A_xC)^{-1}\beta(y)s(x, A_xC) = \beta(y)^{s(x, A_xC)}.$$

Assume $y = [(x, C)] \neq *$; then one can find an open neighborhood U of x in X missing $*$ with $A_x = A_{x'}$ and $A_xC = A_{x'}C$ whenever $x' \in U$, and such that if $y' = [(x', C)]$ belongs to the image of $U \times \{C\}$ in Y, then $\beta(y') = \beta(y)$ under the equality $A_x = A_{x'}$. (To find U one uses Lemma 4.3, part (5), Lemma 4.2(II), and by a now familiar argument; see, e.g., the proof of Lemma 4.2.) Thus the function $y \mapsto \beta(y)$ from Y to $\{\beta(y) | y \in Y\}$ is locally constant, except perhaps at $*$, where $\beta(*) = 1$. Also, if $y' = [(x', C)]$ belongs to the image of $U \times \{C\}$ in Y, then B_y and $B_{y'}$ are isomorphic by an inner automorphism of A^d defined by $s(x', A_{x'}C) = s(x, A_xC)$. Let B stand for the free product of the pro-$\mathcal{C}$-groups $\beta(y)$, over the compact Hausdorff totally disconnected pointed space Y. We identify B_y with the image of $\beta(y)$ in B, and we also write $B = \coprod_{y \in Y} B_y$.

Every equivalence relation $R \in \mathfrak{R}_1$ induces an open equivalence relation on Y, denoted again by R, and defined by

$$[(x, C)]\, R\, [(x', C')] \quad \text{iff} \quad x\, R\, x' \ \text{ and } A_xC = A_{x'}C'.$$

Hence by Proposition 2.1 we have $B = \varprojlim_{R \in \mathfrak{R}_1} B_R$, where $B_R = \coprod_{yR \in Y/R} B_{yR}$, $B_{yR} = B_y$ if $y \notin *R$ and $B_{*R} = 1$. Now, it is plain that Y/R and Y_R (as defined on p. 325) are homeomorphic pointed finite discrete spaces under the map $[(x, C)]R \to [(xR, C_R)]$ (where $C_R = \pi_R(C)$). Moreover, if $[(x, C)]R = *R$ then $xR = *R$ and $B_{[(x,C)]} = \{1\} = H_R^d \cap A_{xR}^{s^R(xR, A_{xR}C_R)}$. Also if $[(x, C)]R \neq *R$ then $xR \neq *R$, and we have that $\pi_R^d\colon A^d \to A_R^d$ restricts to an isomorphism

$$B_{[(x, C)]R} = H^d \cap A_x^{s(x, A_xC)} \to H_R^d \cap A_{xR}^{s^R(xR, A_{xR}C_R)}.$$

So

$$H_R^d \approx \left(\coprod_{[(x, C)] \in Y/R}^{d} B_{[(x, C)]R} \right) \amalg^d F_R^d,$$

and, by Theorem 3.2,

$$H_R \approx \left(\coprod_{[(x, C)] \in Y/R} B_{[(x, C)]R} \right) \amalg F_R = B_R \amalg F_R$$

where F_R is the pro-$\mathcal{C}$-completion of F_R^d. Moreover this isomorphism commutes with the maps $\pi_{R,R'}$ as defined in §2. Finally $\varprojlim$ commutes with coproducts and thus

$$H = \varprojlim_{R \in \mathfrak{R}_1} H_R \approx \Big(\varprojlim_{R \in \mathfrak{R}_1} B_R\Big) \amalg \Big(\varprojlim_{R \in \mathfrak{R}_1} F_R\Big) \approx B \amalg F.$$

5. **Consequences.** We keep the assumption that $\mathcal{C}$ is extension closed.

Theorem 5.1. *Let $(X, *)$ be a pointed topological space and let $\tilde{F} = \tilde{F}(X, *)$ be the free pro-$\mathcal{C}$-group on $(X, *)$. Let H be an open subgroup of F. Then H is a free pro-$\mathcal{C}$-group on a certain compact, totally disconnected, Hausdorff pointed topological space $(T, *)$.*

Proof. By an argument similar to the one used in Gildenhuys and Lim [3, Proposition 1.7], we may assume, without loss in generality, that X is compact Hausdorff, totally disconnected. By Proposition 1.4, $\tilde{F}$ is the free pro-$\mathcal{C}$-product of the groups $A_x = \hat{Z}_{\mathcal{C}}$, $x \in X \backslash \{*\}$, $A_* = \{1\}$, indexed by X. By Theorem 4.1, $H = B \amalg F$ where F is a free pro-$\mathcal{C}$-group on a compact, totally disconnected Hausdorff pointed topological space $(Z, *)$, and B is a free pro-$\mathcal{C}$-product of conjugates of open subgroups of the A_x's. If a prime p divides the order of some group in $\mathcal{C}$, then $\mathcal{C}$ contains all finite p-groups, since $\mathcal{C}$ is extension closed. Hence $A_x = \hat{Z}_{\mathcal{C}}$ is a product of the additive groups of the p-adic integers, where p ranges over all those primes that divide the order of some group in $\mathcal{C}$. It follows that the open subgroup $A_x^s \cap H$ is isomorphic to $\hat{Z}_{\mathcal{C}}$ (see also Ribes [13, Theorem 6.5]), and B is a free pro-$\mathcal{C}$-product of copies of $\hat{Z}_{\mathcal{C}}$, indexed by a compact Hausdorff totally disconnected space $(Y, *)$. Hence by Proposition 1.4, $B = \tilde{F}(Y, *)$. Put $(T, *) = (Z, *) \vee (Y, *)$ (the coproduct in the category of pointed topological spaces, i.e., T is the union of Z and Y with only the distinguished points identified). Then it is plain from the definition of free pro-$\mathcal{C}$-groups that $H = B \amalg F = \tilde{F}(T, *)$. □

Using Theorem 3.2 one proves the following result by an argument similar to the one used in Theorem 5.1.

Theorem 5.2. *Let F be a free pro-$\mathcal{C}$-group on a set X, and let H be an open subgroup of F. Then H is a free pro-$\mathcal{C}$-group on a certain set Y.* □

Theorem 5.3. *Let A_x, $x \in X$, be pro-$\mathcal{C}$-groups indexed by a set X. Let A be the restricted free pro-$\mathcal{C}$-product of the A_x's (cf. §1) and let H be an open subgroup of A. Then $H = B \amalg F$ where F is a finitely generated free pro-$\mathcal{C}$-group and B is a restricted free pro-$\mathcal{C}$-product of conjugates in A of the groups A_x.*

Proof. Let $\bar{X} = X \cup \{*\}$ be the one-point compactification of the discrete space X, and put $A_* = 1$. Then, by Proposition 1.5, $A = \coprod_{x \in X} A_x$. Thus we can apply Theorem 4.1 to obtain $H = B \amalg F$. To show that F is finitely generated we will prove that Z'', as defined in the proof of Theorem 4.1, is finite in this case. By an argument used in Corollary 4.4 there is an open neighborhood W of $*$ in $\bar{X}$

such that the set $\{r_x(C) | x \in W,\ C \in A^d/H^d\}$ is finite. Since $\overline{X}$ is the one-point compactification of a discrete space, we have that $\overline{X}\backslash W$ is finite. Hence the set $\{r_x(C) | x \in \overline{X},\ C \in A^d/H^d\}$ is finite. Thus Z'' is finite. Using still the notation of the proof of Theorem 4.1, notice that

$$Y'\backslash(\{*\} \times A^d/H^d) = (\overline{X} \times A^d/H^d)\backslash(\{*\} \times A^d/H^d)$$

is a discrete space; so $(Y'/\sim)\backslash\{*\}$ is discrete and the open neighborhoods of $*$ in $Y'/\sim$ are precisely the complements of finite sets; hence Y is the one-point compactification of $(Y'/\sim)\backslash\{*\}$. Thus according to Proposition 1.5, B is a restricted free pro-$\mathcal{C}$-product of groups indexed by the set $(Y'/\sim)\backslash\{*\}$. □

The following result is now clear.

Corollary 5.4. *Let H be an open subgroup of a restricted free pro-$\mathcal{C}$-group F. Then H is a restricted free pro-$\mathcal{C}$-group.*

Finally we prove an analog of a well-known result for discrete free products (Magnus-Karrass-Solitar [10, p. 243]).

Theorem 5.5. *Let A be as in Theorem 4.1, and let H be an open normal subgroup of A such that $H \cap A_x = 1$ for every $x \in X$. Then H is a free pro-$\mathcal{C}$-group.*

Proof. We use the notation of Theorem 4.1. Notice that each B_y has the form $H \cap A_x^s$ for some $x \in X$ and $s \in A$. So, since H is normal, we have $B_y = (H \cap A_x)^s = 1$. Thus, $H = F$.

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