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JORDAN ALGEBRAS AND CONNECTIONS ON HOMOGENEOUS SPACES

BY

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ABSTRACT. We use the correspondence between G-invariant connections on a reductive homogeneous space G/H and certain nonassociative algebras to explicitly compute the pseudo-Riemannian connections in terms of a Jordan algebra J of endomorphisms. It is shown that if G and H are semisimple Lie groups, then J is a semisimple Jordan algebra. Also a general method for computing examples of J is given.

1. Introduction. In this paper we present many algebraic results that are related to the differential geometry of homogeneous spaces and we now give some background results. Thus, let G be a connected Lie group with Lie algebra g and H a closed (Lie) subgroup with Lie subalgebra h. The pair (G, H) on (g, h) is called a *reductive pair* if there exists a subspace m of g such that $g = m + h$ (direct sum) with $(\mathrm{Ad}\ H)m \subset m$; that is, in terms of the algebras $[h, m] \subset m$. The corresponding homogeneous space G/H is called a *reductive* homogeneous space. In this case if we also have $[m, m] \subset h$, then (g, h) is a *symmetric pair* and G/H is a symmetric space. We shall be most interested in the case when (g, h) is not a symmetric pair.

In [7] there was established a correspondence between G-invariant connections ∇ on a reductive space G/H and certain nonassociative algebras (m, α) where α is a bilinear multiplication function on m as follows.

Theorem. *Let G/H be a reductive homogeneous space with fixed Lie algebra decomposition $g = m + h$ with $(\mathrm{Ad}\ H)m \subset m$. Then there is a one-to-one correspondence between the set of all G-invariant connections ∇ on G/H and the set of nonassociative algebras (m, α) so that* Ad H *is contained in the automorphism group of (m, α); that is,* $(\mathrm{Ad}\ P)\alpha(X, Y) = \alpha((\mathrm{Ad}\ P)X, (\mathrm{Ad}\ P)Y)$ *for all $P \in H$ and X, $Y \in m$.*

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If the G-invariant connection ∇ on G/H is a pseudo-Riemannian connection that is induced by a nondegenerate form C on m, then the algebra satisfies the following conditions; see [4], [7]. The mappings $a(X): m \to m: Y \to \alpha(X, Y)$ are C-skew symmetric for all $X \in m$ and the mappings $\mathrm{Ad}\, P: m \to m: Y \to (\mathrm{Ad}\, P)Y$ satisfy $C((\mathrm{Ad}\, P)X, (\mathrm{Ad}\, P)Y) = C(X, Y)$ for all $X, Y \in m$ and $P \in H$. Thus by "differentating" the automorphism and isometry conditions for $\mathrm{Ad}\, H$ we investigate this situation in terms of nonassociative algebras (m, α) which are determined as follows. Let (g, h) be a reductive pair with a fixed direct sum decomposition $g = m + h$ and $[h, m] \subset m$. Let $\alpha: m \times m \to m$ be a bilinear multiplication function which defines an algebra (m, α) such that $\mathrm{ad}\, U: m \to m: Y \to [U, Y]$ for all $U \in h$ is in the derivation algebra of (m, α). Let C be a nondegenerate symmetric bilinear form on m such that the mappings $a(X)$ and $\mathrm{ad}\, U$ for all $X \in m$, $U \in h$ are C-skew symmetric. We now denote such an algebra (m, α) by (m, α, C) and note that in terms of algebras that (m, α, C) determines the corresponding pseudo-Riemannian G-invariant connections.

A well behaved example of this situation is when a pseudo-Riemannian connection ∇ is of the first kind; that is, for all $X \in m$ the maps $t \to \pi \exp tX$ are geodesics where $\pi: G \to G/H$ is the canonical projection. In this case $\alpha(X, Y) = \frac{1}{2}XY$ where $XY = [X, Y]_m$ is the projection of $[X, Y]$ in g into the m-component of the decomposition $g = m + h$; see [7]. For this case we change the notation and let B denote the corresponding nondegenerate form and let $(m, \frac{1}{2}XY, B)$ denote the corresponding algebra. The case where g and h are semisimple gives a reductive pair (g, h) where $g = m + h$ with $m = h^{\perp}$ relative to the Killing form of g and B is the Killing form restricted to $m \times m$; see Corollary 2, §2.

Jordan algebras enter into the analysis of the connection ∇ given by (m, α, C) by comparing ∇ with a given connection of the first kind determined by $(m, \frac{1}{2}XY, B)$ as follows. Thus assuming a pseudo-Riemannian connection exists (which we do throughout this paper), we have since B and C are nondegenerate forms that there exists a unique $S \in GL(m)$ such that

(1) $$C(X, Y) = B(SX, Y).$$

Since B is a symmetric form, S satisfies

(2) $$S^b = S$$

where b denotes the adjoint relative to B. Since $\mathrm{ad}\, U$ is B- and C-skew symmetric, S also satisfies

(3) $$[\mathrm{ad}\, U, S] = 0$$

for all $U \in h$. For $B(S \text{ ad } UX, Y) = C(\text{ad } UX, Y) = -C(X, \text{ad } UY) = -B(SX, \text{ad } UY) = B(\text{ad } USX, Y)$.

The set of endomorphisms of m satisfying (2) and (3) form a Jordan algebra J relative to the usual operation $S_1S_2 = \frac{1}{2}(S_1 \circ S_2 + S_2 \circ S_1)$ where $\circ$ is composition of endomorphisms. Thus the connection ∇ determined by (m, α, C) corresponds to a unique invertible element S in J. Now from results in [4], [9] the formula for α is given by

$$(4) \qquad 2\alpha(X, Y) = XY + S^{-1}[X(SY) - (SX)Y]$$

where $\frac{1}{2}XY$ is the multiplication in $(m, \frac{1}{2}XY, B)$. Thus given an invertible element in J we can define the connection ∇ corresponding to (m, α, C) by using formulas (1) and (4) to determine C and α. However for any $T \in J$, $S = \exp T$ is invertible in J. Thus in this way there is a correspondence between the set of connections (determined above by (m, α, C)) and the elements of J. Furthermore, using (4) these connections are parameterized by elements of the symmetric space $M = J \cap GL(m)$; see [5]. Note that for $S = I \in J \cap GL(m)$ we obtain the original algebra $(m, \frac{1}{2}XY, B)$ so that M is nonempty.

We now note that the relationship between Jordan algebras and connections is rather general. For in the case of a Lie group G (with $H = \{e\}$) if we assume the algebra $(g, \frac{1}{2}[X, Y], B)$ exists where ad X is B-skew symmetric, then the corresponding Jordan algebra is $J = \{S \in \text{End}(g)\colon S^b = S\}$. Conversely, given any Jordan algebra J of endomorphisms of a vector space V which are symmetric relative to a nondegenerate form B on V, then we can regard V as an abelian Lie algebra and form $(g, \frac{1}{2}[X, Y], B)$. Thus the corresponding Jordan algebra of this triple is just J.

In the remaining sections we discuss mainly reductive pairs (g, h) with g and h semisimple. We compare (m, α, C) with $(m, \frac{1}{2}XY, B)$ where B is the Killing form restricted to $m \times m$ as mentioned above. Thus we find a direct sum decomposition $m = K_0 + K_1 + K_2$ into ad h-invariant subspaces and show J is semisimple. We give examples of this decomposition and J, but leave the complete classification of J as an open problem. This is a hard problem taking into account the classification of semisimple subalgebras of a semisimple Lie algebra [1]. Finally in §5 we give some sample applications involving the connection and curvature of G/H regarded as functions of $M = J \cap GL(m)$.

2. Semisimplicity of J. Let (g, h) be a reductive pair with decomposition $g = m + h$ and $(\text{ad } h)m \subset m$. From the Lie algebra identities for g, it is easy to see that ad $h \subset \text{Der}(m, \frac{1}{2}XY)$ which is the derivation algebra of $(m, \frac{1}{2}XY)$. We now assume there is a symmetric nondegenerate form B on m so that $(m, \frac{1}{2}XY, B)$

exists; that is, for all $X \in m$, $U \in h$ the mappings

$$L(X): m \longrightarrow m: Y \longrightarrow XY \quad \text{and} \quad \text{ad}\, U: m \longrightarrow m: Y \longrightarrow [U, Y]$$

are B-skew symmetric. Now let (m, α, C) be another algebra as discussed in §1; specifically, (m, α) is an algebra and C is a symmetric nondegenerate form so that for all $X \in m$, $U \in h$ the mappings $a(X): m \longrightarrow m: Y \longrightarrow \alpha(X, Y)$ and ad $U: m \longrightarrow m: Y \longrightarrow [U, Y]$ are C-skew symmetric where ad $h \subset \text{Der}(m, \alpha)$ which is the derivation algebra of (m, α). Let $J = \{S \in \text{End}(m): [S, \text{ad}\, U] = 0$ all $U \in h$ and $S^b = S\}$ be the Jordan algebra which relates all the possible algebras (m, α, C) to $(m, \tfrac{1}{2}XY, B)$ as previously discussed. We now have the following.

Theorem 1. *Let* (g, h) *and* $(m, \tfrac{1}{2}XY, B)$ *be as above and let* ad h *be completely reducible in* m. *Then* J *is a semisimple Jordan algebra.*

Proof. Let A be the associative subalgebra of $\text{End}(m)$ given by

$$A = \{T \in \text{End}(m): [T, \text{ad}\, U] = 0 \quad \text{all} \quad U \in h\}.$$

Then A is closed under the involution b induced by B; that is, $B(TX, Y) = B(X, T^b Y)$. For, from $0 = [T, \text{ad}\, U]$ we see $0 = [T^b, \text{ad}\, U]$ using $(\text{ad}\, U)^b = -\text{ad}\, U$. Thus A is an associative algebra with involution and the set $H(A, b) = \{T \in A: T^b = T\}$ of B-symmetric elements is a Jordan algebra relative to the usual multiplication. From [6] we see that the Jordan radical is given by $\text{Rad}\, H(A, b) = H(A, b) \cap \text{Rad}\, A$ where $\text{Rad}\, A$ is the radical of the associative algebra A. Therefore noting that $J = H(A, b)$ it suffices to show A is semisimple.

Thus since ad h is completely reducible on m, we see from [3, p. 102] that the centralizer

$$C = \{T \in gl(m): [T, \text{ad}\, U] = 0 \text{ all } U \in h\}$$

is completely reducible on m. But identifying $gl(m)$ with $\text{End}(m)$, this means A is completely reducible on m. This implies A is semisimple.

Corollary 2. *Let* h *be a semisimple subalgebra of the semisimple Lie algebra* g. *Then* (g, h) *is a reductive pair with decomposition* $g = m + h$ *where* $m = h^{\perp}$ *relative to the Killing form,* Kill, *of* g. *Also for* $X, Y \in m$, *the form* $B(X, Y) = \text{Kill}(X, Y)$ *is nondegenerate on* m *and the algebra* $(m, \tfrac{1}{2}XY, B)$ *satisfies* $L(X)$ *and* ad U *are* B*-skew symmetric for all* $X \in m$, $U \in h$. *In this case the Jordan algebra* J *is semisimple.*

Proof. The last statement follows by noting ad h is semisimple and therefore is completely reducible in m. The first parts were proven in [10] which we now sketch. Since h is semisimple, the Killing form of g restricted to h is nondegenerate. Thus we can decompose $g = m + b$ with $m = b^{\perp}$. With B as above we see ad U is B-skew symmetric for $U \in b$ and the map $L(X)$: $m \longrightarrow m$: $Y \longrightarrow XY$ is B-skew symmetric as follows. Let $b(X, Y) = [X, Y]$ be the projection of $[X, Y]$ in g into b, then since $m = b^{\perp}$ we have $B(L(X)Y, Z) = B(XY + b(X, Y), Z) = \text{Kill}([X, Y], Z) = -\text{Kill}(Y, [X, Z]) = -B(Y, XZ + b(X, Z)) = -B(Y, L(X)Z)$.

In the next sections we obtain a more detailed decomposition for m and J in the case g is semisimple and b is semisimple.

3. **Decompositions for m and J.** We continue the discussion of the previous section and obtain a decomposition for m (Theorem 3). Some of these results are buried in [9] and we repeat them for convenience. For any reductive pair (g, b) with decomposition $g = m + b$ and $(\text{ad } b)m \subset m$, the identities of g induce identities on (m, XY) as follows. As before for $X, Y \in m$ let $[X, Y] = XY + b(X, Y)$ where $XY = [X, Y]_m$ is the projection of $[X, Y]$ into m and similarly $b(X, Y) = [X, Y]_b$. For $U \in b$ we also use the notation $D(U) = \text{ad } U$ restricted to m and $D(X, Y) = D(b(X, Y))$. Then we have

(3.1) $\quad XY = -YX$, bilinear

(3.2) $\quad D(X, Y) = -D(Y, X)$, bilinear

(3.3) $\quad D(X, Y)Z + D(Y, Z)X + D(Z, X)Y = X(YZ) + Y(ZX) + Z(XY)$

(3.4) $\quad D(XY, Z) + D(YZ, X) + D(ZX, Y) = 0$

(3.5) $\quad [D(U), D(X, Y)] = D(D(U)X, Y) + D(X, D(U)Y)$

(3.6) $\quad D(U)(XY) = (D(U)X)Y + X(D(U)Y)$.

Now assume g and b are semisimple; then from Corollary 2 we see that $g = m + b$ with $m = b^{\perp}$ relative to the Killing form on g and that $B(X, Y) = \text{Kill}(X, Y)$ is nondegenerate and $L(X)$ and $D(U)$ are B-skew symmetric. Next let

$$K_0 = \{X \in m : (\text{ad } U)X = 0 \text{ for all } U \in b\}$$

then using (3.6) and (3.3) we see that K_0 is a Lie subalgebra of (m, XY). We also have

$$b(K_0, m) = 0 \tag{3.7}$$

for if $W \in K_0$, $Z \in m$ and $U \in b$, then

$$0 = \text{Kill}_g((\text{ad } U)W, Z) = \text{Kill}_g([U, W], Z) = \text{Kill}_g(U, [W, Z])$$
$$= \text{Kill}_g(U, WZ + b(W, Z)) = \text{Kill}_g(U, b(W, Z)), \quad \text{using } m = b^{\perp}$$

and since $\mathrm{Kill}\,|\,h \times h$ is nondegenerate, $h(W, Z) = 0$. From this we see $[K_0, m] = K_0 m \subset m$ so that m is a $\mathrm{ad}_g K_0$-module and also $L: K_0 \to L(K_0)$ is a representation of K_0 on m where $L(K_0) = \{L(V): V \in K_0\}$. Thus let $V, W \in K_0$ on $Z \in m$, then from (3.3) and (3.7) we obtain $0 = V(WZ) + W(ZV) + Z(VW) = ([L(V), L(W)] - L(VW))\,Z$ which gives the result

$$L(VW) = [L(V), L(W)]. \tag{3.8}$$

A similar computation shows that $L(K_0) \subset \mathrm{Der}(m, XY)$. Also since $D(U)$ is a derivation of (m, XY), noting (3.6), we have $[D(U), L(V)] = L(D(U)\,V) = 0$ for $V \in K_0$ and $U \in h$.

Next we use some results on centralizers to show $K_0 = c \oplus K_0'$ where c is the center of K_0 and K_0' is zero or a semisimple Lie algebra in (m, XY). Since g and h are semisimple, $L = \mathrm{ad}\, g$ and $L_1 = \mathrm{ad}\, h$ are completely reducible on g. Therefore the centralizer, $C_L(L_1)$, of L_1 in L is completely reducible on g [3, p. 102]. But

$$C_L(L_1) = \{\mathrm{ad}\, W \in \mathrm{ad}\, g : [\mathrm{ad}\, h, \mathrm{ad}\, W] = 0\} = \{\mathrm{ad}\, W \in \mathrm{ad}\, g : \mathrm{ad}[h, W] = 0\}.$$

Since g is semisimple, $\mathrm{ad}[h, W] = 0$ implies $[h, W] = 0$. We now show $W \in K_0$ as follows. Let $W = W_1 + W_2$ where $W_1 \in m$ and $W_2 \in h$, then for $U \in h$, $0 = [U, W] = [U, W_1] + [U, W_2]$. From the ad h-invariant direct sum $g = m + h$, this implies $[h, W_i] = 0$. But since h is semisimple and $[h, W_2] = 0$ we see $W_2 = 0$; thus $W = W_1 \in m$ which means $W_1 \in K_0$. This implies $C_L(L_1) = \mathrm{ad}_g K_0$ is completely reducible in g and therefore completely reducible on $\mathrm{ad}_g K_0$-submodules of g.

Next for $X \in m$ and $V \in K_0$ we see

$$\mathrm{ad}_g V(X) = [V, X] = VX + h(V, X) = L(V)X, \quad \text{using (3.7)}$$

so that the action of $\mathrm{ad}_g K_0$ and $L(K_0)$ on m are the same. Since m is an $\mathrm{ad}_g K_0$-submodule, this implies $L(K_0)$ is completely reducible on m. Thus from [3] we may write $L(K_0) = L(c) \oplus L(K_0')$ for suitable subsets c and K_0' of K_0 and where $L(c)$ is the center of $L(K_0)$ and $L(K_0')$ is zero or semisimple. Now the map $K_0 \to L(K_0)$: $V \to L(V)$ is an isomorphism of Lie algebras. For it is a homomorphism using (3.8) and $L(V) = 0$ implies $\mathrm{ad}_g V = 0$ so that $V = 0$. From this isomorphism we obtain the desired result

$$K_0 = c \oplus K_0'.$$

Next we consider a direct sum decomposition for m and show $B\,|\,K_0 \times K_0$ is nondegenerate (recall $B(X, Y) = \mathrm{Kill}(X, Y)$). Since ad h is completely reducible on m and K_0 is ad h-invariant we have a direct sum $m = K_0 + b$ where b is ad h-invariant. Furthermore $(\mathrm{ad}\, h)b = b$. For if $p = (\mathrm{ad}\, h)b$ is a proper ad h-submodule of b, then by the complete reducibility of ad h on b, we can find an ad h-invariant complement p' with $b = p' + p$ (direct sum). Thus $(\mathrm{ad}\, h)p' \subset p' \cap (\mathrm{ad}\, h)b = p' \cap p = 0$ so that $p' \subset K_0$, a contradiction. Thus $m = K_0 + b$ with $(\mathrm{ad}\, h)b = b$.

To see that $B \mid K_0 \times K_0$ is nondegenerate, let $U \in K_0$ be such that $B(U, K_0) = 0$. Then

$$B(U, m) = B(U, K_0 + b) = B(U, (\mathrm{ad}\, h)b) = -B((\mathrm{ad}\, h)U, b) = 0$$

and since $B \mid m \times m$ is nondegenerate, $U = 0$. We also have $B \mid c \times c$ and $B \mid K_0' \times K_0'$ are nondegenerate.

Since $B \mid K_0 \times K_0$ is nondegenerate we can decompose $m = K_0 + b$ where we can assume $b = K_0^{\perp}$ relative to B, $(\mathrm{ad}\, h)b = b$ and $B \mid b \times b$ is nondegenerate. Using $b = K_0^{\perp}$ note that $L(K_0)b \subset b$ since $B(K_0, K_0 b) = B(K_0 K_0, b) = 0$ using K_0 is a subalgebra of (m, XY). Now let

$$K_1 = \{Y \in b : L(U)Y = 0 \text{ all } U \in K_0\}$$

then since $L(K_0) \subset \mathrm{Der}(m, XY)$ (remarks following (3.8)), we see K_1 is a subalgebra of (m, XY). Also from the remarks following (3.8) we see $[L(U), \mathrm{ad}\, P] = 0$ for $U \in K_0$, $P \in h$ so that K_1 is ad h-invariant.

Since $L(K_0)$ is completely reducible in m and b is an $L(K_0)$-submodule, we can decompose $b = K_1 + K_2$ into $L(K_0)$-submodules. In a manner similar to the proof that $B \mid K_0 \times K_0$ is nondegenerate, we see $B \mid K_1 \times K_1$ is nondegenerate. Thus we can assume $K_2 = K_1^{\perp}$ relative to $B \mid b \times b$ and we have that K_2 is $L(K_0)$- and ad h-invariant and $B \mid K_2 \times K_2$ is nondegenerate. We summarize some of these results.

Theorem 3. (a) *Let g be a semisimple Lie algebra and h a semisimple subalgebra, then there is a decomposition $g = m + h$ where $m = h^{\perp}$ relative to the Killing form of g and (g, h) is a reductive pair.*

(b) *The algebra (m, XY) is such that $m = c + K_0' + K_1 + K_2$ as an* ad h-*invariant direct sum and satisfies the multiplicative relations*

$$cc = cK_0' = cK_1 = 0, \quad cK_2 \subset K_2,$$
$$K_0' K_0' = K_0', \quad K_0' K_1 = 0, \quad K_0' K_2 \subset K_2,$$
$$K_1 K_1 \subset K_1, \quad K_1 K_2 \subset K_2, \quad K_2 K_2 \subset m.$$

(c) $K_0 = c + K_0'$ *is such that* $(\mathrm{ad}\, h)K_0 = 0$ *and* K_0' *is zero or a semisimple Lie subalgebra of* (m, XY). *Also* $(\mathrm{ad}\, h)K_i = K_i$ *for* $i = 1, 2$.

(d) $h + K_0$ *is a Lie subalgebra of* g *and* m *is an* ad$(h + K_0)$*-module.* $L(K_0) \subset$ Der(m, XY) *and* $[L(K_0), \text{ad } h] = 0$. *Also* $L(K_0)K_2 = K_2$ *and* K_2 *is* ad$(h + K_0 + K_1)$*-invariant.*

Proof. We have previously discussed (a) and most of (b) except $K_1K_2 \subset K_2$. To see this we let $U \in K_1$, $V \in K_2$ and $UV = X + Y + Z$ where $X \in K_0$, $Y \in K_1$ and $Z \in K_2$. Now $X = 0$ as follows. For any $P \in K_0$,

$$B(P, X) = B(P, UV - Y - Z) = B(P, UV), \quad \text{orthogonality}$$
$$= B(PU, V) = 0$$

using $K_0K_1 = 0$. Thus since $B \mid K_0 \times K_0$ is nondegenerate, $X = 0$. Similarly let $Q \in K_1$, then

$$B(Q, Y) = B(Q, UV) = B(QU, V) = 0$$

using $QU \in K_1K_1 \subset K_1$ and $B(K_1, K_2) = 0$. Since $B \mid K_2 \times K_2$ is nondegenerate, $Y = 0$.

Parts (c) and (d) have been discussed except showing K_2 is ad$(h + K_0 + K_1)$-invariant. To see this we note that ad$(h + K_0)K_2 = (\text{ad } h)K_2 + (\text{ad } K_0)K_2 \subset K_2$, recalling $(\text{ad } K_0)K_2 = L(K_0)K_2 \subset K_2$. Thus we must show $(\text{ad } K_1)K_2 = K_1K_2 + h(K_1, K_2)$ is in K_2 which follows from (b) and $h(K_1, K_2) = 0$. To see this note

$$0 = \text{Kill}([h, K_1], K_2), \quad \text{using } K_2 = K_1^{\perp}$$
$$= \text{Kill}(h, [K_1, K_2])$$
$$= \text{Kill}(h, h(K_1, K_2)), \quad \text{using } m = h^{\perp}.$$

Thus since Kill $\mid h \times h$ is nondegenerate, $h(K_1, K_2) = 0$.

We shall give some examples later, but it should be noted that the problem of finding "holonomy irreducible" connections determined by (m, α, C) leads to the problem of finding algebras (m, α, C) which have no left ideals which are ad h-invariant. In particular these algebras must be simple; see [9]. The simplest examples of this type were discussed in [9, Proposition 3] where $S \mid K_i = \lambda_i I$ and relations between the λ_i's were determined. A general analysis of the algebras (m, α, C) probably must follow the lines of Lie algebras or extended Lie algebras [8] since identity elements or idempotents need not be in (m, α, C). Thus using $C(\alpha(X, Y), Z) = -C(Y, \alpha(X, Z))$ we see that if C is nondegenerate and E is an identity element, then

$$C(X, Y) = C(\alpha(E, X), Y) = -C(X, \alpha(E, Y)) = -C(X, Y)$$

a contradiction. If C is positive definite and E is an idempotent we obtain

$$C(E, E) = C(\alpha(E, E), E) = -C(E, \alpha(E, E)) = -C(E, E),$$

a contradiction.

Proposition 4. *Let g be a semisimple Lie algebra with h a semisimple subalgebra so that (g, h) is a reductive pair with $g = m + h$ where $m = h^{\perp}$. If $m = K_0 + b$ is the decomposition as before, then $JK_0 = K_0$ and $Jb = b$. Thus $J = J_0 \oplus J(b)$ is the direct sum of ideals where $J_0 = J| K_0$ and $J(b) = J | b$. Furthermore J_0 is isomorphic to the Jordan algebra of $r \times r$ B-symmetric matrices where r is the dimension of K_0.*

Proof. Let $U \in K_0$ and $S \in J$, then since $[\mathrm{ad}\, h, S] = 0$ we have $\mathrm{ad}\, h(SU) = S(\mathrm{ad}\, hU) = 0$ so that by definition $SK_0 \subset K_0$. Since $I \in J$, we have $JK_0 = K_0$. Since $b = K_0^{\perp}$ and S is B-symmetric we also have $Jb = b$.

Next relative to $m = K_0 + b$ we see that $\mathrm{ad}\, h | m$ and $S \in J$ have the matrices of the form

$$\begin{bmatrix} 0 & \\ & h_{22} \end{bmatrix} \text{ and } \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}.$$

Thus the condition $[\mathrm{ad}\, h, S] = 0$ gives no restriction on S_{11}. Therefore the set of matrices S_{11} which is isomorphic to J_0 has the only restrictions $S_{11}^{b} = S_{11}$. That is, J_0 is isomorphic to $\{T \in \mathrm{End}(K_0)\colon T^{b_0} = T\}$ where b_0 is the adjoint relative to the nondegenerate form $B_0 = B | K_0 \times K_0$.

Examples in the next section lead to the following.

Conjecture. Let $m = K_0 + K_1 + K_2$ as before with $K_0 \neq 0$, then $JK_i = K_i$ for $i = 0, 1, 2$.

If this is the case, then $J = J_0 \oplus J_1 \oplus J_2$ where $J_i = J | K_i$ are ideals in J.

4. **Examples.** We now consider the decomposition of the previous sections for various simple matrix algebras g where the semisimple subalgebra h is imbedded in g in a natural way.

To see easily that $JK_i = K_i$ we recall some facts on homogeneous components for an associative algebra [2, p. 124]. The associative algebra A we want to consider is the algebra generated by $\mathrm{ad}\, h | m$. Since we are assuming h is semisimple, it is completely reducible; thus A is completely reducible and semisimple. Furthermore $AK_0 = 0$ and $Ab = b$ so that we shall restrict our attention to $A | b \equiv \overline{A}$.

Now let $\Gamma = \{T \in \mathrm{End}(b)\colon [T, P] = 0 \text{ for all } P \in \overline{A}\}$ be the centralizer of $\overline{A}$.

Then we see $J(b) \subset \Gamma$ and $J(b)$ consists of the $B_1 = B|b \times b$ symmetric elements. Let n be an $\bar{A}$-irreducible submodule of b, then the homogeneous component p determined by n is $p = \Sigma n_i$ summed over all $\bar{A}$-irreducible submodules n_i which are $\bar{A}$-isomorphic to n. In this case we have $\Gamma p = p$ and p is a Γ-completely reducible submodule of b. In the examples we shall show that the $\bar{A}$-homogeneous components of $b = K_1 + K_2$ are either in K_1 or K_2; thus $\Gamma K_1 = K_1$ and $\Gamma K_2 = K_2$.

The general idea behind the examples is as follows. Let g be represented by matrices and suppose $h = h_1 \oplus h_2 \oplus \cdots \oplus h_t$ as a direct sum of simple ideals; consider the case $h = h_1 \oplus h_2$. In this case represent g by matrices

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and imbed h as matrices of the form

$$\begin{bmatrix} 0 & & 0 \\ & h_{22} & \\ 0 & & h_{33} \end{bmatrix}.$$

There is a 0 in the (1,1) position because K_0 is not zero unless $A_{11} = 0$. From matrix computations we roughly obtain

$$K_0 \text{ is the set of matrices } \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$K_1 \text{ is the set of matrices } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix};$$

$$K_2 \text{ is the set of matrices } \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix}.$$

Example. (1) Let $g = so(n)$ be the $n \times n$ skew-symmetric matrices and $h = so(k)$ the $k \times k$ skew-symmetric matrices where $k < n - 1$. As above regard g as matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{ii} are skew-symmetric and imbed b as matrices of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}.$$

Then a straightforward computation, using $\mathrm{Kill}(P, Q) = \lambda \operatorname{tr} PQ$ for $P, Q \in g$, shows $m = b^{\perp}$ is of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}.$$

From this we see K_0 are matrices of the form

$$\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

and $b = K_0^{\perp}$ are matrices of the form

$$\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}.$$

Now for $U \in K_0$, $V \in b$ we compute $L(U)V = [U, V] = 0$ to obtain

$$(*) \qquad A_{11}A_{12} = A_{21}A_{11} = 0.$$

Let $r = n - k$, then since K_0 consists of all $r \times r$ skew-symmetric matrices we see that (*) implies $A_{21} = 0$; that is, $K_1 = 0$ and $b = K_2$. Thus $m = K_0 + K_2$ with the above matrix description.

Now we see that $J = J_0 \oplus J_2$ as given at the end of §3. In this case J_0 is the simple Jordan algebra of symmetric $r \times r$ matrices. We now consider J_2.

Since $A_{12}^t = -A_{21}$ where ^{t} is the transpose, we set $A = A_{12}$, then

$$\begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix} \in K_2$$

and we let A_i be a $r \times k$ matrix of the form

$$A_i = \begin{bmatrix} & 0 & \\ a_{i1} a_{i2} & \cdots & a_{ik} \\ & 0 & \end{bmatrix}$$

and let p_i be the set of matrices of the form

$$\begin{bmatrix} 0 & A_i \\ -A_i^t & 0 \end{bmatrix}.$$

Then a computation shows ad $hp_i = p_i$, using (ad $h)^t = -$ ad h. Furthermore the p_i are ad h-irreducible and $K_2 = p_1 + \cdots + p_r$. We have p_i and p_j are ad h-isomorphic using the obvious map. Thus let $(\alpha_1, \cdots, \alpha_k)$ be an arbitrary k-tuple in R^k and let

$$A_i = \begin{bmatrix} & 0 & \\ \alpha_1 & \cdots & \alpha_k \\ & 0 & \end{bmatrix} \text{ ith row, and}$$

$$A_j = \begin{bmatrix} & 0 & \\ \alpha_1 & \cdots & \alpha_k \\ & 0 & \end{bmatrix} \text{ jth row.}$$

Then the map $p_i \to p_j$ given by

$$\begin{bmatrix} 0 & A_i \\ -A_i^t & 0 \end{bmatrix} \to \begin{bmatrix} 0 & A_j \\ -A_j^t & 0 \end{bmatrix}$$

is an ad h-isomorphism. Thus since $K_2 = \Sigma p_i$ we see K_2 is a homogeneous component and this is an orthogonal decomposition.

Use the decomposition $K_2 = \Sigma_{i=1}^r p_i$ to obtain a basis for K_2 as follows. Let $E_1, \cdots, E_k$ be a basis of p_1 and let ad $hE_s = \Sigma_j a_{js} E_j$ define a matrix h_{11} for ad h on p_1. Let $\phi: p_1 \to p_i$ be an ad h- (or an $\bar{A}$)-isomorphism and let $E_s^i = \phi(E_s)$ give a corresponding basis of p_i. Then ad $hE_s^i =$ ad $h\phi(E_s) = \phi($ad $hE_s) = \Sigma_j a_{js}(\phi E_j) = \Sigma_j a_{js} E_s^i$. This defines a matrix h_{ii} for ad h on p_i and $h_{ii} = h_{11}$. Thus relative to a basis of K_2 chosen this way ad $h|K_2$ has matrix

$$\begin{bmatrix} h_{11} & & 0 \\ & \ddots & \\ 0 & & h_{rr} \end{bmatrix}$$

where $h_{11} = h_{ii}$ for $i = 1, \cdots, r$.

Next relative to this basis write the matrix for $S|K_2$ in block form of $k \times k$ matrices

$$\begin{bmatrix} S_{11} & \cdots & S_{1r} \\ & \vdots & \\ S_{r1} & \cdots & S_{rr} \end{bmatrix}.$$

Then $[\mathrm{ad}\, h, S] = 0$ gives the relations

$$[h_{ii}, S_{ij}] = h_{ii}S_{ij} - S_{ij}h_{jj} = 0.$$

Since $h_{ii} = h_{11}$ we obtain $[h_{11}, S_{ij}] = 0$. But h_{11} represents any $k \times k$ skew-symmetric matrix and this implies $S_{ij} = \lambda_{ij}I$ for $\lambda_{ij} \in R$ and I is the $k \times k$ identity matrix. (If necessary, just compute.) Thus since $S^t = S$, the ideal $J_2 = J|K_2$ is isomorphic to the $r \times r$ symmetric matrices, $r = n - k$.

Example. (2) We now consider when h is actually semisimple. Thus let $g = so(n)$ be given by block matrices

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

and let $h = so(k_1) \oplus so(k_2) \oplus so(k_3)$ be imbedded as matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix}$$

similar to the first example and let $n = r + k_1 + k_2 + k_3$ where $r > 1$. As in the preceding example we have m given by matrices of the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix}$$

and $m = K_0 + K_1 + K_2$ where the K_i are given by matrices of the form

$$K_0 : \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_1 : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} \\ 0 & A_{32} & 0 & A_{34} \\ 0 & A_{42} & A_{43} & 0 \end{bmatrix},$$

$$K_2 : \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & 0 & 0 \\ A_{31} & 0 & 0 & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}.$$

As in the preceding example we use the skew-symmetry of matrices in $so(n)$ to obtain the following decomposition $K_1 = p_1 + p_2 + p_3$ where the p_i are given by

$$p_1 : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & 0 \\ 0 & A_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$p_2 : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \\ 0 & A_{42} & 0 & 0 \end{bmatrix},$$

$$p_3 : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} \\ 0 & 0 & A_{43} & 0 \end{bmatrix}.$$

Next $K_2 = q_1 + q_2 + q_3$ where the q_i are given by

$$q_1: \begin{bmatrix} 0 & A_{12} & 0 & 0 \\ A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$q_2: \begin{bmatrix} 0 & 0 & A_{13} & 0 \\ 0 & 0 & 0 & 0 \\ A_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$q_3: \begin{bmatrix} 0 & 0 & 0 & A_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}.$$

The homogeneous components of ad h (or $\overline{A}$) in b are contained in $p_1 + p_2 + p_3 = K_1$ or $q_1 + q_2 + q_3 = K_2$. This is a straightforward computation (discussed below) which shows ad h acts differently in K_1 and K_2 so that there can be no isomorphic irreducible ad h-submodules in K_1 and K_2.

Thus we see for $S \in J$ that $SK_i \subset K_i$ and J_0 consists of $r \times r$ symmetric matrices. For $J_2 = J|K_2$ we note that

$$U = \begin{bmatrix} 0 & & & 0 \\ & h_{22} & & \\ & & h_{33} & \\ 0 & & & h_{44} \end{bmatrix} \text{ in } h, \quad \text{and}$$

$$X = \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12}^t & 0 & 0 & 0 \\ -A_{13}^t & 0 & 0 & 0 \\ -A_{14}^t & 0 & 0 & 0 \end{bmatrix} \text{ in } K_2$$

gives

$$(1) \qquad [U, X] = \begin{bmatrix} 0 & -A_{12}b_{22} & -A_{13}b_{33} & -A_{14}b_{44} \\ -b_{22}A_{12}^t & 0 & 0 & 0 \\ -b_{33}A_{13}^t & 0 & 0 & 0 \\ -b_{44}A_{14}^t & 0 & 0 & 0 \end{bmatrix}$$

so that ad b acts differently on the q's and consequently the q_i are homogeneous components (also see end of this example). Thus S: $q_i \longrightarrow q_i$ and $S|K_2$ has matrix

$$\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}.$$

To find the S_i's is essentially the same as in Example (1); thus we see $J_2 = J_2(q_1) \oplus J_2(q_2) \oplus J_2(q_3)$ where the $J_2(q_i)$ are isomorphic to the Jordan algebra of $r \times r$ symmetric matrices.

To find $J_1 = J|K_1$ we note for U given as above and for

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} \\ 0 & -A_{23}^t & 0 & A_{34} \\ 0 & -A_{24}^t & -A_{34}^t & 0 \end{bmatrix}$$

in $K_1 = p_1 + p_2 + p_3$ that

$$(2) \qquad [U, Y] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & b_{22}A_{23} - A_{23}b_{33} & b_{22}A_{24} - A_{24}b_{44} \\ & * & 0 & b_{33}A_{34} - A_{34}b_{44} \\ & & & 0 \end{bmatrix}$$

where * is given by skew-symmetry. By making various choices for the b_{ii} we see that the p_i do not contain ad b-isomorphic irreducible submodules. Thus the p_i are homogeneous components and are actually irreducible. Thus we see S: $p_i \longrightarrow p_i$ so S on K_1 has matrix

$$\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

where the S_i are symmetric on p_i. Therefore S_i has a real characteristic root λ_i on p_i and $\{X \in p_i : S_i X = \lambda_i X\}$ is ad b-invariant. Thus by irreducibility, $S_i = \lambda_i I$ on p_i; that is, J_1 is isomorphic to $RI_1 \oplus RI_2 \oplus RI_3$ where I_i is the identity on p_i. Finally comparing the equations (1) and (2) above we see that we can choose the b_{ii} so that ad b acts differently on K_1 and K_2 so that the homogeneous components in b are either in K_1 or K_2.

Example. (3) We now consider a case where the forms are not positive definite. Thus let g be the $n \times n$ matrices of trace zero (that is, A_{n-1}-type) and let b be the $k \times k$ matrices of trace zero imbedded as follows. Represent elements of g by a sum of block matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + R \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

where trace A_{22} = trace $A_{11} = 0$. Let b be represented by the $k \times k$ block of matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & b_{22} \end{bmatrix}$$

where we use the notation $b_{22} = A_{22}$ and let

$$V = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

Again using $\mathrm{Kill}(P, Q) = \lambda \operatorname{tr} PQ$, we see that $m = b^{\perp}$ consists of matrices of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} + RV$$

and K_0 is given by matrices

$$\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} + RV.$$

Thus $K_0 = c \oplus K_0'$ where the center $c = RV$ and K_0' is isomorphic to the $r \times r$ matrices of trace zero where $n = r + k$.

Next $m = K_0 + b$ and b is given by matrices

$$\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

and a computation shows $h = K_2$. Let p_1 be matrices of the form

$$\begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}$$

and let p_2 be matrices of the form

$$\begin{bmatrix} 0 & 0 \\ A_{12} & 0 \end{bmatrix}.$$

Then $(\text{ad } h)(p_i) = p_i$ and $K_2 = p_1 + p_2$. The action of ad h is given by

$$\text{ad } h \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -A_{12}h_{22} \\ h_{22}A_{21} & 0 \end{bmatrix}.$$

Let A_i be a matrix of the form

$$\begin{bmatrix} & 0 & \\ \alpha_{i1} & \cdots & \alpha_{ik} \\ & 0 & \end{bmatrix};$$

then matrices of form

$$(*) \qquad \begin{bmatrix} 0 & A_i \\ 0 & 0 \end{bmatrix}$$

yield isomorphic irreducible ad h-submodules of p_1. Similarly for p_2 using column matrices. However the p_1 and p_2 do not contain ad h-isomorphic irreducible submodules. Briefly, suppose $n_1 \to n_2$: $X \to \overline{X}$ is such an isomorphism of irreducible submodules where

$$X = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{X} = \begin{bmatrix} 0 & 0 \\ \overline{A} & 0 \end{bmatrix}.$$

Then we obtain $h_{22}\overline{A} = -\overline{Ah}_{22}$ and by suitably choosing the elements in h_{22} (using 0's and 1's in the correct places) we obtain contradictions. Thus p_1 and p_2 are ad h-homogeneous components.

Using the decomposition $m = K_0 + p_1 + p_2$ we see that $S \in J$ has matrix

$$\begin{bmatrix} S_0 & 0 & 0 \\ 0 & \overline{S}_1 & 0 \\ 0 & 0 & \overline{S}_2 \end{bmatrix}$$

so that J_0 is isomorphic to the $t \times t$ B-symmetric matrices where $t = r^2$ is the dimension of K_0.

Next on $b = p_1 + p_2$, ad $h|b$ has matrix

$$\begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$$

where h_i is the matrix of ad $h|p_i$; thus [ad h, S] = 0 gives $[h_i, \overline{S}_i] = 0$. Now consider $[h_1, \overline{S}_1] = 0$ on p_1; the case for p_2 is similar. Let $p_1 = q_1 + \cdots + q_r$ where the q_i are the submodules given by (*) above and $r = n - k$. By choosing a "natural" bases for each q_i we see h_1 has matrix of the form

$$\begin{bmatrix} H_{11} & & 0 \\ & \ddots & \\ 0 & & H_{rr} \end{bmatrix}$$

where $H_{ii} = H_{jj}$ is $k \times k$ of trace zero. Now relative to this same basis, write

$$\overline{S}_1 = \begin{bmatrix} S_{11} & \cdots & S_{rr} \\ \vdots & & \vdots \\ S_{r1} & \cdots & S_{rr} \end{bmatrix}$$

then $[h_1, \overline{S}_1] = 0$ gives $H_{ii} S_{ij} - S_{ij} H_{jj} = 0$. But $H_{ii} = H_{jj}$ so that each S_{ij} commutes with $k \times k$ matrices of trace zero. Since S_{ij} commutes with the identity, we see S_{ij} commutes with all $k \times k$ matrices; thus $S_{ij} = \lambda_{ij} I$ so that $\overline{J}_1 = J|p_1$ is isomorphic to the Jordan algebra of $r \times r$ matrices.

Similarly for $\overline{S}_2 \in \overline{J}_2 = J|p_2$ we obtain a matrix as follows. We note from Kill$(P, Q) = \lambda \operatorname{tr} PQ$ that $B(p_i, p_i) = 0$ and $B(p_1, p_2) \neq 0$. That is, p_1 and p_2 are related by the nondegenerate form B and considered dual. Choosing a basis for p_1 from the decomposition $p_1 = q_1 + \cdots + q_r$ as above, we get a decomposition $p_2 = q_1' + \cdots + q_r'$ such that $B(q_i, q_j') = 0$ if $i \neq j$ and q_i and q_i' are related by the nondegenerate form B and considered dual. Choosing a natural basis for q_i as in (*), the corresponding dual basis for q_i' is obtained by using the corresponding column matrices.

Now let $\overline{S}_2 \in \overline{J}_2$ have matrix $[\mu_{ij}I]$ relative to $p_2 = q_1' + \cdots + q_r'$ decomposition (as in the case for p_1). Then for $X_i \in q_i$ we have $\overline{S}_1 X_i = \Sigma \lambda_{ki} X_k$ with $X_k \in q_k$ and for $X_i' \in q_i'$ we have $\overline{S}_2 X_i' = \Sigma \mu_{ki} X_k'$ with $X_k' \in q_k'$. Using S is B-symmetric and the orthogonal relations of the q_i and q_j' we have by the usual computations that $\lambda_{ji} = \mu_{ij}$. This determines J_2.

Even though the algebras (m, α, C) can be determined by finding J, these algebras are difficult to compute. In the next section we shall see how to obtain restrictions on J by considering geometric formulas.

5. **Some applications.** We shall give some sample applications by considering various "geometrical" formulas which are functions of the connection by considering them as functions of elements of $M = J \cap GL(m)$. The formulas show these functions are analytic on the open set M (in J) and we can compute their Taylor's series, find critical points, etc. Since we will always assume $I \in M$, we are considering a "perturbation" problem to see how formulas for a connection given by $S = I + tT \in M$ for $T \in J$ and t near $0 \in R$ differ from formulas determined by a connection of the first kind.

First let ∇ be a G-invariant pseudo-Riemannian connection on G/H given by the algebra (m, α, C) as in §1. Thus we are assuming a pseudo-Riemannian connection of the first kind exists, given by $(m, \frac{1}{2}XY, B)$, and we are comparing the connections via $C(X, Y) = B(SX, Y)$ which gives the elements $S \in M$. The connection formula for ∇ given by α is

$$2\alpha(X; Y) = XY + S^{-1}[X(SY) - (SX)Y],$$

for $X, Y \in m$ with $g = m + h$.

Let $M = J \cap GL(m)$ and let $L^2(m, m)$ be the space of bilinear maps on m, then define the function

$$a : M \longrightarrow L^2(m, m) : S \longrightarrow a(S)$$

where $a(S)(X, Y) = \alpha(X, Y)$ as given above. From the formula for α, we see a is analytic on M; and in particular a is analytic at I, which gives the original connection of the first kind.

We now compute the Taylor's series for the function a expanded about a point $P \in M$. Thus let $T \in J$ and $t \in R$ so that $P + tT \in M$ (and so that the following computations holds). From formulas,

$$\begin{aligned}2a(P + tT)(X, Y) &= XY + (P + tT)^{-1}[X \cdot (P + tT)Y - (P + tT)X \cdot Y] \\ &= XY + P^{-1}A + t(P^{-1}B - UP^{-1}A) \\ &\quad + \cdots + (-1)^{k-1} t^k U^{k-1}(P^{-1}B - UP^{-1}A) + \cdots\end{aligned}$$

where $U = P^{-1}T$, $A = X \cdot PY - PX \cdot Y$, and $B = X \cdot TY - TX \cdot Y$. From this we compute the higher derivatives $2k!(D^k a(P))\, T^{(k)}$ operating on (X, Y) to be $(-1)^{k-1}U^{k-1}(P^{-1}B - UP^{-1}A)$. In particular if $T \in J$ is such that $(Da(P))(T) = 0$, then $P^{-1}B - UP^{-1}A = 0$ so that

$$(*) \qquad 2a(P + tT)(X, Y) = XY + P^{-1}A = XY + P^{-1}(X \cdot PY - PX \cdot Y)$$

which is the connection determined by $P \in M$.

Now suppose P is a critical point of the function a; that is, $(Da(P))T = 0$ for all $T \in J$. Then from the above computations we see the Taylor's series is of the form

$$a(P + tT)(X, Y) = (a(P) + (Da(P))T + \cdots)(X, Y) = a(P)(X, Y)$$
$$= \tfrac{1}{2}XY + \tfrac{1}{2}P^{-1}(X \cdot PY - PX \cdot Y)$$

from (*). But by choosing $T = I \in J$ we see

$$0 = 2[Da(P)I](X, Y) = P^{-1}B - UP^{-1}A, \quad \text{in terms of } I$$
$$= P^{-1}[X \cdot IY - IX \cdot Y - P^{-1}(X \cdot PY - PX \cdot Y)]$$

which implies $X \cdot PY - PX \cdot Y = 0$; thus we see $a(P)(X, Y) = \tfrac{1}{2}XY$. That is a critical point of a gives a connection of the first kind.

Next we consider the perturbation of curvature. The curvature evaluated at $H \in G/H$ for the G-invariant connection ∇ determined by (m, α, C) is given by

$$R(X, Y)Z = \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - \alpha(XY, Z) - D(X, Y)Z$$

(see [7] and note §3 for $D(X, Y)Z$). Thus regarding R as a function on M we have an analytic map

$$r: M \longrightarrow L^3(m, m): S \longrightarrow r(S)$$

where $r(S)(X, Y, Z) = R(X, Y)Z$. Since we know $I \in J$, let $T \in J$ and $t \in R$ near enough 0 in R so that $S = I + tT \in M$ and so that we can perform the following series computations.

From preceding computations we see

$$2\alpha(X, Y) = 2a(I + tT)(X, Y)$$
$$= XY + (I + tT)^{-1}[X \cdot (I + tT)Y - (I + tT)X \cdot Y]$$
$$= XY + \sum_{k=1}^{\infty} (-1)^{k-1} t^k T^{k-1}(X \cdot TY - TX \cdot Y)$$

and using this in the above formula for $R(X, Y)Z$ we obtain the following for $r(S)(X, Y, Z)$:

$$4r(S)(X, Y, Z) = 2\alpha(X, 2\alpha(Y, Z)) - 2\alpha(Y, 2\alpha(X, Z)) - 4\alpha(XY, Z) - 4D(X, Y)Z$$

$$= X \cdot YZ - Y \cdot XZ - 2XY \cdot Z - 4D(X, Y)Z$$

$$+ \sum_{P=1}^{\infty} (-1)^{P-1} t^P \{X \cdot T^{P-1}(Y \cdot TZ - TY \cdot Z) - Y \cdot T^{P-1}(X \cdot TZ - TX \cdot Z)$$

$$+ T^{P-1}[X \cdot T(YZ) - TX \cdot YZ$$

$$- Y \cdot T(XZ) + TY \cdot XZ - 2XY \cdot TZ + 2T(XY) \cdot Z]\}$$

$$+ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{k+n} t^{k+n} \{T^{k-1}[X \cdot T^n(Y \cdot TZ - TY \cdot Z)$$

$$- Y \cdot T^n(X \cdot TZ - TX \cdot Z)$$

$$- TX \cdot T^{n-1}(Y \cdot TZ - TY \cdot Z)$$

$$+ TY \cdot T^{n-1}(X \cdot TZ - TX \cdot Z)]\}.$$

From this we can compute the derivatives $[D^k r(I) T^{(k)}](X, Y, Z)$; in particular

$$4[Dr(I)T](X, Y, Z) = X(Y \cdot TZ - TY \cdot Z) - Y(X \cdot TZ - TX \cdot Z) + X \cdot T(YZ) - TX \cdot YZ$$

$$- Y \cdot T(XZ) + TY \cdot XZ - 2XY \cdot TZ + 2T(XY) \cdot Z.$$

If I is a critical point of r, then we obtain a restriction on J given by $[R_1(X, Y), T] = 0$ for all $T \in J$ where $R_1(X, Y) = \frac{1}{4}[L(X), L(Y)] - \frac{1}{2}L(XY) - D(X, Y)$ is the curvature endomorphism for a connection of the first kind; recall $L(X)\colon m \longrightarrow m\colon Y \longrightarrow XY$. To see this we have from $0 = [Dr(I)T](X, Y, Z)$ as an operator on Z that

$$0 = [L(X), L(Y)]T - [L(X), L(TY)] + [L(Y), L(TX)]$$

$$+ L(X)TL(Y) - L(Y)TL(X) + 2L(T(XY)) - 2L(XY)T.$$

Taking the transpose of this equation, recalling $L(X)^b = -L(X)$ and $T^b = T$, we have

$$0 = -T[L(X), L(Y)] + [L(X), L(TY)] - [L(Y), L(TX)]$$

$$+ L(Y)T\,L(X) - L(X)T\,L(Y) - 2L(T(XY)) + 2T\,L(XY).$$

Adding these equations gives

$$0 = [[L(X), L(Y)], T] + 2[T, L(XY)] = 4[R_1(X, Y), T].$$

Now let $H = \{e\}$ and let G be semisimple, then $g = m$ and $L(X) = \mathrm{ad}\, X$. This gives $R_1(X, Y) = -\frac{1}{4}\mathrm{ad}([X, Y])$ and $0 = [\mathrm{ad}([X, Y]), T]$. But $g = [g, g]$ so we have $[\mathrm{ad}\, Z, T] = 0$ for all $Z \in g$. This implies $[X, TY] - [TX, Y] = 0$ and therefore from the formula, $\alpha(X, Y) = \frac{1}{2}[X, Y]$. This proves the following.

Proposition 5. *Let G be a semisimple connected Lie group with Lie algebra g. Let $r\colon M \to L^3(g, g)$ where $r(S)(X, Y, Z) = R(X, Y)Z$ is the curvature determined by (g, α, C). If $I \in M$ is a critical point of r, then the connection determined by (g, α, C) is of the first kind; that is $\alpha(X, Y) = \frac{1}{2}[X, Y]$.*

Examples indicate similar results hold for reductive pairs (g, h) with g and h semisimple where $g = m + h$ with $m = h^{\perp}$ and $mm \not\subseteq 0$; that is, (g, h) is not a symmetric pair.

REFERENCES

1. E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sb. 30 (72) (1952), 349–462; English transl., Amer. Math. Soc. Transl. (2) 6 (1957), 111–244. MR 13, 904.

2. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1956. MR 18, 373.

3. ———, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962. MR 26 #1345.

4. B. Kostant, *On holonomy and homogeneous spaces*, Nagoya Math. J. 12 (1957), 31–54. MR 21 #6003.

5. O. Loos, *Symmetric spaces*. II, Benjamin, New York, 1969. MR 39 #365b.

6. K. McCrimmon, *On Herstein's theorem relating Jordan and associative algebras*, J. Algebra 13 (1969), 382–392. MR 40 #2721.

7. K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. Math. J. 76 (1954), 33–65. MR 15, 468.

8. A. Sagle, *On simple extended Lie algebras over fields of characteristic zero*, Pacific J. Math. 15 (1965), 621–648. MR 32 #7612.

9. ———, *Some homogeneous Einstein manifolds*, Nagoya Math. J. 39 (1970), 81–106. MR 42 #6748.

10. A. Sagle and D. Winter, *On homogeneous spaces and reductive subalgebras of simple Lie algebras*, Trans. Amer. Math. Soc. 128 (1967), 142–147. MR 37 #2910.

11. A. Sagle and J. Schumi, *Multiplications on homogeneous spaces, nonassociative algebras and connections*, Pacific J. Math. (to appear).

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