

## SOME POSITIVE TRIGONOMETRIC SUMS

BY

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**ABSTRACT.** Vietoris found an interesting generalization of the classical inequality  $\sum_{k=1}^n (\sin k\theta/k) > 0$ ,  $0 < \theta < \pi$ . A simplified proof is given for his inequality and his similar inequality for cosine series. Various new results which follow from these inequalities include improved estimates for the location of the zeros of a class of trigonometric polynomials and new positive sums of ultraspherical polynomials which extend an old inequality of Fejér. Both of Vietoris' inequalities are special cases of a general problem for Jacobi polynomials, and a summary is given of known results on this problem.

1. Vietoris' theorems. In a little known paper [18] Vietoris proved the following theorems.

**Theorem 1.** If  $a_0 \geq a_1 \geq \dots \geq a_n > 0$  and  $(2k)a_{2k} \leq (2k-1)a_{2k-1}$ ,  $k \geq 1$ , then

$$s_n(x) = \sum_{k=1}^n a_k \sin kx > 0, \quad 0 < x < \pi,$$

and

$$t_n(x) = \sum_{k=0}^n a_k \cos kx > 0, \quad 0 < x < \pi.$$

**Theorem 2.** If

$$(1.1) \quad c_{2k} = c_{2k+1} = 2^{-2k} \binom{2k}{k}, \quad k \geq 0,$$

then

$$(1.2) \quad \sigma_n(x) = \sum_{k=1}^n c_k \sin kx > 0, \quad 0 < x < \pi,$$

and

$$(1.3) \quad \tau_n(x) = \sum_{k=0}^n c_k \cos kx > 0, \quad 0 < x < \pi.$$

These two theorems are equivalent. The second is obviously a special case of the first. On the other hand, Theorem 1 follows from Theorem 2. For if

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$d_0 \geq d_1 \geq \dots \geq d_n > 0$ , a summation by parts shows that

$$\sum_{k=1}^n c_k d_k \sin kx > 0, \quad 0 < x < \pi,$$

and

$$\sum_{k=0}^n c_k d_k \cos kx > 0, \quad 0 < x < \pi.$$

Letting  $a_k = c_k d_k$ ,  $0 \leq k \leq n$ , gives Theorem 1.

For completeness we will give a proof of Theorem 2. Some of Vietoris' ideas will be used, but many of the difficulties of his proof have been replaced by easier arguments. However the reader should be aware that greater elegance often means diminished power, and Vietoris' original arguments may turn out to be useful for the more complicated problems which are mentioned at the end of this paper. The applications include estimates for the zeros of certain trigonometric polynomials and an improvement of a forty-year old result of Fejér on sums of ultraspherical polynomials.

For the proof we need three lemmas.

**Lemma 1.** For  $m \geq 1$  we have  $\binom{2m}{m} < 2^{2m}(\pi m)^{-1/2}$ .

**Proof.** Let  $\gamma_m = m^{1/2} 2^{-2m} \binom{2m}{m}$ . Then  $\gamma_m < \gamma_{m+1}$  for  $m \geq 1$ ; and by Stirling's formula,  $\gamma_m \rightarrow \pi^{-1/2}$  as  $m \rightarrow \infty$ .

**Lemma 2.** If the sequence  $\{c_k\}_{k=0}^{\infty}$  is defined by (1.1), then for  $0 < x < \pi$ ,

$$(1.4) \quad \sum_{k=1}^{\infty} c_k \sin kx = \sum_{k=0}^{\infty} c_k \cos kx = \left( \frac{1}{2} \cot \frac{x}{2} \right)^{1/2}.$$

**Proof.** For  $|z| \leq 1$ ,  $z \neq 1$ , we have  $(1-z)^{-1/2} = \sum_{k=0}^{\infty} c_{2k} z^k$ . Since  $c_{2k} = c_{2k+1}$ , it follows that

$$(1+z)(1-z^2)^{-1/2} = \sum_{k=0}^{\infty} c_k z^k$$

for  $|z| \leq 1$ ,  $z \neq \pm 1$ . On setting  $z = e^{ix}$ ,  $0 < x < \pi$ , and separating real and imaginary parts, we get (1.4).

**Lemma 3.** Let  $P_r(x) = \sum_{k=0}^r b_k e^{ikx}$ , where  $b_0 \geq b_1 \geq \dots \geq b_r > 0$ . Then for  $n > m \geq 0$  we have

$$(1.5) \quad |P_n(x) - P_m(x)| \leq b_{m+1} / \sin(x/2), \quad 0 < N < 2\pi.$$

**Proof.** Sum by parts and use the standard estimate  $|\sum_{k=0}^n e^{ikx}| \leq 1/\sin(x/2)$ . We are now ready to prove Theorem 2.

**Proof.** Consider  $\sigma_n$  first. We may assume  $n \geq 2$ . Different arguments are needed for each of the intervals  $0 < x \leq \pi/n$ ,  $\pi/n < x < \pi - \pi/n$  and  $\pi - \pi/n \leq x < \pi$ .

For  $0 < x \leq \pi/n$ , all terms in the sum are nonnegative, and the first is strictly positive.

For  $\pi - \pi/n \leq x < \pi$ , set  $x = \pi - y$ , so that  $0 < y \leq \pi/n$ . If  $n$  is even, say  $n = 2m$ , we have

$$\begin{aligned}\sigma_n(x) &= \sum_{k=1}^{2m} (-1)^{k-1} c_k \sin ky = \sum_{k=1}^m [c_{2k-1} \sin (2k-1)y - c_{2k} \sin 2ky] \\ &= \sum_{k=1}^m (2k-1)c_{2k-1} \left[ \frac{\sin (2k-1)y}{2k-1} - \frac{\sin 2ky}{2k} \right].\end{aligned}$$

This last sum has positive terms, since  $t^{-1} \sin t$  is decreasing in  $t$  on  $(0, \pi]$  and  $2ky \leq 2my = ny \leq \pi$ . And if  $n$  is odd there is an extra term,  $c_n \sin ny$ , which is positive for  $0 < y < \pi/n$ .

If  $n \geq 3$ , we must still consider the interval  $\pi/n < x < \pi - \pi/n$ . There we have  $\sin x > \sin(\pi/n) \geq (\pi/n)(1 - \pi^2/6n^2)$ . Now by Lemmas 2 and 3,

$$\sigma_n(x) \geq (\tfrac{1}{2} \cot(x/2))^{1/2} - c_{n+1}/\sin(x/2);$$

hence, for  $\pi/n < x < \pi - \pi/n$ , we have

$$(1.6) \quad 2 \sin(x/2) \sigma_n(x) \geq [(\pi/n)(1 - \pi^2/6n^2)]^{1/2} - 2c_{n+1}.$$

Here, the term in square brackets is decreasing in  $n$  for  $n \geq 3$ , and  $c_{2m} = c_{2m+1}$  for  $m \geq 0$ . Hence, the right hand side of (1.6) is positive for  $n = 2m - 1$ , if it is positive for  $n = 2m$ . And for  $n = 2m$  it follows from Lemma 1 that the right hand side of (1.6) is at least equal to

$$(2\pi m)^{-1/2} \{ \pi(1 - \pi^2/24m^2) \}^{1/2} - 2\sqrt{2};$$

this is positive for  $m \geq 2$ . Therefore  $\sigma_n(x) > 0$  for  $\pi/n < x < \pi - \pi/n$ . The theorem is now proven for  $\sigma_n$ .

For  $r_n$  the proof is similar to that for  $\sigma_n$ , but the details are slightly more involved. The result is obvious for  $n = 0$  and  $n = 1$ , and an elementary computation shows that  $r_2(x) = \cos^2 x + \cos x + \tfrac{1}{2} > 0$ . We may therefore assume  $n \geq 3$ .

Firstly, we observe that  $r_n(x) > 0$  for  $0 < x \leq \pi/n$ , since

$$\frac{dr_n}{dx} = - \sum_{k=1}^n k c_k \sin kx < 0, \quad 0 < x < \pi/n,$$

and

$$r_n\left(\frac{\pi}{n}\right) = \sum_{k=0}^{[n/2]} (c_k - c_{n-k}) \cos \frac{k\pi}{n} > 0.$$

Secondly, we show that  $r_n(x) > 0$  for  $\pi - \pi/(n+1) < x < \pi$ . We set  $y = \pi - x$ , and write

$$r_n(x) = \sum_{k=0}^{[(n-1)/2]} c_{2k} [\cos 2ky - \cos (2k+1)y] + \epsilon_n,$$

where  $\epsilon_n = 0$  if  $n = 2m - 1$  and  $\epsilon_n = c_{2m} \cos 2my$  if  $n = 2m$ . When  $\epsilon_n = 0$ , the monotonicity of  $\cos x$ ,  $0 \leq x \leq \pi$ , shows that  $r_n(x) > 0$  for  $0 < y < \pi/n$ . When  $n = 2m$ , we have

$$\begin{aligned} r_n(x) &\geq c_{2m} (1 - \cos y + \cos 2y - \cos 3y + \cdots + \cos 2my) \\ &= c_{2m} (1 + \cos x + \cos 2x + \cdots + \cos 2mx). \end{aligned}$$

Summing this series gives

$$r_n(x) \geq c_{2m} \frac{\sin(m + \frac{1}{2})x \cos mx}{\sin(x/2)} = c_{2m} \frac{\cos(m + \frac{1}{2})y \cos my}{\cos(y/2)}.$$

It follows that  $r_n(x) > 0$  for  $0 < (m + \frac{1}{2})y < \pi/2$ , that is, for  $0 < y < \pi/(n+1)$ .

Lastly, we consider the interval  $\pi/(n+1) \leq x \leq \pi - \pi/(n+1)$  for  $n \geq 3$ . The same argument as for  $\sigma_n(x)$  on  $\pi/n < x < \pi - \pi/n$  shows that it is enough if

$$[\pi/(n+1)(1 - \pi^2/6(n+1)^2)]^{1/2} - 2c_{n+1} > 0.$$

Here again, it suffices to consider even values of  $n$ , say  $n = 2m$ . Computation shows that this inequality holds for  $n = 4$  and  $6$ . For  $m \geq 4$ , the stronger inequality

$$\left[ \frac{\pi}{2m+1} \left( 1 - \frac{\pi^2}{6(2m+1)^2} \right) \right]^{1/2} - \frac{2}{\sqrt{\pi m}} > 0$$

holds, since it holds for  $m = 4$  and since its left hand side, when multiplied by  $\sqrt{m}$ , is an increasing function of  $m$ .

This concludes the proof for  $r_n$ .

**2. Applications to other trigonometric sums.** A trigonometric polynomial with real coefficients, of degree  $n$ , has exactly  $2n$  zeros (modulo  $2\pi$ ) [12, §VI, Problem 14]. Pólya [11] proved that a sine polynomial, or a cosine polynomial, with positive and monotonically increasing coefficients, has all its zeros real and simple. Then Szegő [15] showed how one could obtain estimates for the location of the zeros of such polynomials; this is the problem which we now consider.

We require the following corollary of Theorem 1.

**Corollary 1.** *If  $(2k-1)A_{k-1} \geq 2kA_k > 0$  for  $k \geq 1$ , and if  $0 < x < 2\pi$ , then*

$$\sum_{k=0}^n A_k \sin(k + \frac{1}{4})x > 0 \quad \text{and} \quad \sum_{k=0}^n A_k \cos(k + \frac{1}{4})x > 0.$$

**Proof.** Set  $\alpha_k = 2^{-2k} \binom{2k}{k}$ . It suffices to establish the particular case  $A_k = \alpha_k$ ,  $k = 0, \dots, n$ ; the general case then follows by partial summation.

Now by Theorem 2, we have

$$\sum_{k=0}^n \alpha_k [\sin 2kx + \sin (2k+1)x] > 0, \quad 0 < x < \pi,$$

$$\sum_{k=0}^n \alpha_k [\cos 2kx + \cos (2k+1)x] > 0, \quad 0 < x < \pi.$$

Hence,

$$\sum_{k=0}^n \alpha_k \sin (2k + \frac{1}{2})x > 0 \quad \text{and} \quad \sum_{k=0}^n \alpha_k \cos (2k + \frac{1}{2})x > 0$$

for  $0 < x < \pi$ , and the proof is complete.

**Corollary 2.** Let  $A_1, \dots, A_n$  satisfy the conditions of Corollary 1. If  $0 \leq \nu \leq \frac{1}{4}$  and  $0 < x < 2\pi$ , or  $-\frac{1}{4} \leq \nu \leq \frac{1}{4}$  and  $0 < x < \pi$ , then  $\sum_{k=0}^n A_k \cos (k + \nu)x > 0$ .

**Proof.** Use Corollary 1, and the identity

$$\cos (k + \nu)x = \cos (k + \frac{1}{4})x \cos (\nu - \frac{1}{4})x - \sin (k + \frac{1}{4})x \sin (\nu - \frac{1}{4})x.$$

Similarly, one proves

**Corollary 3.** Let  $A_1, \dots, A_n$  satisfy the conditions of Corollary 1. If  $\frac{1}{4} \leq \nu \leq \frac{1}{2}$  and  $0 < x < 2\pi$ , or  $\frac{1}{4} \leq \nu \leq \frac{3}{4}$  and  $0 < x < \pi$ , then  $\sum_{k=0}^n A_k \sin (k + \nu)x \geq 0$ .

We shall combine Corollary 1 with an argument due to Szegő, and obtain bounds for the zeros of a wide class of trigonometric polynomials. Szegő [15] applied the classical inequality of Fejér

$$(2.1) \quad \sum_{k=0}^n \sin (k + \frac{1}{2})x \geq 0, \quad 0 \leq x \leq 2\pi,$$

to prove, among other things, the following two theorems.

**Theorem A.** If  $\lambda_0 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , and if  $s_k, t_k$  denote the zeros of

$$p(t) = \lambda_0 \cos nt + \lambda_1 \cos (n-1)t + \dots + \lambda_{n-1} \cos t + \lambda_n$$

and of

$$q(t) = \lambda_0 \sin nt + \lambda_1 \sin (n-1)t + \dots + \lambda_{n-1} \sin t,$$

respectively, on  $0 < t < \pi$ , then

$$(2.2) \quad (k - \frac{1}{2})\pi / (n + \frac{1}{2}) < s_k < (k + \frac{1}{2})\pi / (n + \frac{1}{2}), \quad k = 1, \dots, n,$$

$$(2.3) \quad k\pi / (n + \frac{1}{2}) < t_k < (k + 1)\pi / (n + \frac{1}{2}), \quad k = 1, \dots, n-1.$$

**Theorem B.** *If, in addition to the conditions in Theorem A, we have*

$$(2.4) \quad 2\lambda_0 - \lambda_1 > \lambda_1 - \lambda_2 \geq \lambda_2 - \lambda_3 \geq \dots \geq \lambda_{n-1} - \lambda_n \geq \lambda_n \geq 0,$$

*then the right hand sides of (2.2) and (2.3) can be replaced by  $k\pi/n$  and  $(k + \frac{1}{2})\pi/n$ , respectively.*

It is interesting to remark at this juncture that one may not replace  $\sin(k + \frac{1}{2})x$  by  $\cos(k + \frac{1}{2})x$  in (2.1), since

$$\sum_{k=0}^n \cos(k + \frac{1}{2})x = \frac{\sin(n+1)x}{2 \sin(x/2)}, \quad 0 < x < 2\pi.$$

In contrast, we have at our disposal in Corollary 1 a sine polynomial and the corresponding cosine polynomial, both positive on  $(0, 2\pi)$ . This allows us to strengthen both sides of the inequalities in Theorem A under a supplementary condition on the  $\lambda$ 's different from the one in Theorem B. We shall prove

**Theorem 3.** *Let  $p(t)$ ,  $q(t)$ ,  $s_k$  and  $t_k$  be defined as in Theorem A. If*

$$(2.5) \quad (2k-1)\lambda_{k-1} \geq 2k\lambda_k > 0, \quad k \geq 1,$$

*then*

$$(2.6) \quad (k - \frac{1}{2})\pi/(n + \frac{1}{4}) < s_k < k\pi/(n + \frac{1}{4}), \quad k = 1, \dots, n,$$

$$(2.7) \quad k\pi/(n + \frac{1}{4}) < t_k < (k + \frac{1}{2})\pi/(n + \frac{1}{4}), \quad k = 1, \dots, n-1.$$

**Proof.** Since  $p(t) + iq(t) = \sum_{k=0}^n \lambda_k e^{i(n-k)t}$ , we have

$$e^{-i(n+1/4)t} [p(t) + iq(t)] = \sum_{k=0}^n \lambda_k \cos(k + \frac{1}{4})t - i \sum_{k=0}^n \lambda_k \sin(k + \frac{1}{4})t,$$

whence

$$p(t) \cos(n + \frac{1}{4})t + q(t) \sin(n + \frac{1}{4})t = \sum_{k=0}^n \lambda_k \cos(k + \frac{1}{4})t,$$

$$p(t) \sin(n + \frac{1}{4})t - q(t) \cos(n + \frac{1}{4})t = \sum_{k=0}^n \lambda_k \sin(k + \frac{1}{4})t.$$

By Corollary 1, therefore,

$$(2.8) \quad p(t) \cos(n + \frac{1}{4})t + q(t) \sin(n + \frac{1}{4})t > 0, \quad 0 < t < 2\pi,$$

$$(2.9) \quad p(t) \sin(n + \frac{1}{4})t - q(t) \cos(n + \frac{1}{4})t > 0, \quad 0 < t < 2\pi.$$

Letting  $t = k\pi/(n + \frac{1}{4})$  in (2.8) and  $t = (k + \frac{1}{2})\pi/(n + \frac{1}{4})$  in (2.9) leads to

$$(-1)^k p(k\pi/(n + \frac{1}{4})) > 0, \quad k = 0, 1, \dots, n,$$

$$(-1)^k p((k + \frac{1}{2})\pi/(n + \frac{1}{4})) > 0, \quad k = 0, 1, \dots, n-1;$$

and these inequalities imply (2.6). Similarly, we see that (2.7) holds, and the theorem is proved.

The other zeros of  $p(t)$  are at  $t = 2m\pi \pm s_k$  and those of  $q(t)$ , at  $t = 2m\pi \pm t_k$  and at  $t = m\pi$  ( $m = 0, \pm 1, \pm 2, \dots$ ).

The left hand inequalities in (2.6) and (2.7) are only slight improvements over the corresponding inequalities in Theorem A, but the right hand inequalities are slight improvements even on Theorem B. The condition (2.4) implies  $\lambda_k \geq c(n-k)$  for some  $c > 0$ , while the condition (2.5) in Theorem 3 implies only  $\lambda_k \geq c'(n-k)^{1/2}$ . This gives some indication of the depth of Vietoris' theorem.

3. Applications to ultraspherical polynomials. Both the Fejér-Jackson-Gronwall inequality,

$$(3.1) \quad \sum_{k=1}^n \frac{\sin kx}{k} > 0, \quad 0 < x < \pi,$$

and W. H. Young's inequality,

$$1 + \sum_{k=1}^n \frac{\cos kx}{k} > 0, \quad 0 < x < \pi,$$

are contained in Theorem 1.

Since (3.1) implies or is related to many other inequalities (see for example [2], [8] and the references given there), the existence of an extension of (3.1) suggests that some of these other inequalities can also be extended. This is true to some extent, as we proceed to show.

Certain power series occur so frequently that their coefficients have acquired names and have been studied in detail. One such series is

$$(1 - 2xr + r^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)r^n,$$

where  $C_n^{\nu}(x)$  is a polynomial of degree  $n$  in  $x$ . These are the ultraspherical polynomials; many of their properties are discussed in Chapter 4 of [16] and Chapter 10 of [6].

Fejér [7] proved that, for  $n = 1, 2, \dots$ ,

$$(3.2) \quad \sum_{k=0}^n C_k^{\lambda}(x) > 0, \quad -1 < x < 1, \quad 0 < \lambda \leq \frac{1}{2};$$

in other words, the power series coefficients of  $(1-r)^{-1}(1-2xr+r^2)^{-\lambda}$  are positive if  $0 < \lambda \leq \frac{1}{2}$ . One extension of this result was proved in [4]:

**Theorem 4.** *The function  $(1-r)^{-2\nu}(1-2xr+r^2)^{-\nu}$  has positive power series coefficients for  $-1 < x < 1$ ,  $\nu > 0$ .*

This theorem is equivalent to

$$\sum_{k=0}^n \frac{(2\nu)_{n-k}}{(n-k)!} C_k^{\nu}(x) > 0, \quad -1 < x < 1, \quad \nu > 0.$$

Here,  $(a)_k$  is defined by  $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$ ; observe that  $\alpha_k = 2^{-2k} \binom{2k}{k} = (\frac{1}{2})_k / k!$ .

A different extension of Fejér's inequality (3.2) is

**Theorem 5.** For  $0 < \nu \leq \frac{1}{4}$ ,  $\sum_{k=0}^n (\frac{1}{2})_k C_k^\nu(x) / (2\nu)_k > 0$ ,  $-1 < x < 1$ .

**Proof.** Combine Corollary 2 with the following Mehler-type integral representation for  $C_n^\nu(x)$  [5, 3.15.2(23)]:

$$(\sin \theta)^{2\nu-1} C_n^\nu(\cos \theta) = \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\nu)} \frac{(2\nu)_n}{n!} \int_0^\theta [\cos \phi - \cos \theta]^{\nu-1} \cos(n+\nu)\phi d\phi, \\ \nu > 0, 0 < \theta < \pi.$$

A summation by parts shows that Theorem 5 is stronger than (3.2) for  $0 < \nu < \frac{1}{4}$ .

A consequence of the case  $\lambda = \frac{1}{2}$  of (3.2) is

$$(3.3) \quad \sum_{k=0}^n \frac{C_k^\nu(x)}{C_k^\nu(1)} > 0, \quad -1 < x < 1, \nu \geq \frac{1}{2}.$$

This follows from Feldheim's integral [8],

$$\frac{C_n^\nu(\cos \theta)}{C_n^\nu(1)} = \frac{2\Gamma(\nu + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(\nu - \lambda)} \int_0^{\pi/2} \sin^{2\lambda} \phi \cos^{2\nu-2\lambda-1} \phi (1 - \sin^2 \theta \cos^2 \phi)^{n/2} \\ \cdot \frac{C_n^\lambda(\cos \theta (1 - \sin^2 \theta \cos^2 \phi)^{-1/2})}{C_n^\lambda(1)} d\phi, \\ \nu > \lambda > -\frac{1}{2}, 0 < \theta < \pi,$$

which shows that if (3.3) holds for some value of  $\nu$ ,  $\nu > -\frac{1}{2}$ , then it holds for all larger values of  $\nu$ . For  $\nu \geq 1$ , (3.3) follows from Feldheim's integral and (3.1), since

$$(3.4) \quad C_n^1(\cos \theta) / C_n^1(1) = (\sin(n+1)\theta) / (n+1) \sin \theta.$$

Using Feldheim's integral, (1.2) and (3.4) give

$$(3.5) \quad \sum_{k=0}^n (k+1) c_{k+1} \frac{C_k^\nu(x)}{C_k^\nu(1)} > 0, \quad -1 < x < 1, \nu \geq 1,$$

where  $c_{2k} = c_{2k+1} = (\frac{1}{2})_k / k!$ . Since  $k c_k < (k+1) c_{k+1}$  for  $k \geq 1$ , (3.5) implies (3.3) for  $\nu \geq 1$ , by partial summation.

An inequality in the same vein as (3.3) is

**Theorem 6.** For  $\nu > 0$  and  $-1 < x < 1$ , we have

$$\sum_{k=0}^n c_k \frac{C_k^\nu(x)}{C_k^\nu(1)} > 0,$$

where  $c_{2k} = c_{2k+1} = (\frac{1}{2})_k / k!$



The proof follows immediately from Feldheim's integral and Theorem 2, since  $\lim_{\lambda \rightarrow 0} (C_n^\lambda(\cos \theta)/C_n^\lambda(1)) = \cos n\theta$ . This result was pointed out to us by George Gasper. It is weaker than (3.3) for  $\nu \geq \frac{1}{2}$ .

**4. Open problems.** To fit Viëtoris' theorem into a general context it is necessary to introduce Jacobi polynomials,  $P_n^{(\alpha, \beta)}(x)$ , which may be defined by

$$(4.1) \quad (1-x)^\alpha(1+x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{n+\alpha}(1+x)^{n+\beta}], \quad \alpha, \beta > -1.$$

They satisfy the orthogonality relations

$$(4.2) \quad \int_{-1}^1 P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx = 0, \quad m \neq n,$$

$$(4.3) \quad \int_{-1}^1 [P_n^{(\alpha, \beta)}(x)]^2 (1-x)^\alpha (1+x)^\beta dx = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} = [b_n^{(\alpha, \beta)}]^{-1}.$$

The standard references are [16, Chapter 4] and [6, Chapter 10].

Viëtoris' theorem and many other results are special cases of the following problem.

**Problem 1.** Expand  $f(x) = (1-x)^{-\gamma}(1+x)^{-\delta}$  in an orthogonal series of Jacobi polynomials,

$$(4.4) \quad f(x) \sim \sum_{n=0}^{\infty} a_n b_n^{(\alpha, \beta)} P_n^{(\alpha, \beta)}(x)$$

where

$$a_n = \int_{-1}^1 (1-x)^{-\gamma}(1+x)^{-\delta} P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx.$$

For which values of  $\alpha, \beta, \gamma, \delta$  are all the partial sums of (4.4) nonnegative, i.e.

$$\sum_{k=0}^n a_k b_k^{(\alpha, \beta)} P_k^{(\alpha, \beta)}(x) \geq 0, \quad -1 \leq x \leq 1, \quad n = 0, 1, \dots?$$

Since

$$(4.5) \quad P_n^{(-1/2, -1/2)}(\cos \theta) / P_n^{(-1/2, -1/2)}(1) = \cos n\theta$$

and

$$(4.6) \quad P_n^{(1/2, 1/2)}(\cos \theta) / P_n^{(1/2, 1/2)}(1) = (\sin(n+1)\theta) / (n+1) \sin \theta,$$

Lemma 2 and Theorem 2 give a positive solution to Problem 1 for

$$(4.7) \quad \alpha = \beta = -1/2, \quad \gamma = 1/4, \quad \delta = -1/4,$$

$$(4.8) \quad \alpha = \beta = 1/2, \quad \gamma = 3/4, \quad \delta = 1/4.$$

The case  $\alpha = \gamma, \beta = \delta$  has been studied in some detail. Problem 1 has a positive solution for  $\alpha = \beta, -1 < \alpha \leq 3/2$ , [16, Theorem 15.5] and [3], a negative solution for  $\alpha > 3/2$ ; and the general case  $(\alpha, \beta)$  has a positive solution for

$\alpha, \beta \geq 0, \alpha + \beta \leq 1; \alpha = \beta + 1, -1/2 \leq \beta \leq 0; \beta = \alpha + 1, -1/2 \leq \alpha \leq 0$  [1]; and a negative solution for  $\alpha > \beta + 1$  or  $\beta > \alpha + 1$  [1], and also for  $\alpha > 3/2$  or  $\beta > 3/2$  [16, Theorem 15.2.2 and Theorem 15.4].

Other known cases of Problem 1 are

$$(4.9) \quad \alpha = 1/2, \quad \beta = -1/2, \quad \gamma = 1, \quad \delta = 0,$$

which reduces to Fejér's sum

$$(4.10) \quad \sum_{k=0}^n \sin(k + 1/2)\theta \geq 0,$$

and

$$(4.11) \quad \alpha = \beta, \quad -1 < \alpha \leq 0, \quad \gamma = \alpha + 1/2, \quad \delta = 0,$$

which was proven by Fejér [7] for  $-1/2 \leq \alpha \leq 0$  and Szegő [17] for  $-1 < \alpha < -1/2$ .

Most of the methods which have been used to obtain positive results for Problem 1 are special and do not extend to other cases. The one exception is the use of the positive sum [2],

$$(4.12) \quad \sum_{k=0}^n \frac{P_k^{(\alpha, \beta)}(x)}{P_k^{(\beta, \alpha)}(1)} > 0, \quad -1 < x \leq 1, \quad -1 < \alpha \leq \beta + 1, \quad \alpha + \beta \geq 0.$$

This inequality is probably true for  $\beta \geq -1/2, \alpha + \beta \geq 0$ , and if so it would give a new powerful weapon to attack special cases of Problem 1. In particular (4.12) for  $\alpha = 3/2, \beta = -1/2$  is equivalent to

$$(4.13) \quad \sum_{k=0}^n (k+1)^2 \frac{P_{2k}^{(3/2, 3/2)}(x)}{P_{2k}^{(3/2, 3/2)}(1)} \geq \frac{1}{4} \sum_{k=0}^n \frac{P_{2k}^{(3/2, 3/2)}(x)}{P_{2k}^{(3/2, 3/2)}(1)}, \quad -1 \leq x \leq 1,$$

and Problem 1 for  $\alpha = \beta = \gamma = \delta = 3/2$  is

$$(4.14) \quad \sum_{k=0}^n (k+1)^2 \frac{P_{2k}^{(3/2, 3/2)}(x)}{P_{2k}^{(3/2, 3/2)}(1)} \geq 0, \quad -1 \leq x \leq 1.$$

Since

$$\sum_{k=0}^n \frac{P_{2k}^{(3/2, 3/2)}(x)}{P_{2k}^{(3/2, 3/2)}(1)} > 0, \quad -1 \leq x \leq 1,$$

(this is the even part of (3.3) for  $\nu = 2$ ), it is clear that (4.13) is a deeper result than (4.14). However (4.12) is not true for all  $(\alpha, \beta)$ . For instance, it fails for  $\alpha < 1/2, \beta = -1/2$ , when  $n = 2$ .

Rather than leave the impression that all of these problems would be solved if only (4.12) held, we should remark that Problem 1 can contain deeper results than (4.12). For example, when  $\alpha = \beta = 1/2$ , (4.12) is just the Fejér-Jackson-Gronwall inequality, and Vietoris' inequality (1.2) is a deeper result. Inequality (4.12)

is deeper the closer  $\beta$  is to  $-\frac{1}{2}$  but unfortunately it is also harder to prove then. The common source of these two problems seems to be the following problem about Bessel functions:

**Problem 2.** For which values of  $(\alpha, \beta)$  is

$$(4.15) \quad \int_0^x t^{-\beta} J_\alpha(t) dt \geq 0, \quad x \geq 0?$$

The condition  $\beta < \alpha + 1$  is necessary for the convergence of (4.15) at zero, and will not always be mentioned below.

The connection between (4.12) and Problem 2 comes from

$$\lim_{n \rightarrow \infty} \frac{P_n^{(\alpha, \beta)}(\cos(\theta/n))}{P_n^{(\alpha, \beta)}(1)} = 2^\alpha \Gamma(\alpha + 1) \theta^{-\alpha} J_\alpha(\theta).$$

If (4.12) holds for a pair  $(\alpha, \beta)$  then Problem 2 has a positive solution for the same  $(\alpha, \beta)$ . The connection between Problems 1 and 2 comes from

$$(4.16) \quad x^{\beta-1} = \frac{\int_0^\infty t^{-\beta} J_\alpha(xt) dt}{\int_0^\infty t^{-\beta} J_\alpha(t) dt}.$$

Problem 2 is much more tractable than either of the problems involving Jacobi polynomials. When  $\alpha \geq \frac{1}{2}$ , (4.15) holds for  $\beta = -\frac{1}{2}$  (and hence for  $\beta \geq -\frac{1}{2}$ , by the second mean value theorem) by a theorem of Makai ([10], [13]); and the left hand side of (4.15) changes sign infinitely often on  $0 < x < \infty$  if  $\beta < -\frac{1}{2}$  [14]. For  $-1 < \alpha < \frac{1}{2}$ , there is a unique solution  $\beta(\alpha)$  ( $-\frac{1}{2} < \beta(\alpha) < 0$ ) to the equation

$$(4.17) \quad \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t) dt = 0,$$

where  $j_{\alpha,2}$  is the second positive zero of  $J_\alpha(t)$ . Then (4.15) holds for  $\beta(\alpha) \leq \beta < \alpha + 1$  and the only cases of equality are  $x = 0$  or  $\beta = \beta(\alpha)$ ,  $x = j_{\alpha,2}$ . This has been proved by E. Makai when  $-\frac{1}{2} < \alpha < \frac{1}{2}$ , it is classical when  $\alpha = -\frac{1}{2}$ , and the proof for  $-1 < \alpha < -\frac{1}{2}$  follows. Makai's proof will appear in the near future.

Let  $j_{\alpha,n}$  denote the  $n$ th positive zero of  $J_\alpha(x)$ . Then Makai's comparison theorem [10] applied to the differential equation

$$y'' + [(4x^2 - 4\alpha^2 + 1)/4x^2]y = 0, \quad y = x^{1/2} J_\alpha(x),$$

implies

$$\left| \int_{j_{\alpha,n-1}}^{j_{\alpha,n}} x^{1/2} J_\alpha(x) dx \right| > \left| \int_{j_{\alpha,n}}^{j_{\alpha,n+1}} x^{1/2} J_\alpha(x) dx \right|, \quad n = 2, 3, \dots, |\alpha| > 1/2.$$

The second mean value theorem then gives

$$(4.18) \quad \left| \int_{j_{\alpha,n-1}}^{j_{\alpha,n}} x^{-\beta} J_\alpha(x) dx \right| > \left| \int_{j_{\alpha,n}}^{j_{\alpha,n+1}} x^{-\beta} J_\alpha(x) dx \right|, \\ n = 2, 3, \dots, -1 < \alpha < -1/2, -1/2 < \beta < 0.$$

The proof of (4.15) for  $-1 < \alpha < -\frac{1}{2}$  will be complete if we show that  $\beta(\alpha)$  defined by (4.17) exists,  $-\frac{1}{2} < \beta(\alpha) < 0$ , and is unique. Let

$$g(\beta) = \int_0^{j_{\alpha,2}} t^{-\beta} J_{\alpha}(t) dt.$$

By Cooke's theorem (see [13]),  $g(0) > 0$ , and we will show that  $g(-1-\alpha) < 0$ . This gives the existence of  $\beta(\alpha)$  and the inequalities  $-\frac{1}{2} < -1-\alpha < \beta(\alpha) < 0$ ,  $-1 < \alpha < -\frac{1}{2}$ .

Recall the classical differentiation formula [6, 7.2.8 (50)]

$$\frac{d}{dt} \{t^{\alpha+1} J_{\alpha+1}(t)\} = t^{\alpha+1} J_{\alpha}(t).$$

Integrating this gives

$$\int_0^{j_{\alpha,2}} t^{\alpha+1} J_{\alpha}(t) dt = (j_{\alpha,2})^{\alpha+1} J_{\alpha+1}(j_{\alpha,2}), \quad \alpha > -1.$$

Now  $J_{\alpha+1}(j_{\alpha,2}) < 0$  since for  $\alpha > -1$  we have  $J_{\alpha+1}(x) < 0$ ,  $j_{\alpha+1,1} < x < j_{\alpha+1,2}$  and  $j_{\alpha+1,1} < j_{\alpha,2} < j_{\alpha+1,2}$ . Thus there is at least one  $\beta(\alpha)$ , and  $\int_0^x t^{-\beta(\alpha)} J_{\alpha}(t) dt > 0$ ,  $x \neq j_{\alpha,2}$ , follows from the above argument. Then an integration by parts shows that  $\int_0^x t^{-\beta} J_{\alpha}(t) dt > 0$ ,  $x > 0$ ,  $\beta > \beta(\alpha)$ , so  $\beta(\alpha)$  is unique.

It is likely that  $\beta(\alpha)$  is a decreasing function of  $\alpha$  for  $-1 < \alpha < -\frac{1}{2}$ . The following values of  $\beta(\alpha)$  were computed by J. Al-Abdulla using an integration package supplied by C. de Boor.

$\alpha$	$\beta(\alpha)$
-.5	-.1915562
-.4	-.2259427
-.3	-.2593436
-.2	-.2918541
-.1	-.3235531
0	-.3545096
.1	-.3847832
.2	-.4144258
.3	-.4434834
.4	-.4719960
.5	-.5000000

Makai's result gives a new proof of a theorem of Szegő [8]. Szegő proved that

$$\int_0^x t^{-\alpha} J_{\alpha}(t) dt \geq 0, \quad \alpha \geq \bar{\alpha}, x \geq 0,$$

where  $\bar{\alpha}$  is the solution of

$$\int_0^{\bar{\alpha}} t^{2-\alpha} J_{\alpha}(t) dt = 0.$$

Let  $R$  be the set of  $(\alpha, \beta)$  for which (4.15) holds. When  $-1 < \alpha \leq \frac{1}{2}$  the boundary of  $R$  is the set of points where (4.17) holds, and from Sonine's first integral  $R$  contains  $(\alpha + \mu, \beta + \mu)$ ,  $\mu > 0$ , when it contains  $(\alpha, \beta)$ . See the argument in [8]. Another proof of Szegő's theorem is given in [9].

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