TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 187, Issue 1, 1974

THE EXISTENCE, CHARACTERIZATION AND ESSENTIAL UNIQUENESS OF SOLUTIONS OF L^∞ EXTREMAL PROBLEMS[1]

BY

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ABSTRACT. Let $I = (a, b)$ be an interval in $\mathbf{R}$ and let $H^{n,\infty}$ consist of those real-valued functions f such that $f^{(n-1)}$ is absolutely continuous on I and $f^{(n)} \in L^\infty(I)$. Let L be a linear differential operator of order n with leading coefficient 1, $a = x_1 < \cdots < x_m = b$ be a partition of I and let the linear functionals L_{ij} on $H^{n,\infty}$ be given by

$$L_{ij}f = \sum_{\nu=0}^{n-1} a_{ij}^{(\nu)} f^{(\nu)}(x_i), \quad j = 1, \cdots, k_i, \; i = 1, \cdots, m,$$

where $1 \leq k_i \leq n$ and the k_i n-tuples $(a_{ij}^{(0)}, \cdots, a_{ij}^{(n-1)})$ are linearly independent. Let r_{ij} be prescribed real numbers and let $U = \{f \in H^{n,\infty}: L_{ij}f = r_{ij}, j = 1, \cdots, k_i, i = 1, \cdots, m\}$. In this paper we consider the extremal problem

$$(*) \qquad \|Ls\|_{L^\infty} = \alpha = \inf\{\|Lf\|_{L^\infty}: f \in U\}.$$

We show that there are, in general, many solutions to (*) but that there is, under certain consistency assumptions on L and the L_{ij}, a fundamental (or core) interval of the form (x_i, x_{i+n_0}) on which all solutions to (*) agree; n_0 is determined by the k_i and satisfies $n_0 \geq 1$. Further, if s is any solution to (*) then on (x_i, x_{i+n_0}), $|Ls| = \alpha$ a.e. Further, we show that there is a uniquely determined solution s_* to (*), found by minimizing $\|Lf\|_{L^\infty}$ over all subintervals (x_j, x_{j+1}), $j = 1, \cdots, m-1$, with the property that $|Ls_*|$ is constant on each subinterval (x_j, x_{j+1}) and Ls_* is a step function with at most $n-1$ discontinuities on (x_j, x_{j+1}). When $L = D^n$, s_* is a piecewise perfect spline. Examples show that the results are essentially best possible.

1. Introduction. Let m points $x_1 < x_2 < \cdots < x_m$ and m interpolation values $r_1, r_2, \cdots, r_m$ be prescribed in $\mathbf{R}$ and let n be a positive integer satisfying $n \leq$

Received by the editors December 12, 1972 and, in revised form, May 29, 1973.

AMS (MOS) subject classifications (1970). Primary 49A35, 49A50, 41A05, 41A15; Secondary 41A50, 34B10.

Key words and phrases. Minimization, interpolation, spline, fundamental interval of uniqueness, Tchebycheff system.

(1) Research sponsored by National Science Foundation grant GP32116.

$m-1$. We denote by $H^{n,\infty}(x_1, x_m)$ the real linear space of functions f such that $f^{(n-1)}$ is absolutely continuous on (x_1, x_m) and $f^{(n)} \in L^\infty(x_1, x_m)$. The minimization problem on (x_1, x_m) given by

$$\begin{aligned} \|s^{(n)}\|_{L^\infty} &= \alpha \\ &= \inf\{\|f^{(n)}\|_{L^\infty}: f \in H^{n,\infty}(x_1, x_m), f(x_i) = r_i,\ 1 \le i \le m\} \end{aligned} \tag{1.1}$$

was shown to have a solution $s \in H^{n,\infty}(x_1, x_m)$ in [8] via the property that $\{f^{(n)}: f \in H^{n,\infty}(x_1, x_m), f(x_i) = r_i, 1 \le i \le m\}$ is weak* closed in $L^\infty(x_1, x_m)$. In fact, a much more general result was obtained for linear inequality constraints and nonsingular linear differential operators of order n. In his dissertation [12], Smith showed that a solution s of (1.1) exists which has the property that $|s^{(n)}|$ is a step function on (x_1, x_m) with discontinuities restricted to $x_2, \cdots, x_{m-1}$ and moreover that $s^{(n)}$ is a step function on (x_i, x_{i+1}), $i = 1, \cdots, m-1$, with at most $n-1$ discontinuities on each such interval. His method consisted of considering limits of certain sequences of L_p extremal solutions as $p \to \infty$, the latter having been characterized analytically by Golomb [5]. In this paper, we show that there exist $n+1$ consecutive points $x_r, \cdots, x_{r+n}$ among $x_1, \cdots, x_m$ such that *any* two solutions of (1.1) agree on the interval $[x_r, x_{r+n}]$ and such that every solution of (1.1) satisfies $|s^{(n)}| = \alpha$ a.e. on (x_r, x_{r+n}). Moreover, we show that there is a unique solution of (1.1) on $[x_1, x_m]$ with the further property that inductively, on each subinterval (x_i, x_{i+1}), $1 \le i \le m-1$, ess sup$|f^{(n)}|$ is minimal; this unique solution s has the property that $|s^{(n)}|$ is equivalent to a step function on (x_1, x_m) with discontinuities restricted to $x_2, \cdots, x_{m-1}$ and moreover that $s^{(n)}$ is equivalent to a step function on (x_i, x_{i+1}), $1 \le i \le m-1$, with at most $n-1$ discontinuities on each such interval.

In fact, our results are much more general and (1.1) is but the prototype of a class of problems which we now describe. We consider again the m points $x_1 < \cdots < x_m$ and, associated with each of these points x_i, we consider the linear functionals L_{ij} on $H^{n,\infty}(x_1, x_m)$ defined by

$$L_{ij}f = \sum_{\nu=0}^{n-1} a_{ij}^{(\nu)} f^{(\nu)}(x_i), \qquad j = 1, \cdots, k_i,\ i = 1, \cdots, m,$$

for prescribed real numbers $a_{ij}^{(\nu)}$, such that, for each i, the k_i n-tuples $(a_{ij}^{(0)}, \cdots, a_{ij}^{(n-1)})$ are linearly independent; here $1 \le k_i \le n$ for $i = 1, \cdots, m$ and, at x_1 and x_m, the derivatives are one-sided or, equivalently, taken in the limiting sense. Let L be a nonsingular linear differential operator on $[x_1, x_m]$ of order n of the form

$$L = D^n + \sum_{j=0}^{n-1} c_j D^j$$

where $c_j \in C[a, b]$, $j = 0, 1, \cdots, n-1$. We consider the generalized minimization problem on (x_1, x_m)

$$\|Ls\|_{L^\infty} = \alpha = \inf\{\|Lf\|_{L^\infty}: f \in U\}, \tag{1.2}$$

$$U = \{f \in H^{n,\infty}(x_1, x_m): L_{ij}f = r_{ij},\ 1 \le j \le k_i,\ 1 \le i \le m\}, \tag{1.3}$$

for prescribed real numbers r_{ij}. U is clearly nonempty because of the linear independence of $\{(a_{ij}^{(0)}, \cdots, a_{ij}^{(n-1)})\}_{j=1}^{k_i}$ for $i = 1, \cdots, m$. We state now

Theorem 1. *The minimization problem* (1.2) *has a solution* $s \in H^{n,\infty}(x_1, x_m)$ *and the class* $S(U)$ *of all such solutions* s *for a fixed choice of* U *is a convex set. Let* $S_1(U) = S(U)$ *and for* $2 \le i \le m$, *let* $S_i(U)$ *consist of all solutions to the minimization problem*

$$\alpha_{i-1} = \inf\{\|Ls\|_{L^\infty(x_{i-1}, x_i)}: s \in S_{i-1}(U)\}.$$

Then each $S_i(U)$ *is nonempty*; *in particular,* $S_m(U) = \bigcap_{i=1}^m S_i(U)$ *is nonempty.*

In order to obtain characterization and uniqueness results we must make additional assumptions regarding the differential operator L and the linear functionals L_{ij}. Regarding L we assume

(I) $c_j \in C^j[a, b]$; the null space of the formal adjoint L^* of L given by

$$L^*f = \sum_{j=0}^{n} (-1)^j D^j(c_j f) \tag{1.4}$$

is spanned by a Tchebycheff system, i.e., if $u \in C^n[x_1, x_m]$ satisfies $L^*u \equiv 0$ on $[x_1, x_m]$ and if $u(y_1) = \cdots = u(y_n) = 0$ for any set of n points $x_1 \le y_1 < \cdots < y_n \le x_m$, then $u \equiv 0$ on $[x_1, x_m]$.

We remark that there exists a positive constant δ such that, if $x_m - x_1 \le \delta$, then the null space of L^* is spanned by a Tchebycheff system [7, p. 346]. Now let $L_1, \cdots, L_N$ denote the lexicographic ordering of the L_{ij}, i.e., if $N_0 = 0$ and $N_i = \sum_{\nu=1}^{i} k_\nu$, $i = 1, \cdots, m$, with $N = N_m$, we define

$$L_{N_{i-1}+j} = L_{ij}, \quad 1 \le j \le k_i,\ i = 1, \cdots, m. \tag{1.5}$$

We *define* n_0 to be the maximum positive integer satisfying the following property: For any n_0 consecutive points among $x_1, \cdots, x_m$ the sum of the integers k_i associated with these points does not exceed n. Clearly, we have $1 \le n_0 \le n$. Then our assumption regarding the functionals $L_1, \cdots, L_N$ is as follows.

(II) $N \ge n+1$ and L and the L_i are consistent:

(a) For every n_0 consecutive points $x_{\lambda_0+1}, \cdots, x_{\lambda_0+n_0}$ and prescribed values y_i there is a function u in the null space of L satisfying $L_i u = y_i$, $i = N_{\lambda_0}+1, \cdots, N_{\lambda_0+n_0}$.

(b) For every n_0+1 consecutive points $x_{\lambda_0}, \cdots, x_{\lambda_0+n_0}$ such that $\sum_{\nu=\lambda_0}^{\lambda_0+n_0} k_\nu \ge n+1$ the equations $L_i u = 0$, $i = N_{\lambda_0-1}+1, \cdots, N_{\lambda_0+n_0}$ for u in the null space of L imply $u \equiv 0$.

Theorem 2. *Suppose* (I) *and* (II) *are satisfied. Then there is a fundamental interval* $J = [x_{\lambda_1}, x_{\lambda_2+n_0}]$ *for some* $1 \le \lambda_1 \le \lambda_2 \le m - n_0$ *satisfying* $\sum_{i=\lambda_1}^{\lambda_2+n_0} k_i \ge n+1$ *such that any two solutions of* (1.2) *agree on* J. *Moreover, if* $s \in S(U)$, *then* $|Ls| = \alpha$ *a.e. on* J. *The class of solutions* $S_m(U)$, *as defined in Theorem* 1, *contains a single element* s_*. *Moreover,* s_* *satisfies the property that* $|Ls_*|$ *is equivalent to a step function on* (x_1, x_m) *with discontinuities restricted to* $x_2, \cdots, x_{m-1}$ *and, on* (x_i, x_{i+1}), $i = 1, \cdots, m-1$, Ls_* *is equivalent to a step function with at most* $n-1$ *discontinuities on each such interval.*

We have restricted our attention in this theorem to the case $N \ge n+1$ as is stated in (II). For $N = n$ and linear functionals consistent with respect to the null space of L the unique solution of (1.2) is defined by (generalized) interpolation and $\alpha = 0$ in this case. The case $\alpha = 0$ is possible when $N \ge n+1$ but is of little interest. In §2 we present a complete proof of Theorem 1, though the existence of a solution of (1.2) has been established with more general linear functional constraints in [8]. Then, in §3, we give a proof of Theorem 2 and show, through two examples, that the result is essentially best possible.

It is of considerable interest, we feel, that for $L = D^n$, spline solutions arise naturally as unique solutions s_* of (1.2) which are singled out so that $s_*^{(n)}$ has minimum L^∞ norm on (x_i, x_{i+1}), $i = 1, \cdots, m-1$. The emergence of splines as certain L^∞ extremal solutions is not new and appears elsewhere, viz., in the Favard-Achieser-Krein theorem [1], [2] on the best L^∞ approximation of periodic functions by trigonometric polynomials, in the elegant solution by Schoenberg and Cavaretta [11] of Landau's problem on the half-line regarding the best estimation of derivatives and in the very general results of Golomb [6] concerning the estimation of uniform norms of periodic functions, with certain zero Fourier coefficients, in terms of higher order derivatives. Also of mention here are the papers of Favard [3] and Glaeser [4].

Finally, we mention that the L^∞ extremal solutions have certain optimal approximation properties in the L^∞ norm [9] in analogy with the generalized spline extremals which arise in L^2 minimization problems. When $L = D^n$, the L^∞ splines are piecewise polynomials of degree n, whereas the L^2 splines are piecewise of degree $2n-1$.

2. **Existence.** In this section, we shall present a proof of Theorem 1. For this, we require some notation which we now introduce. $H^{n,\infty}(x_1, x_m)$ is a Banach space under the norm

$$\|u\| = \sum_{j=0}^{n} \|u^{(j)}\|_{L^\infty(x_1,x_m)}. \tag{2.1}$$

Let N_L denote the n dimensional null space of L of $C^n[x_1, x_m]$ functions. For the kernel $\theta(\cdot, \xi) \in N_L$, $\xi \in [x_1, x_m]$, defined by

$$[D^j\theta(\cdot, \xi)]_\xi = \delta_{j,n-1}, \qquad j = 0, \cdots, n-1,$$

we have the well-known representation, for $f \in H^{n,\infty}(x_1, x_m)$,

$$f(x) = u(x) + \int_{x_1}^{x_m} \hat{\theta}(x, \xi) Lf(\xi)\, d\xi, \qquad x_1 \le x \le x_m, \tag{2.2}$$

where $u \in N_L$ and

$$\hat{\theta}(x, \xi) = \begin{cases} \theta(x, \xi) & \text{if } x \ge \xi \\ 0 & \text{otherwise} \end{cases}.$$

Now let U_0 be the linear subspace of N_L consisting of those functions ϕ satisfying $L_{ij}\phi = 0$, $j = 1, \cdots, k_i$, $i = 1, \cdots, m$. U_0 clearly has dimension k satisfying $0 \le k \le n$. Let $\mu_1, \cdots, \mu_k$ be continuous linear functionals on $H^{n,\infty}(x_1, x_m)$ satisfying $\mu_i(v_j) = \delta_{ij}$, $1 \le i, j \le k$, where $v_1, \cdots, v_k$ are a basis for U_0. For convenience of notation, we order the L_{ij} and r_{ij} as $L_1, \cdots, L_N$ and $r_1, \cdots, r_N$. It is an elementary fact of linear algebra that there exist $n-k$ members $L_{j_1}, \cdots, L_{j_{n-k}}$ of $\{L_1, \cdots, L_N\}$ (we allow $k = 0$) such that $\mu_1, \cdots, \mu_k, L_{j_1}, \cdots, L_{j_{n-k}}$ are linearly independent over N_L, i.e., if $u \in N_L$ and $\mu_i u = 0$, $1 \le i \le k$, and $L_{j_\nu} u = 0$, $1 \le \nu \le n-k$, then $u = 0$.

Proof of Theorem 1. In order to show that the minimization problem (1.2) has a solution, it suffices to show that every nonempty intersection of LU with a closed ball B in $L^\infty(x_1, x_m)$ contains all its weak* sequential limit points. Indeed, this reduction is possible since, if $\|Lu_\nu\|_{L^\infty} \searrow \alpha$, $\{Lu_\nu\} \subset LU \cap B$, then Lu_ν contains a weak* convergent subsequence [13, p. 137] converging to an element which must have norm less than or equal to α by the lower semicontinuity of the norm with respect to weak* convergence. Thus, let $\{f_\nu\} \subset U$ be such that $\{Lf_\nu\}$ is weak* convergent to $y \in L^\infty(x_1, x_m)$ with $\|Lf_\nu\| \le C$. We immediately deduce that $\|y\| \le C$ from the relations

$$\int_\beta^\gamma Lf_\nu(\xi)d\xi \to \int_\beta^\gamma y(\xi)d\xi \tag{2.3}$$

for all $x_1 \le \beta < \gamma \le x_m$. We now set

$$f(x) = u(x) + \int_{x_1}^{x_m} \hat\theta(x, \xi)y(\xi)d\xi \tag{2.4}$$

where $u \in N_L$ is to be determined in such a way that $f \in U$. Now by the weak* convergence of Lf_ν and the Peano-type representations

$$L_i f_\nu = r_i = \int_{x_1}^{x_m} L_i\hat\theta(\cdot, \xi)Lf_\nu(\xi)d\xi + L_i u_\nu \tag{2.5}$$

for $i = 1, \cdots, N$ and $u_\nu \in N_L$, we deduce that $L_i u_\nu$ is convergent to

$$l_i = r_i - \int_{x_1}^{x_m} \hat\theta(\cdot, \xi)y(\xi)d\xi \tag{2.6}$$

for $i = 1, \cdots, N$. Here $L_i\hat\theta(\cdot, \xi)$ is, of course, an integrable function of ξ on (x_1, x_m). Define now $u \in N_L$ by

$$\mu_j u = 0, \quad 1 \le j \le k, \qquad L_{j_\nu} u = l_{j_\nu}, \quad 1 \le \nu \le n - k. \tag{2.7}$$

We now claim that

$$L_i f = r_i, \qquad i = 1, \cdots, N, \tag{2.8}$$

where f is defined by (2.4). Indeed, if we define functions $\tilde u_\nu \in N_L$ satisfying $\tilde u_\nu - u_\nu \in U_0$ and $\mu_i \tilde u_\nu = 0$, $1 \le i \le k$, then the sequence $\{\tilde u_\nu\}$ is convergent to u in the norm on N_L given by

$$\|u\|_{N_L} = \sum_{i=1}^{k} |\mu_i(u)| + \sum_{\nu=1}^{n-k} |L_{j_\nu}(u)|. \tag{2.9}$$

Now if λ is any of the linear functionals $L_1, \cdots, L_N$ then λ is continuous on N_L in the norm defined by (2.9) and hence $\lambda(\tilde u_\nu) \to \lambda(u)$. But $\lambda\tilde u_\nu = \lambda u_\nu$, so that, if $\lambda = L_{i_0}$, then

$$L_{i_0} u = l_{i_0}, \qquad 1 \le i_0 \le N, \tag{2.10}$$

since i_0 was arbitrary. Now by (2.4) we have

$$L_i f = L_i u + \int_{x_1}^{x_m} L_i\hat\theta(\cdot, \xi)y(\xi)d\xi \tag{2.11}$$

so that (2.8) follows from (2.6), (2.10) and (2.11). This completes the proof of the assertion that (1.2) has a solution. Since it is clear that $S(U)$ is convex, we need only verify the last statement of the theorem. To do this, we define, inductively, convex sets $S_i(U)$, $1 \le i \le m$, as follows. $S_1(U) = S(U)$. If $S_i(U)$ has been defined for $1 \le i \le m-1$, and is nonempty, we consider the minimization problem

$$\|Ls\|_{L^\infty(x_i,x_{i+1})} = \alpha_i = \inf\{\|Lf\|_{L^\infty(x_i,x_{i+1})}: f \in S_i(U)\}. \tag{2.12}$$

$S_{i+1}(U)$ is defined to be the set of all solutions of (2.12). Since any member of $S_m(U)$ clearly satisfies the second statement of Theorem 1, it suffices to show that (2.10) has a solution for each i, $i = 1, \cdots, m-1$. We sketch here the proof for an arbitrary $i = i_0$. Let α_{i_0} be given by (2.12) and choose a sequence $\{f_\nu\} \subset S_{i_0}(U)$ satisfying ess $\sup_{x \in (x_{i_0},x_{i_0+1})} |Lf_\nu(x)| \searrow \alpha_{i_0}$ which is possible since $LS_{i_0}(U)$ is bounded and nonempty in $L^\infty(x_{i_0}, x_{i_0+1})$. By the weak* relative compactness of $\{Lf_\nu\}$ in $L^\infty(x_1, x_m)$, we can select a subsequence $\{f_{\nu_j}\}$ of $\{f_\nu\}$ such that Lf_{ν_j} is weak* convergent to $y \in L^\infty(x_1, x_m)$. Defining f by (2.4), where $u \in N_L$ is defined, as before, by (2.7), we deduce that $f \in S_{i_0}(U)$. Here the only change is the replacement of f_ν by f_{ν_j} in (2.5). Finally, the lower semicontinuity of the norm with respect to weak* convergence implies that ess $\sup_{x \in (x_{i_0},x_{i_0+1})} |Lf(x)| = \alpha_{i_0}$ which completes the proof of the theorem.

3. **Characterization and uniqueness.** We assume throughout this section that hypotheses (I) and (II) are satisfied. We begin with three closely related propositions which are the essential ingredients in the proof of Theorem 2. The term interpolation of data will mean the specification of values of $L_1, \cdots, L_N$.

Proposition 1. *Let n_0 be given as in §1. Let E be a closed set in $[x_1, x_m]$ with the property that E intersects in a set of positive measure any collection of n_0 consecutive intervals in the collection $\{[x_i, x_{i+1}]\}_{i=1}^{m-1}$. Then, given data at the points $x_1, \cdots, x_m$, there is a function $f \in H^{n,\infty}(x_1, x_m)$ which interpolates those data such that Lf is supported in E.*

Proof. By the definition of n_0 and the hypothesis, there is a closed subset E_1 of E of positive measure lying in an interval $I = (x_l, x_{l+1})$ where there are at most n of the functionals L_{ij} associated with the points $x_{l+1}, \cdots, x_m$.

Let $L^\infty(E_1)$ denote those bounded measurable functions on $\mathbf{R}$ which vanish a.e. off E_1, that is, which are supported on E_1. If $g \in L^\infty(E_1)$, then

$$f(x) = \int_{x_l}^x \theta(x; y) g(y)\, dy, \qquad x \ge x_l, \tag{3.1}$$

gives the unique function in $H^{n,\infty}(x_1, x_m)$ which vanishes for $x \le x_l$ and for which $Lf = g$. Because of our assumption (I),

$$\theta(x; y) = \sum_{j=1}^n \phi_j(x) \phi_j^*(y) \tag{3.2}$$

for some choice $\{\phi_1, \cdots, \phi_n\}$ of a basis of the null space N_L of L and some choice $\{\phi_1^*, \cdots, \phi_n^*\}$ of a basis of the null space of L^* [10, pp. 75–78]; in addition, (I) implies that $\{\phi_1^*, \cdots, \phi_n^*\}$ is a Tchebycheff system. If we substitute the expression (3.2) for θ into (3.1) we find that

$$f(x) = \sum_{j=1}^{n} \phi_j(x) \int_I g(y)\phi_j^*(y)\,dy, \quad x \geq x_{l+1}. \tag{3.3}$$

We wish to show that the coefficients of $\phi_1, \cdots, \phi_n$ in (3.3) may be any n-tuple of real numbers for an appropriate choice of $g \in L^\infty(E_1)$. Hence, consider the map T defined by

$$Tg = \left\{\int_I g(y)\phi_j^*(y)\,dy\right\}_{j=1}^{n}, \quad g \in L^\infty(E_1). \tag{3.4}$$

T is clearly linear; if T were not onto $\mathbf{R}^n$, then there would be scalars $\beta_1, \cdots, \beta_n$, not all zero, with

$$0 = \sum_{j=1}^{n} \beta_j \int_I g(y)\phi_j^*(y)\,dy = \int_I g(y) \left(\sum_1^n \beta_j\phi_j^*(y)\right) dy, \quad \text{for all } g \in L^\infty(E_1).$$

This implies that $\Sigma_1^n \beta_j \phi_j^*$ vanishes a.e. on E_1 and since E_1 has positive measure and $\{\phi_1^*, \cdots, \phi_n^*\}$ is Tchebycheff, we learn $\Sigma_1^n \beta_j \phi_j^* = 0$ which in turn implies all the β_j are zero. Hence, T is onto and we deduce the following:

(3.5) Given data at $x_{l+1}, \cdots, x_m$, there is an element f of $H^{n,\infty}(x_1, x_m)$ which vanishes identically on $[x_1, x_l]$ and which interpolates the given data at $x_{l+1}, \cdots, x_m$ such that Lf is supported on E_1.

Again, by the definition of n_0 and the hypotheses, there is a closed subset E_2 of E of positive measure in some interval $I' = (x_k, x_{k+1})$ to the left of E_1 where there are no more than n of the functionals L_{ij} associated with the points $x_{k+1}, \cdots, x_l$. Repeating the argument above we see that any set of data at the points x_i between E_2 and E_1 may be interpolated by a function g in $H^{n,\infty}(x_1, x_m)$ which vanishes identically to the left of E_2 and for which Lg is supported on E_2. Adding to g an appropriate f from (3.5) we see that the interpolation can actually be accomplished by a function h which vanishes identically to the left of E_2, for which $L_{ij}h = 0$ for all those functionals L_{ij} associated with the points $x_{l+1}, \cdots, x_m$, and for which Lh is supported on $E_1 \cup E_2$.

Suppose now the sets $E_1, \cdots, E_t$ have been constructed as above, $t \geq 2$. If there are n_0 or fewer of the points x_i to the left of E_t, then there are n or fewer of the functionals L_{ij} associated with these points $x_1, \cdots, x_q$, $q \leq n_0$; hence, given data at $x_1, \cdots, x_q$ there is, by hypothesis, an element f of the null space of L which interpolates those data at $x_1, \cdots, x_q$. Adding to f an appropriate element g of $H^{n,\infty}(x_1, x_m)$ with Lg supported on $E_1 \cup \cdots \cup E_t$ we see that the data at $x_1, \cdots, x_q$ may be interpolated by a function h in $H^{n,\infty}(x_1, x_m)$ with Lh supported in E and for which $L_{ij} h = 0$ for all the functionals associated with the points $x_{q+1}, \cdots, x_m$. If there are more than n_0 of the points x_i to the left of E_t, then by hypothesis there is a closed subset E_{t+1} of E lying in one of the intervals $(x_\nu, x_{\nu+1})$ to the left of E_t where there are no more than n of the functionals L_{ij} associated with the points x_i which lie between E_{t+1} and E_t.

Now we repeat the argument which led to (3.5) to obtain a function $h \in H^{n,\infty}(x_1, x_m)$ which interpolates arbitrary prescribed data between E_{t+1} and E_t, which vanishes to the left of E_{t+1}, for which Lh is supported on $E_1 \cup \cdots \cup E_{t+1}$ and which satisfies $L_{ij} h = 0$ for all the functionals associated with the points to the right of E_t. We continue in this fashion until there are n_0 or fewer points remaining, at which time we terminate the process by using the argument given above. Any data can be interpolated by a finite sum of the functions just constructed. This completes the proof.

Proposition 2. *Let A and B be two disjoint sets of positive measure in any interval $I = [x_i, x_{i+1}]$, $1 \le i \le m-1$. Then given $h \in H^{n,\infty}(x_1, x_m)$ with Lh supported on A, there is a $g \in H^{n,\infty}$ with Lg supported on B such that g interpolates the same data as h at $x_1, \cdots, x_m$.*

Proof. We know that h is given by, after suitable modification,

$$h(x) = \sum_{j=1}^{n} \phi_j(x) \int_I (Lh)(y)\phi_j^*(y)\,dy \quad \text{for } x \ge x_{i+1}. \tag{3.6}$$

(By subtracting an appropriate element of the null space of L we may assume that h vanishes identically on $[x_1, x_i]$.) The proof that a g exists with $Lg \in L^\infty(B)$ and $g(x) = h(x)$, $x \notin (x_i, x_{i+1})$, is identical with the proof that leads to (3.5) and need not be repeated.

The next proposition is a sharpening of Proposition 2 which allows us to control the sign of Lf and yet still interpolate any given data. We will need the notion of two sets being interspersed.

Definition. Let A and B be two sets of positive measure. We say A and B intersperse at least k times if there are subsets $A_1, \cdots, A_k$ of A of positive measure and subsets $B_1, \cdots, B_k$ of B if k is odd and $B_1, \cdots, B_{k-1}$ if k is even, of positive measure which satisfy the following inequalities for all j for which they are meaningful:

$$a_j < b_j < a_{j+1} \quad \text{for all } a_j \in A_j,\ b_j \in B_j. \tag{3.7}$$

Equivalently, A and B intersperse at least k times if the function which is 1 on A and -1 on B and 0 elsewhere changes sign at least k times on $\mathbf{R}$.

Proposition 3. *Let A and B be disjoint sets of positive measure in any interval $I = [x_i, x_{i+1}]$, $i = 1, \cdots, m-1$, which intersperse at least n times. Then given $h \in H^{n,\infty}(x_1, x_m)$ with Lh supported on I, there is a function $g \in H^{n,\infty}(x_1, x_m)$ with Lg supported on I which interpolates the same data as h at $x_1, \cdots, x_m$ and further, g may be chosen so that Lg is nonnegative on A and nonpositive on B.*

Proof. Again we may subtract an element of the null space of L from h and thereby assume that h vanishes identically on $[x_1, x_i]$. For $x \geq x_{i+1}$, we know that $h(x)$ is given by (3.6) and, as before, we need only show that there is a function $\phi = Lg \in L^\infty(x_1, x_m)$ having the desired support and sign properties and with

$$\int_I \phi_j^*(y)(Lg)(y)\,dy = \int_I \phi_j^*(y)(Lh)(y)\,dy, \qquad j = 1, \cdots, n.$$

g, of course, is then given $g(x) = \int_{x_1}^x \theta(x; \xi)\,\phi(\xi)\,d\xi$ for $x_1 \leq x \leq x_m$.

Let V be the set of bounded measurable functions on I which are nonnegative on A and nonpositive on B. V is a convex cone. Let T be given by (3.4); $T(V)$ is then a convex cone in $\mathbf{R}^n$; if the origin were not an interior point of $T(V)$, then there would be scalars $\beta_1, \cdots, \beta_n$, not all zero, with

$$\sum_1^n \beta_j \int_I \phi_j^*(y) p(y)\,dy \geq 0, \quad \text{for all } p \in V.$$

Equivalently,

$$\int_I p(y) \left(\sum_1^n \beta_j \phi_j^*(y) \right) dy \geq 0, \quad \text{for all } p \in V.$$

This clearly implies that $\Phi = \Sigma\, \beta_j \phi_j^*$ is nonnegative on A and nonpositive on B. Since A and B intersperse at least n times, Φ must have at least n zeros; since $\{\phi_1^*, \cdots, \phi_n^*\}$ is a Tchebycheff system, Φ must vanish identically and hence all the β_i are zero. This contradiction shows that the origin is an interior point of the cone $T(V)$ and hence $T(V) = \mathbf{R}^n$ and the proof is completed.

Proof of Theorem 2. Let f be any solution of (1.2) with $\alpha > 0$ and for $\delta > 0$ let E be the set where $|Lf| \leq \alpha - \delta$. Suppose that E intersects in a set of positive measure any collection of n_0 consecutive intervals in the collection $\{[x_i, x_{i+1}]\}_{i=1}^{m-1}$. By Proposition 1, then, there is a function $g \in H^{n,\infty}(x_1, x_m)$ with Lg supported on E such that g interpolates the same data as f. Consider for $\epsilon > 0$ the function $h = (1+\epsilon)^{-1}(f + \epsilon g)$. Clearly $h \in H^{n,\infty}(x_1, x_m)$ and h lies in U defined in (1.3); further, when ϵ is sufficiently small, $\|Lh\|_{L^\infty}$ is strictly less than α, a contradiction. It follows that there is a collection of n_0 consecutive intervals on which $|Lf| = \alpha$ a.e. for this solution f.

Let $I_1, \cdots, I_p$ be those intervals among $\{[x_i, x_{i+1}]\}_{i=1}^{m-1}$ with the property that for each I_j, $j = 1, \cdots, p$, there is some solution f_j to (1.2) with $|Lf_j| < \alpha$ on a set $E_j \subseteq I_j$ of positive measure. If this set of intervals is empty, then $J = [x_1, x_m]$. Thus, assume $p \geq 1$. Now let $g = p^{-1}(f_1 + \cdots + f_p)$; then g is a solution to (1.2) and by construction $|Lg| < \alpha$ on a set of positive measure in each of the intervals I_j, $j = 1, \cdots, p$. However, we have already proved that there are n_0 consecutive intervals on which $|Lg| = \alpha$ a.e. Hence, the collection $\{I_j\}_{j=1}^{p}$ omits n_0 consecutive intervals in $[x_1, x_m]$ and hence $|Lf| = \alpha$ a.e. on these n_0 consecutive intervals for any solution f to (1.2).

Let J_1 be the union of the n_0 intervals found above; let J be the largest interval of the form $[x_k, x_{k+r}]$ which contains J_1 and such that $|Lf| = \alpha$ a.e. on J for all solutions f of (1.2); it is possible that $J = J_1$. We claim there are $n+1$ or more of the functionals L_{ij} associated with the points $x_k, \cdots, x_{k+r}$. Suppose this is false; we shall construct a solution f of (1.2) with $|Lf| \leq \alpha' < \alpha$ a.e. on J, a contradiction. Let us suppose for simplicity that $k > 1$; minor modifications of the following take care of the case $k = 1$. Suppose also that $k + r < m$; let $I_1 = (x_{k-1}, x_k)$ and $I_2 = (x_{k+r}, x_{k+r+1})$. By the definition of I_1, there is a solution f_1 of (1.2) with $|Lf_1| \leq \alpha - \delta$ on a closed set E_1 of positive measure in I_1 and a solution f_2 of (1.2) with $|Lf_2| \leq \alpha - \delta$ on a closed set E_2 of positive measure in I_2; let $g = \frac{1}{2}(f_1 + f_2)$ so that g is a solution of (1.2) and $|Lg| \leq \alpha - \delta$ (some $\delta > 0$) on both E_1 and E_2. Let h be the element of $H^{n,\infty}(x_1, x_m)$ which vanishes identically on $[x_1, x_{k-1}]$ and for which Lh is Lg on J and zero elsewhere. Since the functionals L_{ij} are consistent and since there are n or fewer of them associated with the points $x_k, \cdots, x_{k+r}$ there is an $h_1 \in H^{n,\infty}(x_1, x_m)$ which vanishes identically on $[x_1, x_{k-1}]$, which interpolates h at $x_k, \cdots, x_{k+r}$, and for which Lh_1 is supported on E_1. (This is just the content of (3.5).) Further, there is an $h_2 \in H^{n,\infty}(x_1, x_m)$ which vanishes identically on $[x_1, x_{k+r}]$, which interpolates $h - h_1$ at $x_{k+r+1}, \cdots, x_m$, and for which Lh_2 is supported on E_2, since $h - h_1$ agrees with a member of N_L on $[x_{k+r+1}, x_m]$. Consider now the function $f = g - \epsilon(h - h_1 - h_2)$; $f \in H^{n,\infty}(x_1, x_m)$ and, by construction, $f \in U$. Further,

$$Lf = \begin{cases} (Lg)(1-\epsilon) & \text{on } J, \\ Lg + \epsilon Lh_1 & \text{on } E_1, \\ Lg + \epsilon Lh_2 & \text{on } E_2, \\ Lg & \text{elsewhere.} \end{cases}$$

Hence, for sufficiently small ϵ, Lf is a solution to (1.2); but $|Lf| \le \alpha' < \alpha$ a.e. on J, a contradiction. Consequently, we learn that there are $n+1$ or more of the functionals L_{ij} associated with the points $x_k, \cdots, x_{k+r}$. Now let f and g be any two solutions to (1.2). Then $Lf = Lg$ a.e. on J by convexity; hence by (3.1) $f - g = \phi$ on J where ϕ is in the null space of L. However, $L_{ij}(f) = L_{ij}(g)$ for all the functionals L_{ij} and thus $L_{ij}(\phi) = 0$ for those $n+1$ or more functionals associated with the points $x_k, \cdots, x_{k+r}$. This implies $\phi = 0$ by the consistency of the functionals and hence $f = g$ on J, as desired.

Let $I = [x_{i-1}, x_i]$ be one of the subintervals and let $s \in S_i(U)$. If $\alpha_{i-1} = 0$, then go onto $[x_i, x_{i+1}]$. Otherwise, an application of Proposition 2, with B consisting of those points in I with $|Ls_*| \le \alpha_{i-1} - \delta$ (or some subset of that set) with $A = I - B$, with h any function in $H^{n,\infty}(x_1, x_m)$ such that $Lh = 0$ off A and $Lh = Ls_*$ on A, and g an interpolant of h such that Lg is supported in B, shows that $|Ls_*| = \alpha_{i-1}$ a.e. on I through a consideration of $s_* - \epsilon(h - g)$. Hence, if s_* is any element in $S_m(U)$, then $|Ls_*| = \alpha_{j-1}$ a.e. on each $I_j = [x_{j-1}, x_j]$, $j = 2, \cdots, m$. If s is any other element of $S_m(U)$, then so is $\frac{1}{2}(s + s_*)$ and hence $|L(s_* + s)| = 2\alpha_{j-1}$ a.e. on I_j. Consequently, $Ls_* = Ls$ on I_j and therefore $Ls_* = Ls$ a.e. on (x_1, x_m). Thus $s_* = s$ on $[x_1, x_m]$, as desired.

Finally, we show that Ls_* is a step function on each interval I; to do this we use Proposition 3. Assume first that $\alpha_I > 0$. In Proposition 3 take A to be the set where $Ls_* = -\alpha_I$ and B to be the set where $Ls_* = \alpha_I$ and h to be any function in $H^{n,\infty}(x_1, x_m)$ with $Lh = 0$ off I and $Lh = Ls_*$ on I. Suppose that A and B intersperse n or more times on I. Then by Proposition 3 there is a function g in $H^{n,\infty}(x_1, x_m)$ with Lg supported on I, $Lg \ge 0$ on A and $Lg \le 0$ on B such that g interpolates the same data as h. Consider the function $f = s_* - \epsilon(h - g)$; $f \in U$ and $Lf = Ls_*$ off I and $Lf = (1-\epsilon)Ls_* + \epsilon Lg$ on I. Since Lg has the opposite sign of Ls_*, when ϵ is sufficiently small, $\|Lf\|$ will be strictly less than α_I on I, a contradiction. This proves the final assertion of Theorem 2 when $\alpha_I > 0$. If $\alpha_I = 0$, then the assertion is trivial.

Corollary 1. *There is a spline solution to* (1.2) *when* $L = D^n$.

Corollary 2. *If* $n_0 = n = m - 1$, *then the solution to* (1.2) *is uniquely determined in* $[x_1, x_m]$.

For example, if we specify only the values of the function at the points $x_1, \cdots, x_{n+1}$ as discussed in the introduction, then $n_0 = n$ and since there are only n intervals, the solution must be unique in $[x_1, x_{n+1}]$. We give a few simple examples below to illustrate the lack of uniqueness in general and the fact that the knots in the spline solution need not fall at the interpolation nodes.

Example 1. Let $L = D$ and take $m = 3$ with the data (0, 0), (1, 1) and (2, 1). Then n_0 is 1 and it is simple to see that any solution of (1.2) must coincide on [0, 1] with the line segment joining (0, 0) to (1, 1). However, on [1, 2] there are many possibilities; one can take any C^1 function whose graph joins (1, 1) to (2, 1) and whose slope is always between -1 and 1. The spline solution, of course, contains the horizontal segment joining (1, 1) to (2, 1).

Example 2. Let $L = D^2$ and $m = 4$ with the following data: $(-2, 0)$, $(-1, -1)$, (1, 1) and (2, 0). Here $n_0 = 2$ and there are 3 intervals. Let F be the solution on the fundamental interval extended to $[-2, 2]$ so that F solves (1.1) and let α_1 be the sup norm of $|F''|$ on the remaining interval so that $\alpha_1 \leq \alpha$. Let $G(x) = -F(-x)$ for $-2 \leq x \leq 2$. Then $G \in H^{2,\infty}(-2, 2)$, $G(x_i) = r_i$ for $i = 1, \cdots, 4$, and $\|G''\| = \alpha$. Hence, by uniqueness, $G(x) = F(x)$ on the fundamental interval. However, either $[-2, -1]$ or $[1, 2]$ together with $[-1, 1]$ constitutes the fundamental interval and hence $\alpha_1 = \alpha$ and $G = F$ on $[-2, 2]$. Thus the solution F is uniquely determined on $[-2, 2]$ and is odd about $x = 0$; in particular, $F(0) = 0$. This implies that F'' changes sign at $x = 0$ and hence the spline solution has a knot at $x = 0$, which is not a node.

The authors have recently shown, in an article to appear in the Journal of Approximation Theory, that problem (*) of the abstract for $L = D^n$ admits a perfect spline solution under hypothesis II of this paper. That is, there is a solution s for which $D^n s = \pm\alpha$, with at most n discontinuities of $D^n s$ between consecutive nodes.

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