TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 186, December 1973

WEIGHTED GROTHENDIECK SUBSPACES

BY

JOÃO B. PROLLA AND SILVIO MACHADO

ABSTRACT. Let V be a family of nonnegative upper semicontinuous functions on a completely regular Hausdorff space X. For a locally convex Hausdorff space E, let $CV_\infty(X; E)$ be the corresponding Nachbin space, that is, the vector space of all continuous functions f from X into E such that vf vanishes at infinity for all $v \in V$, endowed with the topology given by the seminorms of the type $f \mapsto \sup \{v(x)\, p(f(x));\ x \in X\}$, where $v \in V$ and p is a continuous seminorm on E. Given a vector subspace L of $CV_\infty(X; E)$, the set of all pairs $x, y \in X$ such that either $0 = \delta_x|L = \delta_y|L$ or there is $t \in \mathbf{R}$, $t \neq 0$, such that $0 \neq \delta_x|L = t\,\delta_y|L$, is an equivalence relation, denoted by G_L, and we define for $(x, y) \in G_L$, $g(x, y) = 0$ or t, accordingly. The subsets KS_L, resp. WS_L, where $g(x, y) \geq 0$, resp. $g(x, y) \in \{0, 1\}$, are likewise equivalence relations. The G-hull (resp. KS-hull, WS-hull) of L is the vector subspace $\{f \in CV_\infty(X; E);\ f(x) = g(x, y) f(y)$ for all $(x, y) \in G_L$ (resp. KS_L, WS_L)$\}$, and L is a G-space (resp. KS-space, WS-space) if its G-hull (resp. KS-hull, WS-hull) is contained in its closure. This paper is devoted to the characterization, by invariance properties, of the G-spaces resp. KS-spaces and WS-spaces of a given Nachbin space $CV_\infty(X; E)$. As an application we derive an infinite-dimensional Weierstrass polynomial approximation theorem, and a Tietze extension theorem for Banach space valued compact mappings.

1. **Introduction.** Let X be a completely regular Hausdorff space, and let E be a locally convex Hausdorff space, not reduced to $\{0\}$. The vector space of all E-valued continuous functions on X will be denoted by $C(X; E)$, while $C_b(X; E)$ denotes the subspace of all $f \in C(X; E)$ which are bounded. A *weight* on X is a nonnegative upper semicontinuous function on X. A set V of weights on X is directed if, for all $u, v \in V$, there are $w \in V$ and $t > 0$ such that $u \leq tw$ and $v \leq tw$ pointwise. All sets of weights considered will be directed and $V > 0$, i.e., given $x \in X$ there is $v \in V$ such that $v(x) > 0$. Given such a V the *Nachbin space* $CV_\infty(X; E)$ is the vector subspace of all $f \in C(X; E)$ such that vf vanishes at infinity, for every $v \in V$, endowed with the topology determined by the seminorms $f \mapsto \sup \{v(x)\, p(f(x));\ x \in X\}$, where $v \in V$, and $p \in s(E)$, a fixed

Received by the editors April 7, 1973.

AMS (MOS) subject classifications (1970). Primary 46A05, 46E10; Secondary 46E40, 46M05.

Key words and phrases. Nachbin spaces of continuous vector-valued functions, Grothendieck spaces, Kakutani-Stone spaces, Weierstrass-Stone spaces, polynomial algebras, latticial subspaces, Lindenstrauss-Wulbert subspaces, compact mappings.

determining set of continuous seminorms on E. The following definitions are from Blatter [2].

Let L be a vector subspace of $CV_\infty(X; E)$. The set G_L of all pairs $x, y \in X$ such that either $0 = \delta_x|L = \delta_y|L$ or there exists $t \in \mathbf{R}$, $t \neq 0$, such that $0 \neq \delta_x|L = t\delta_y|L$, is an equivalence relation for X, and we define a map $g: G_L \longrightarrow \mathbf{R}$ as follows:

$$g(x, y) = 0, \quad \text{if } 0 = \delta_x \,|\, L = \delta_y \,|\, L;$$

$$g(x, y) = t, \quad \text{if } 0 \neq \delta_x \,|\, L = t\delta_y \,|\, L.$$

The sets KS_L and WS_L of all pairs $(x, y) \in G_L$ such that $g(x, y) \geq 0$ and $g(x, y) \in \{0, 1\}$, respectively, are likewise equivalence relations for X. Let $\Delta \in \{G, KS, WS\}$. The vector subspace $\Delta(L) = \{f \in CV_\infty(X; E);\ f(x) = g(x, y) f(y)$ for all $(x, y) \in \Delta_L\}$ is called the Δ-*hull* of L in $CV_\infty(X; E)$, and L is called a Δ-*space* is $\Delta(L)$ is the closure of L in $CV_\infty(X; E)$. Notice that, by the hypothesis made on V, the closure of L is always contained in $\Delta(L)$. The letters G, KS, and WS stand for Grothendieck, Kakutani-Stone, and Weierstrass-Stone respectively.

The object of the present paper is to characterize, by invariance properties, those subspaces L which are Δ-spaces. Our results are as follows. Let $A(\Delta)$ be the algebra of all $h \in C_b(X; \mathbf{R})$ which are constant on the equivalence classes modulo Δ_L. Assume that $A(\Delta)$ separates the equivalence classes modulo Δ_L. (This is automatically satisfied in the case $\Delta = WS$.) Let $M = \{u^*(f);\ u^* \in E^*, f \in L\}$. Then:

(1) L is a WS-space, if L is a polynomial algebra such that M is localizable under itself in $CV_\infty(X; \mathbf{K})$.

(2) L is a KS-space, if L is a latticial subspace.

(3) L is a G-space, if L is a Lindenstrauss-Wulbert subspace (see Definition 3.10).

In the spirit of Stone's paper [11], we apply some of these results to reobtain vector-valued versions of some classical theorems.

The following elementary result will be used many times.

1.1 Lemma. *Let L be a vector subspace of $CV_\infty(X; E)$. The following statements are equivalent.*

(1) *$M = \{u^*(f);\ u^* \in E^*, f \in L\}$ is a vector subspace of $CV_\infty(X; \mathbf{K})$ and $M \otimes E \subset L$. If L is closed, M is closed, too.*

(2) *Given $u^* \in E^*$, $f \in L$, and $u \in E$, then $u^*(f)u \in L$.*

Proof. It is obvious that (1) implies (2). Conversely, assume (2). M is clearly invariant under scalar multiplication. Let $u^*, v^* \in E^*$, and $f, h \in L$ be given. We may assume that u^* and v^* are linearly independent. Choose $u, v \in E$ such that $u^*(u) = v^*(v) = 1$, $u^*(v) = v^*(u) = 0$. By (2), $k = u^*(f)u + v^*(h)v$ belongs to L. Let $w^* = u^* + v^*$. Then $u^*(f) + v^*(b) = w^*(k) \in M$. Obviously $M \otimes E \subset L$. Suppose now that L is closed. Let f be in the closure of M. Choose $u^* \in E^*$ and $u \in E$ such that $u^*(u) = 1$. Set $b = f \otimes u$. Given $v \in V$, $p \in s(E)$, and $\epsilon > 0$, there exists $w^*(k) \in M$ such that $v(x)|f(x) - w^*(k(x))| < \epsilon/(1 + p(u))$ for all $x \in X$. Then $w^*(k)u \in L$ and $v(x)p(b(x) - w^*(k(x))u) < \epsilon$ for all $x \in X$. Since L is closed, $b \in L$. Therefore, $f = u^*(b) \in M$, and M is closed.

2. Weierstrass-Stone spaces.

2.1 **Definition.** Let E and F be two nonzero topological vector spaces. For each integer $n \geq 1$, $P_f(^nE; F)$ denotes the vector subspace of $C(E; F)$ generated by the set of all maps of the form $x \mapsto u^*(x)^n u$, where $u^* \in E^*$, the topological dual of E, and $u \in F$. The elements of $P_f(^nE; F)$ are called *n-homogeneous continuous polynomials of finite type* from E into F. The vector subspace generated by the union of all $P_f(^nE; F)$, $n \geq 1$, and the constant maps, is denoted by $P_f(E; F)$. Similarly, $L_f(^nE; F)$, for each integer $n \geq 1$, denotes the vector space generated by the set of all maps of the form $(x_1, \cdots, x_n) \mapsto u_1^*(x_1) \cdots u_n^*(x_n)u$ where $u_1^*, \cdots, u_n^* \in E^*$ and $u \in F$. The elements of $L_f(^nE; F)$ are called n-linear continuous mappings of finite type from E^n into F.

2.2 **Lemma.** *Let L be a vector subspace of $CV_\infty(X; E)$. The following statements are equivalent.*

(1) *Given $b_1, \cdots, b_n \in L$ and $T \in L_f(^nE; E)$, where $n \geq 1$, then $T(b_1, \cdots, b_n) \in L$.*

(2) *Given $b \in L$ and $p \in P_f(^nE; E)$, where $n \geq 1$, then $p(b) \in L$.*

(3) *$M = \{u^*(b); u^* \in E^*, b \in L\}$ is an algebra contained in $CV_\infty(X; \mathbf{K})$ such that $M \otimes E \subset L$.*

(4) (a) *Given $b \in L$, $u^* \in E^*$, and $u \in E$, then $u^*(b)u \in L$.*

(b) *There exists a continuous map $T: E \times E \to E$ such that $T(v, v) \neq 0$ for some $v \in E$ for which $T(av, bv) = abT(v, v)$ for all $a, b \in \mathbf{K}$ and $T(h, k) \in L$ for all $h, k \in L$.*

Proof. It is obvious that (1) implies (2). Assume (2). By the Lemma 1.1 the set M is a vector subspace of $CV_\infty(X; \mathbf{K})$ such that $M \otimes E \subset L$. Since $4u^*(b)v^*(k) = [u^*(b) + v^*(k)]^2 - [u^*(b) - v^*(k)]^2$, all that remains to prove is that $u^*(b)^2$ belongs to M for any $u^* \in E^*$ and $b \in L$. If $u^* = 0$, there is nothing to prove. Let $u \in E$ be such that $u^*(u) = 1$, and define $p(t) = u^*(t)^2 u$ for all $t \in E$. By (2), $k = p(b) \in L$. Since $u^*(k) = u^*(b)^2$, the set M is an algebra. Assume now (3). Part (a) follows immediately. To prove part (b) of (4), consider $u^* \in E^*$ and $v \in E$ with $u^*(v) = 1$. Define $T(s, t) = u^*(s)\, u^*(t)v$ for all $s, t \in E$. Then T satisfies all requirements. Finally, assume (4). Let $b_1, \cdots, b_n \in L$, $u_1^*, \cdots, u_n^* \in E^*$, and $u \in E$ be given. We claim that $u_1^*(b_1) \cdots u_n^*(b_n)u \in L$.

The proof is by induction. If $n = 1$, the conclusion follows from part (a). Assume that the conclusion is true for m. Then, given h_i, u_i^*, $i = 1, \cdots, m+1$, we have $u_1^*(h_1) \cdots u_m^*(h_m)v \in L$. Call it h. Let $k = u_{m+1}^*(h_{m+1})v$. Then $k \in L$, by part (a), and $T(h, k) \in L$, by part (b). Choose $v^* \in E^*$ such that $v^*(T(v, v)) = 1$. By part (a), $v^*(T(h, k))u = u_1^*(h_1) \cdots u_{m+1}^*(h_{m+1})u \in L$, and the proof is complete.

2.3 **Definition.** A vector subspace $L \subset CV_\infty(X; E)$ is called a *polynomial algebra* if any of the equivalent statements (1)–(4) is true.

2.4 **Remark.** The name "polynomial algebra" was introduced by Wulbert (unpublished) for vector subspaces satisfying (2), when one allows all continuous polynomials and not just those of finite type. Subspaces L of $C(X; E)$, for X compact and E a real Banach space, satisfying (1) were considered by Pełczyński [9], but he allowed T to be any multilinear continuous mapping and L contained the constants. Blatter, in [2], considered subspaces of $C_\infty(X; E)$, for X locally compact and E a real Banach space, satisfying (4).

2.5 **Remark.** When $E = \mathbf{K}$, $L \subset CV_\infty(X; \mathbf{K})$ satisfies (1)–(4) if, and only if, L is an algebra under pointwise operations, and the equivalence relation WS_L is just the usual equivalence relation X/L defined by $x \sim y$ if, and only if, $h(x) = h(y)$ for all $h \in L$. We recall that, in this case, L is called localizable under itself in $CV_\infty(X; \mathbf{K})$ (see Nachbin [7]) when the following condition holds true: A function $f \in CV_\infty(X; \mathbf{K})$ is in the closure of L if (and always if), for any $v \in V$, any $\epsilon > 0$, and any equivalence class $Y \subset X$ modulo X/L, there exists $h \in L$ such that $v(x)|f(x) - h(x)| < \epsilon$ for all $x \in Y$. It is immediate that L is localizable if, and only if, L is a WS-space. Theorem 4 of Nachbin [7] gives a sufficient condition for L to be localizable in $CV_\infty(X; \mathbf{K})$.

2.6 **Proposition.** *Suppose that $CV_\infty(X; E)$ is a polynomial algebra and let L be a vector subspace of $CV_\infty(X; E)$. The WS-hull of L is a polynomial algebra such that $M = \{u^*(h); u^* \in E^*, h \in WS(L)\}$ is localizable in $CV_\infty(X; \mathbf{K})$.*

Proof. Let $h \in WS(L)$, $u^* \in E^*$, $u \in E$, and $n \geq 1$ be given. Let $k = u^*(h)^n u$. Since $CV_\infty(X; E)$ is a polynomial algebra, $k \in CV_\infty(X; E)$. Let $(x, y) \in WS_L$. Then $k(x) = u^*(h(x))^n u = u^*(g(x, y)h(y))^n u = g(x, y)k(y)$, since $g(x, y) \in \{0, 1\}$. Hence $k \in WS(L)$, and therefore $WS(L)$ is a polynomial algebra. On the other hand, let $f \in CV_\infty(X; \mathbf{K})$ belong to $WS(M)$. Choose $u \in E$ and $u^* \in E^*$ such that $u^*(u) = 1$. Then $h = f \otimes u$ belongs to $WS(L)$, because the formation of WS-hulls is idempotent. Therefore, $u^*(h) = f \in M$, and M is localizable in $CV_\infty(X; \mathbf{K})$.

2.7 **Corollary.** *Suppose that $CV_\infty(X; E)$ is a polynomial algebra and let L be a Weierstrass-Stone space contained in $CV_\infty(X; E)$. Then its closure is a poly-*

nomial algebra and $M = \{u^*(h);\ u^* \in E^*,\ h \in L\}$ *is localizable in* $CV_\infty(X; \mathbf{K})$.

2.8 **Lemma.** *Let* A *be a selfadjoint subalgebra of* $C(X; \mathbf{K})$ *such that every* $h \in A$ *is bounded on the support of every* $v \in V$, *and let* L *be a vector subspace of* $CV_\infty(X; E)$ *which is an* A*-module. Then,* $f \in CV_\infty(X; E)$ *is in the closure of* L *if (and always only if), for every equivalence class* $Y \subset X$ *modulo* X/A, *every* $v \in V$, *every* $p \in s(E)$, *and every* $\epsilon > 0$, *there is* $k \in L$ *such that for all* $x \in Y$, $v(x)p(f(x) - k(x)) < \epsilon$.

Proof. This is an immediate corollary to Theorem 2 of [8].

2.9 **Theorem.** *Let* L *be a vector subspace of* $CV_\infty(X; E)$ *which is a polynomial algebra such that* $M = \{u^*(h);\ u^* \in E^*,\ h \in L\}$ *is localizable in* $CV_\infty(X; \mathbf{K})$. *Then* L *is a WS-space.*

Proof. Let $A = A(WS)$ be the algebra of all $f \in C_b(X; \mathbf{R})$ which are constant on the equivalence classes modulo WS_L. Notice that $WS(M)$ is an A-module. Therefore, since M is localizable, $\overline{M}$ is an A-module. Let $h \in WS(L)$. Let $Y \subset X$ be an equivalence class modulo X/A, let $v \in V$, $p \in s(E)$, and $\epsilon > 0$ be given. Since M and A define the same equivalence relation on X, namely WS_L, h is constant on Y. Let $u \in E$ be this constant value. If $u = 0$, then $0 \in M \otimes E$ is such that $v(x)p(h(x) - 0) = 0 < \epsilon$ for all $x \in Y$. If $u \neq 0$, there exists $f \in L$ whose constant value on Y, say t, is not zero. Let $u^* \in E^*$ be such that $u^*(t) = 1$. Then $k = u^*(f)u$ belongs to $M \otimes E$ and $v(x)p(h(x) - k(x)) = 0 < \epsilon$ for all $x \in Y$. By Lemma 2.8, h belongs to the closure of $\overline{M} \otimes E$. Since $\overline{M} \otimes E$ is contained in the closure of L, L is a Weierstrass-Stone space.

2.10 **Remark.** A subset L of $CV_\infty(X; E)$ is called *separating* if for any $x, y \in X$ with $x \neq y$, there is $h \in L$ such that $h(x) \neq h(y)$, i.e., if WS_L reduces to the diagonal. In this case, $WS(L)$ is precisely the set of all $h \in CV_\infty(X; E)$ which vanish on the subset $Z(L) = \{x \in X;\ f(x) = 0 \text{ for all } f \in L\}$. When $Z(L) = \emptyset$, we say that L is *everywhere different from zero.*

2.11 **Corollary.** *Let* L *be as in Theorem 2.9. Suppose that* L *is separating. Then,* $f \in CV_\infty(X; E)$ *belongs to the closure of* L *if, and only if,* f *vanishes on the set* $Z(L)$.

2.12 **Corollary.** *Let* L *be as in Theorem 2.9. Suppose that* L *is separating and everywhere different from zero. Then* L *is dense in* $CV_\infty(X; E)$.

3. **Kakutani-Stone and Grothendieck spaces.** In this paragraph E is a *real* locally convex Hausdorff space.

3.1 **Definition.** A subset $M \subset C(X; \mathbf{R})$ is a lattice if $\sup(f, h) \in M$ and $\inf(f, h) \in M$ whenever $f, h \in M$.

3.2 **Theorem** (Nachbin [7]). *Let M be a sublattice of $CV_\infty(X; \mathbf{R})$. A function $f \in CV_\infty(X; \mathbf{R})$ is in the closure of M if, and only if, for every pair $x, y \in X$ and every $\epsilon > 0$, there exists $h \in M$ such that $|h(x) - f(x)| < \epsilon$ and $|h(y) - f(y)| < \epsilon$.*

3.3 **Remark.** It follows from Definition 3.1 that the KS-hull of any vector subspace M of $CV_\infty(X; \mathbf{R})$ is a vector lattice.

3.4 **Theorem.** *Any vector sublattice of $CV_\infty(X; \mathbf{R})$ is a Kakutani-Stone space.*

Proof. Let M be a vector sublattice of $CV_\infty(X; \mathbf{R})$ and let $h \in KS(M)$. Let $x, y \in X$ be given arbitrarily.

Case I. $0 = \delta_x|M$ *or* $0 = \delta_y|M$. Suppose $0 = \delta_x|M$. Then $g(x, x) = 0$ and $h(x) = 0$. If $(x, y) \in KS_M$, then $h(y) = 0$ too, and $0 \in M$ agrees with h at x and y. If $(x, y) \notin KS_M$, then there is some $f \in M$ such that $f(x) = 0 = h(x)$ and $f(y) = h(y)$. The case $0 = \delta_y|M$ is analogous.

Case II. $0 \neq \delta_x|M$ *and* $0 \neq \delta_y|M$. If $(x, y) \in KS_M$, then $0 \neq \delta_x|M = r\delta_y|M$ for some $r > 0$. Choose $f \in M$ such that $f(x) = h(x)$. Then $f(y) = h(y)$, because both f and h belong to $KS(M)$. If $(x, y) \notin KS_M$, $\delta_x|M \neq r\delta_y|M$ for all $r > 0$. Choose $f_x \in M$ such that $f_x(x) = 1$. If $f_x(y) \leq 0$, we can in fact suppose $f_x(y) = 0$, by taking $\sup(f_x, 0) \in M$, if necessary. Choose $f_y \in M$ with $f_y(y) = 1$. Then, the function $f = (h(x) - h(y)f_y(x))f_x + h(y)f_y$ belongs to M and is such that $f(x) = h(x)$ and $f(y) = h(y)$. If $f_x(y) > 0$, there is $h_y \in M$ such that $h_y(x) \neq (f_x(y))^{-1}h_y(y)$. Hence the system

$$af_x(x) + bh_y(x) = h(x), \qquad af_x(y) + bh_y(y) = h(y)$$

can be solved for a and b. Then $f = af_x + bh_y \in M$ agrees with h at x and y. From Theorem 3.2, h belongs to the closure of M, and therefore, M is a Kakutani-Stone space.

3.5 **Corollary.** *Let M be a closed vector subspace of $CV_\infty(X; \mathbf{R})$. Then, M is a KS-space if, and only if, M is a vector sublattice.*

3.6 **Proposition.** *A vector subspace M of $CV_\infty(X; \mathbf{R})$ is a G-space if, and only if, $\overline{M}$ is determined by its restrictions to the two-point subsets of X.*

Proof. For any vector subspace M of $CV_\infty(X; \mathbf{R})$, it follows from the definition of $G(M)$ that $G(M)$ is determined by its restrictions to the two-point subsets of X. Conversely, if $\overline{M}$ is determined by its restrictions to the two-point subsets of X, then an obvious modification of the argument presented in the proof of Theorem 3.4 proves that M is a G-space.

3.7 **Remark.** For any vector subspace M of $CV_\infty(X; \mathbf{R})$, it follows from the definition of $G(M)$ that $G(M)$ contains the function $\sup(0, f, h) + \inf(0, f, h)$, whenever it contains f and h.

3.8 Theorem. *Let M be a vector subspace of $CV_\infty(X; \mathbf{R})$ such that M contains* $\sup(0, f, h) + \inf(0, f, h)$ *whenever it contains f and h. Then M is a Grothendieck space.*

Proof. This result can be proved by modifying conveniently the argument of Lindenstrauss and Wulbert presented in the proof of Theorem 2′ of [5], making use of the fact that, for each $f \in CV_\infty(X; \mathbf{R})$, $v \in V$, and $\epsilon > 0$, the set $K = \{x \in X;\ v(x)|f(x)| \geq \epsilon\}$ is compact.

3.9 **Remark.** It follows from the above results that, if M is an algebra contained in $CV_\infty(X; \mathbf{R})$ such that its closure is a lattice, then M is localizable. More generally, if the closure of M is a G-space, then M is localizable.

3.10 **Definition.** A vector subspace $L \subset CV_\infty(X; E)$ is called a *latticial subspace* (respectively a *Lindenstrauss-Wulbert subspace*) if it satisfies the following conditions:

(a) For each $u^* \in E^*$, $u \in E$, and $h \in L$, then $u^*(h)u \in L$.

(b) There exists a continuous mapping $T\colon E \times E \to E$ such that for some $v \in E$, $T(v, v) \neq 0$ and, for all pairs $a, b \in \mathbf{R}$, $T(av, bv) = \max(a, b)\,T(v, v)$ (respectively $T(av, bv) = (\max(0, a, b) + \min(0, a, b))T(v, v)$) and $T(f, h) \in L$, for all $f, h \in L$.

3.11 **Proposition.** *Let L be a (closed) vector subspace of $CV_\infty(X; E)$ and let $M = \{u^*(f);\ u^* \in E^*,\ f \in L\}$. Consider the following statements.*

(1) *L is a latticial subspace.*

(2) *M is a (closed) KS-space such that $M \otimes E \subset L$.*

(3) *L is a Lindenstrauss-Wulbert subspace.*

(4) *M is a (closed) G-space such that $M \otimes E \subset L$.*

Then, (1) *is equivalent to* (2), *and* (3) *is equivalent to* (4).

Proof. The proofs are similar to the proof of Lemma 2.2, and make use of Theorems 3.4 and 3.8.

3.12 **Proposition.** *Let L be a vector subspace of $CV_\infty(X; E)$. Then $KS(L)$ is a latticial subspace, and $G(L)$ is a Lindenstrauss-Wulbert subspace.*

Proof. Let $u^* \in E^*$, $u \in E$, and $f \in KS(L)$ be given. Let $h = u^*(f)u$. For any pair $(x, y) \in KS_L$ we have: $h(x) = u^*(f(x))u = g(x, y)u^*(f(y))u = g(x, y)h(y)$. Hence $h \in KS(L)$. Now choose $v^* \in E^*$ and $v \in E$ such that $v^*(v) = 1$. Define $T\colon E \times E \to E$ by setting $T(s, t) = \max(v^*(s), v^*(t))v$ for all $s, t \in E$. Then T is continuous and $T(v, v) = v \neq 0$, and $T(av, bv) = \max(a, b)\,T(v, v)$. Let now $f, h \in KS(L)$. It follows that $T(f(x), h(x)) = \max(v^*(f(x)), v^*(h(x)))v = \max(g(x, y)v^*(f(y)), g(x, y)v^*(h(y)))v = g(x, y)\,T(f(y), h(y))$, for all $(x, y) \in KS_L$, so $T(f, h)$ belongs to $KS(L)$. Therefore $KS(L)$ satisfies (a) and (b) of Definition 3.10. The proof that $G(L)$ is a Lindenstrauss-Wulbert subspace is similar.

3.13 Corollary. *Let L be a vector subspace of $CV_\infty(X; E)$. If L is a KS-space (respectively a G-space), then its closure is a latticial subspace (respectively a Lindenstrauss-Wulbert subspace).*

3.14 Theorem. *Let L be a vector subspace of $CV_\infty(X; E)$, which is a latticial subspace (respectively a Lindenstrauss-Wulbert subspace). Assume that $A(KS)$ (respectively $A(G)$) separates the equivalence classes modulo KS_L (respectively G_L). Then L is a Kakutani-Stone space (respectively a Grothendieck space).*

Proof. Let $\Delta \in \{KS, G\}$. Let $M = \{u^*(f); u^* \in E^*, f \in L\}$. From Proposition 3.11, M is a Δ-space contained in $CV_\infty(X; \mathbf{R})$ such that $M \otimes E \subset L$. Since $\Delta(M) = \overline{M}$, $\overline{M} \otimes E$ is an $A(\Delta)$-module. Let $f \in CV_\infty(X; E)$ belong to $\Delta(L)$. Let $Y \subset X$ be an equivalence class modulo $X/A(\Delta)$. Since $A(\Delta)$ separates the equivalence classes modulo Δ_L, the set Y is contained in some equivalence class modulo Δ_L. Fix $x_0 \in Y$. If $f(x_0) = 0$, then $f(x) = 0$ for all $x \in Y$, and the function $0 \in M \otimes E$ agrees with f throughout Y. If $f(x_0) \neq 0$, there is some $h \in L$ such that $h(x_0) \neq 0$. Choose $u^* \in E^*$ such that $u^*(h(x_0)) = 1$. Let $k = u^*(h)f(x_0) \in M \otimes E$. Then k agrees with f throughout Y. By Lemma 2.8, f belongs to the closure of $\overline{M} \otimes E$. Since $\overline{M} \otimes E$ is contained in the closure of L, L is a Δ-space

3.15 Remark. The hypothesis that $A(\Delta)$ separates the equivalence classes modulo Δ_L is equivalent to the hypothesis that $C_b(F; \mathbf{R})$ separates points, where F is the quotient space of X by the equivalence relation Δ_L. One instance in which this occurs is the following. Suppose that our completely regular Hausdorff space X is in fact a normal space and that Δ_L is an upper semicontinuous relation, i.e., Δ_L is closed (Willard [12, Theorem 9.9]). Then, F is a normal Hausdorff space too (Engelking [4, Theorem 5, p. 85]). Another instance occurs when Δ_L reduces to the diagonal. Indeed, in this case, $A(\Delta)$ is $C_b(X; \mathbf{R})$. Notice that, in this case, as was pointed out in 2.10, $\Delta(L)$ is the set of all $f \in CV_\infty(X; E)$ which vanish on the set $Z(L) = \{x \in X; h(x) = 0 \text{ for all } h \in L\}$.

3.16 Corollary. *Let L be a vector subspace of $CV_\infty(X; E)$, which is a latticial subspace (respectively a Lindenstrauss-Wulbert subspace). Suppose that KS_L (respectively G_L) reduces to the diagonal. Then, $f \in CV_\infty(X; E)$ belongs to the closure of L, if and only if, f vanishes on the set $Z(L) = \{x \in X; h(x) = 0$ for all $h \in L\}$.*

3.17 Corollary. *Let L be as in Corollary 3.16, and suppose that KS_L (respectively G_L) reduces to the diagonal and that L is everywhere different from zero. Then L is dense in $CV_\infty(X; E)$.*

3.18 Remark. Our Theorem 3.14 does not contain Theorem 1.13 of Blatter

[2]. The argument presented there rests on the fact that $C_\infty(X; \mathbf{R})$ and the KS-hull (respectively the G-hull) of M have the approximation property. However, his arguments can be used to prove the following generalization of Theorem 1.13 of [2].

3.19 **Theorem.** *Let X be a completely regular Hausdorff k-space, and let E be a complete locally convex Hausdorff space. Suppose that for every compact subset $K \subset X$ and every $t > 0$, there exists $v \in V$ such that $v(x) \geq t$ for all $x \in K$. Let L be a vector subspace of $CV_\infty(X; E)$ which is a latticial subspace, respectively a Lindenstrauss-Wulbert subspace. Then L is a KS-space, respectively a G-space.*

Proof. The space $CV_\infty(X; \mathbf{R})$ has the approximation property, by Theorem 3, §5, Bierstedt [1].

4. **Examples and applications.** Let V be the set of characteristic functions of compact subsets of X. The Nachbin space $CV_\infty(X; E)$ is just $C(X; E)$ endowed with the compact-open topology. A subset $L \subset C(X; E)$ is called selfadjoint if $M = \{u^*(f); u^* \in E^*, f \in L\}$ is a selfadjoint subset of $C(X; \mathbf{K})$.

4.1 **Theorem (Weierstrass-Stone).** *Let L be a selfadjoint polynomial subalgebra of $C(X; E)$. A function $f \in C(X; E)$ belongs to the closure of L if, and only if, the following conditions hold:*

(1) *For every $x, y \in X$ such that $f(x) \neq f(y)$, there is $h \in L$ such that $h(x) \neq h(y)$.*

(2) *For every $x \in X$, such that $f(x) \neq 0$, there is $h \in L$ such that $h(x) \neq 0$.*

Proof. Since every selfadjoint subalgebra of $C(X; \mathbf{K})$ is localizable [6, Theorem 1, §30], Theorem 2.9 implies that L is a Weierstrass-Stone space, and therefore its closure is precisely $WS(L)$. It remains only to notice that $f \in WS(L)$ if, and only if, conditions (1) and (2) are satisfied.

4.2 **Corollary.** *Let L be a selfadjoint polynomial subalgebra of $C(X; E)$, separating and everywhere different from zero on X. Then L is dense in $C(X; E)$.*

4.3 **Corollary.** *Let A be a selfadjoint subalgebra of $C(X; \mathbf{K})$, separating and everywhere different from zero on X. Then $A \otimes E$ is dense in $C(X; E)$. In particular, $C(X; \mathbf{K}) \otimes E$ is dense in $C(X; E)$.*

Suppose that X is the Cartesian product of two completely regular Hausdorff spaces X_1 and X_2. Let L be the set of all finite sums of functions of the form $(x_1, x_2) \mapsto f(x_1)h(x_2)$, where $f \in C(X_1; \mathbf{K})$ and $h \in C(X_2; E)$. Dieudonné's approximation theorem on Cartesian products, stating that $C(X_1; \mathbf{K}) \otimes C(X_2; E)$ is dense in $C(X_1 \times X_2; E)$, is now an easy consequence of Corollary 4.2.

4.4 Theorem. *Let E and F be two locally convex Hausdorff spaces over the reals. The closure in the compact-open topology of the set of all continuous polynomials p of finite type from E into F, with $p(0) = 0$, consists of all mappings $f \in C(E; F)$ such that $f(0) = 0$.*

Proof. This follows from a direct application of Theorem 4.1.

4.5 Theorem (Infinite-dimensional Weierstrass polynomial approximation theorem). *Let E and F be two locally convex Hausdorff spaces over the reals. Then $P_f(E; F)$ is dense in $C(E; F)$.*

Proof. Apply Corollary 4.2 to $L = P_f(E; F)$.

4.6 Remark. The space $P_f(E; F)$ is not selfadjoint in the complex case. Instead of $P_f(E; F)$ one has to consider the vector subspace $P_f^*(E; F)$ defined as follows. For each integer $n \geq 1$, $P_f^*(^nE; F)$ denotes the vector subspace of $C(E; F)$ generated by the set of all maps of the form $x \mapsto u^*(x)^n u$, where $u \in F$ and u^* or its complex conjugate belongs to E^*. Then $P_f^*(E; F)$ is, by definition, the vector subspace generated by the union of all $P_f^*(^nE; F)$ with $n \geq 1$ and the constant maps. Theorem 4.5 now holds with $P_f^*(E; F)$ substituted for $P_f(E; F)$.

Let X be a locally compact Hausdorff space and let $V = C_\infty^+(X)$, the set of all positive continuous functions vanishing at infinity on X. The Nachbin space $CV_\infty(X; E)$ is then $C_b(X; E)$ endowed with the strict topology β. Again we are in the bounded case of the approximation problem, and therefore every selfadjoint subalgebra of $C_b(X; \mathbf{K})$ is localizable under itself in the strict topology. Hence Theorem 4.1 and its corollaries hold when $C_b(X; E)$ with the strict topology is substituted for $C(X; E)$ with the compact-open topology.

5. Extension theorems. Let Y be a closed subset of X. Then, for every weight v on X, its restriction to Y is a weight on Y. If V is a set of weights on X, we denote by $V|Y$ the set of weights on Y that are restriction to Y of some element of V. The corresponding Nachbin space $C(V|Y)_\infty(Y; E)$ will be denoted by $CV_\infty(Y; E)$. Let L be a vector subspace of $CV_\infty(X; E)$. A function $f \in CV_\infty(Y; E)$ is said to be extensible in L if there exists $h \in L$ such that $h|Y = f$, i.e. $h(x) = f(x)$ for all $x \in Y$. The set of all such extensible functions is denoted by $L|Y$. Notice that, if L is a polynomial algebra, or a latticial subspace, or a Linderstrauss-Wulbert subspace, the same is true of $L|Y$ as subspace of $CV_\infty(Y; E)$.

5.1 Theorem. *Let Y be a closed subset of a completely regular Hausdorff space X. Let L be a vector subspace of $CV_\infty(X; E)$ which is a polynomial algebra such that every $h \in L$ is bounded on the support of every $v \in V$. Assume that L is selfadjoint and that $L|Y$ is closed in $CV_\infty(Y; E)$. Then $f \in CV_\infty(Y; E)$ is extensible in L if, and only if, the following conditions are satisfied:*

(1) *For any* $x, y \in Y$ *such that* $f(x) \neq f(y)$, *there is* $h \in L$ *such that* $h(x) \neq h(y)$.

(2) *For any* $x \in Y$ *such that* $f(x) \neq 0$, *there is* $h \in L$ *such that* $h(x) \neq 0$.

Proof. Under the hypothesis made, $L|Y$ is a WS-space.

5.2 **Corollary.** *Let* Y *and* L *be as in Theorem* 5.1. *If* $L|Y$ *is separating and everywhere different from zero on* Y, *then every* $f \in CV_{\infty}(Y; E)$ *is extensible in* L.

5.3 **Theorem.** *Let* Y *be a compact subset of a completely regular Hausdorff space* X, *let* E *be a Banach space, and let* $f \in C(Y; E)$. *There is* $h \in C_b(X; E)$ *such that* $h|Y = f$ *and* $\|f\|_Y = \|h\|_X$.

Proof. Let $L = C_b(X; E) \subset C(X; E)$. An obvious modification of the argument given in Stone [11, p. 64] proves that $L|Y$ is closed in $C(Y; E)$. Since X is completely regular, $C_b(X; E)$ is separating and everywhere different from zero on X, in particular on Y. By Corollary 5.2, every $f \in C(Y; E)$ is extensible in $C_b(X; E)$. The final statement follows from the fact that for any function $f \in C(X; E)$, which is bounded on Y, there is $h \in C_b(X; E)$ such that $h|Y = f$ and $\|f\|_Y = \|h\|_X$.

5.4 **Theorem.** *Let* Y *be a closed subset of a normal Hausdorff space* X, *let* E *be a Banach space, and let* $f \in C(Y; E)$ *be a compact mapping. There exists a compact mapping* $h \in C(X; E)$ *such that* $h|Y = f$ *and* $h(X)$ *is contained in the closed convex hull of* $f(Y)$.

Proof. If βX and βY denote the Stone-Čech compactifications of X and Y respectively, it is known that βY identifies with the closure of Y in βX, (see [10]) and that a mapping $f \in C(Y; E)$ has an extension to βY if, and only if, $f(Y)$ is precompact. Hence we can extend f to βY though a function $w \in C(\beta Y; E)$. By Theorem 5.3 we can extend w to βX. If v denotes such an extension, let $u = v|X$. Then $u \in C(X; E)$ and $u|Y = f$. Since E is a Banach space, the closed convex hull K of $f(Y)$ is compact and, therefore, a retract of E (see [3]). Let r be a retraction of E onto K. Then $h = r(u)$ is a compact mapping that answers all the requirements.

5.5 **Theorem.** *Let* Y *be a closed subset of a completely regular Hausdorff space* X, *let* E *be a Banach space. A function* $f \in C_{\infty}(Y; E)$ *is extensible in* $C_{\infty}(X; E)$ *if, and only if, the following conditions hold*:

(1) *For any* $x, y \in Y$ *such that* $f(x) \neq f(y)$, *there is* $h \in C_{\infty}(X; E)$ *such that* $h(x) \neq h(y)$.

(2) *For any* $x \in Y$ *such that* $f(x) \neq 0$, *there is* $h \in C_{\infty}(X; E)$ *such that* $h(x) \neq 0$.

If $f \in C_\infty(Y; E)$ is extensible, we can choose an extension $h \in C_\infty(X; E)$ such that $\|f\|_Y = \|h\|_X$.

Proof. Take $L = C_\infty(X; E)$. As in Theorem 5.3, $L|Y$ is closed in $C_\infty(Y; E)$. Theorem 5.5 follows then from Theorem 5.1.

5.6 **Corollary.** *Let Y be a closed subset of a locally compact Hausdorff space X, let E be a Banach space. Then, every $f \in C_\infty(X; E)$ is extensible in $C_\infty(X; E)$, and we can choose an extension with the same norm.*

Proof. Since X is locally compact, the space $C_\infty(X; E)$ is separating and everywhere different from zero.

BIBLIOGRAPHY

1. K.-D. Bierstedt, *Gewichtete Räume Stetiger Vektorwertiger Funktionen und das Injektive Tensorprodukt*, Ph. D. Dissertation, Johannes Gutenberg-Universität in Mainz, 1970.

2. J. Blatter, *Grothendieck spaces in approximation theory*, Mem. Amer. Math. Soc. No. 120 (1972).

3. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. 1 (1951), 353–367. MR 13, 373.

4 R. Engelking, *Outline of general topology*, PWN, Warsaw, 1965; English transl., North-Holland, Amsterdam; Interscience, New York, 1968. MR 36 #4508; 37 #5836.

5. J. Lindenstrauss and D. E. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Functional Analysis 4 (1969), 332–349. MR 40 #3274.

6. L. Nachbin, *Elements of approximation theory*, Van Nostrand Math. Studies, no. 14, Van Nostrand, Princeton, N. J., 1967. MR 36 #572.

7. ———, *Weighted approximation for function algebras and quasi-analytic mappings*, Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 330–333. MR 33 #7835.

8. L. Nachbin, J. B. Prolla and S. Machado, *Concerning weighted approximation, vector fibrations, and algebras of operators*, J. Approximation Theory 6 (1972), 80–89.

9. A. Pełczyński, *A generalization of Stone's theorem on approximation*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), 105–107. MR 19, 135.

10. M. H. Stone, *On the compactification of topological spaces*, Ann. Soc. Polon. Math. 21 (1948), 153–160. MR 10, 137.

11. ———, *The generalized Weierstrass approximation theorem*. In Buck, R. C., Studies in Modern Analysis, MAA Studies in Mathematics 1 (1962), 30–87.

12. S. Willard, *General topology*, Addison-Wesley, Reading, Mass., 1970. MR 41 #9173.

INSTITUTO DE MATEMATICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, CAIXA POSTAL 1835, ZC-00, 20000 RIO DE JANEIRO, GB, BRASIL