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WEAK COMPACTNESS IN THE ORDER DUAL OF A VECTOR LATTICE

BY

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ABSTRACT. A sequence $\{x_n\}$ in a vector lattice E will be called an l'-sequence if there exists an x in E such that $\sum_{k=1}^n |x_k| \le x$ for all n. Denote the order dual of E by E^b. For a set $A \subset E^b$, let $\|\cdot\|_{A^\circ}$ denote the Minkowski functional on E defined by its polar A° in E. A set $A \subset E^b$ will be called equi-l'-continuous on E if $\lim \|x_n\|_{A^\circ} = 0$ for each l'-sequence $\{x_n\}$ in E.

The main objective of this paper will be to characterize compactness in E^b in terms of the order structure on E and E^b. In particular, the relationship of equi-l'-continuity to compactness is studied. §2 extends to $E^{\sigma c}$ the results in Kaplan [8] on vague compactness in E^c. Then this is used to study vague convergence of sequences in E^b.

1. **Introduction.** The main objective of this paper will be to characterize compactness in the order dual E^b of a vector lattice E in terms of the order structure on E. §2 extends to $E^{\sigma c}$ results in Kaplan [8] on vague compactness in E^c. Then §3 considers the order dual E^b of a vector lattice, and Theorem (3.8) characterizes compactness in E^b in terms of the order structure. These results are then used in §4 to extend those in Schaefer [11] on vaguely convergent sequences. We now give the basic properties of a vector lattice that will be needed.

Throughout this paper, we will always assume that a vector lattice E is archimedean. A set in E will be called *order bounded* if it is contained in some interval $[x, y] = \{z \in E: x \le z \le y\}$. A subset A of E will be called *solid* if it has the property: $x \in A$, $|y| \le |x|$ implies $y \in A$. The *solid envelope* of A is the smallest solid set containing A. In fact, the solid envelope of A is the set $\bigcup_{x\in A} [-|x|, |x|]$.

A vector lattice E will be called *complete* if the sup $\bigvee A$ and inf $\bigwedge A$ of every order bounded set A exist. E will be called *σ-complete* if the sup and inf of every countable order bounded set exist.

A net $\{x_\alpha\}$ in E is ascending (respectively descending) if for every pair of indices, $\alpha \le \beta$ implies $x_\alpha \le x_\beta$ (respectively $x_\alpha \ge x_\beta$). The notation $x_\alpha \uparrow x$ means that x_α is ascending and $x = \bigvee x_\alpha$; similarly for $x_\alpha \downarrow x$. A net $\{x_\alpha\}$ is

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said to *order converge* to x if there exists a net $\{y_\alpha\}$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α. We will denote order convergence by $x_\alpha \to x$. A subset A of E will be called *order closed* if for every net $\{x_\alpha\}$ in A, $x_\alpha \to x$ implies that $x \in A$. Given any set A, the smallest order closed set containing A will be called the *order closure* of A, and denotes by $\overline{A}$.

An *ideal* I of E is a linear subspace with the property that $a \in I$, $|b| \leq |a|$ implies $b \in I$. If an ideal I has a complementary ideal J, that is $E = I \oplus J$, then I will be called a *band*. It follows that there is a canonical projection of E onto I. We will denote the image of a set A under this projection by A_I: $A_I = \{x_I : x \in A\}$. This canonical projection preserves sup's and inf's: $x = \bigvee A$ implies $x_I = \bigvee A_I$ and $x = \bigwedge A$ implies $x_I = \bigwedge A_I$.

Two elements x, y of E are called disjoint if $|x| \wedge |y| = 0$. Given a set A in E we will denote by A' the set $\{x \in E : |x| \wedge |y| = 0$ for all y in $A\}$. It can be shown that A' is a closed ideal and that $(A')'$ is the closed ideal generated by A. It follows that if $E = I \oplus J$, then $J = I'$. Later we will need the following:

Theorem 1.1 (Riesz). *If E is complete, every closed ideal I is a band*: $E = I \oplus I'$.

A real linear functional f on E will be called *bounded* if it is bounded on every order bounded set of E. The vector space of bounded linear functionals on E will be called the *bounded dual* of E and denoted by E^b. Under the definition $f \leq g$ if $\langle x, f\rangle \leq \langle x, g\rangle$ for all x in E^+ (the positive cone of E), E^b is a complete vector lattice.

A linear functional f on E will be called *continuous* if $x_\alpha \to x$ in E implies $\lim_\alpha \langle x_\alpha, f\rangle = \langle x, f\rangle$. We will denote the set of continuous linear functionals on E by E^c. A linear functional f on E will be called *σ-continuous* if $x_n \to x$ in E implies $\lim_n \langle x_n, f\rangle = \langle x, f\rangle$, and the set of σ-continuous linear functionals on E will be denoted by $E^{\sigma c}$. Then $E^c \subset E^{\sigma c} \subset E^b$, and, in fact, E^c and $E^{\sigma c}$ are each a band in E^b.

The weak topology on E defined by E^b will be denoted by $w(E, E^b)$. In this paper E^b will always be taken separating on E, hence the weak topology $w(E, E^b)$ is Hausdorff. E^b also defines a finer topology on E than the weak topology. This topology is given by the family of seminorms $\|\cdot\|_y$, y running through E^b, where $\|x\|_y = \langle |x|, |y|\rangle$ for each x in E. We will denote it by $|w|(E, E^b)$. An equivalent definition of this topology is that it is the topology given by the polars in E of intervals of E^b.

In a similar manner, $|w|(E^b, E)$ is defined on E^b by the family of seminorms $\|\cdot\|_x$, where now x runs through E. Also, E defines the vague (or weak*) topology on E^b, denoted by $w(E^b, E)$.

2. **Compactness in $E^{\sigma c}$ and E^c.** A sequence $\{x_n\}$ in a vector lattice E will be called an *l'-sequence* if there exists an element x in E such that $\Sigma_1^n |x_k| \leq x$ for all n. It is clear that if $\{x_n\}$ is an l'-sequence and $|y_n| \leq |x_n|$, then $\{y_n\}$ is also an l'-sequence.

Any l'sequence $\{x_n\}$ converges to 0 in $|w|(E, E^b)$. For there exists x in E such that $\Sigma_1^n |x_k| \leq x$ for all n. Now consider $y \in E^b$, then $\Sigma_1^n \langle |x_k|, |y|\rangle \leq \langle x, |y|\rangle$, and thus $\lim_n \langle |x_n|, |y|\rangle = 0$.

Given a subset A if E^b we will denote by $\|\cdot\|_{A^\circ}$ the Minkowski functional on E defined by its polar A° in E. Thus for each x in E we have $\|x\|_{A^\circ} = \sup_{y \in A} |\langle x, y\rangle|$.

Consider the sublattices of E^c and $E^{\sigma c}$. Each element of E^c is continuous with respect to order convergence of nets of E, and each element of $E^{\sigma c}$ is continuous with respect to order convergence of sequences of E; whereas, each element of E^b is continuous with respect to convergence (always to 0, of course) of l'-sequence of E. The analogy of this for a set of linear functionals is the following.

Definition 2.1. 1. A subset A of E^c will be called *equicontinuous* on E if $\lim_\alpha \|x_\alpha\|_{A^\circ} = 0$ for each net $x_\alpha \to 0$ in E.

2. A subset A of $E^{\sigma c}$ will be called *equi-σ-continuous* on E if $\lim_n \|x_n\|_{A^\circ} = 0$ for each sequence $x_n \to 0$ in E.

3. A subset A of E^b will be called *equi-l'-continuous* on E if $\lim_n \|x_n\|_{A^\circ} = 0$ for each l'-sequence $\{x_n\}$ in E.

Equivalently, (2.1) says that A is equicontinuous on E if each order convergent net in E converges uniformly on A; A is equi-σ-continuous if each order convergent sequence in E converges uniformly on A; and A is equi-l'continuous if each l'sequence converges to 0 uniformly on A. We now give some basic properties of equi-l'continuous subsets of E^b.

Proposition 2.2. *An equi-l'-continuous set A of linear functionals on E is $|w|(E^b, E)$-bounded.*

Proof. Let x be an element of E and suppose $\sup_{y \in A} \langle |x|, |y|\rangle = \infty$. For each n choose y_n in A such that $\langle |x|, |y_n|\rangle > 2^n$. Now $\langle |x|, |y_n|\rangle = \sup_{|b| \leq |x|} |\langle b, y_n\rangle|$, so choose $|b_n| \leq |x|$ such that $|\langle b_n, y_n\rangle| > 2^n$, thus $|\langle b_n/2^n, y_n\rangle| > 1$. But $\{b_n/2^n\}$ is an l'-sequence in E, and we have a contradiction since A is equi-l'-continuous on E.

Proposition 2.3. *A subset A of E^b is equi-l'-continuous on E if and only if its (convex) solid envelope is equi-l'-continuous on E.*

Proof. We need only consider the solid envelope B of A, since it is clear that equi-l'-continuity is equivalent for a set and its convex envelope.

Suppose B is not equi-l'-continuous, then there exists $\epsilon > 0$ and an l'-sequence

$\{x_n\}$ such that $\|x_n\|_{B^\circ} > \epsilon$. Since $\{x_n\}$ is an l'-sequence there is an element x of E such that $\Sigma_1^n |x_k| \leq x$ for all n. Since $\|x_n\|_{B^\circ} > \epsilon$, choose $\{y_n\} \subset A$ such that $\langle |x_n|, |y_n| \rangle > \epsilon$.

By standard formula $\langle |x_n|, |y_n| \rangle = \sup_{|b| \leq |x_n|} |\langle b, y_n \rangle|$, so choose $|b_n| \leq |x_n|$ such that $|\langle b_n, y_n \rangle| > \epsilon$. But $\{b_n\}$ is also an l'-sequence, and $\|b_n\|_{A^\circ} \geq |\langle b_n, y_n \rangle| \geq \epsilon$. Thus we have a contradiction of A being equi-l'-continuous on E.

Remark. To show that a subset A of E^b is equi-l'-continuous on E, one need only show that $\lim_n \|x_n\|_{A^\circ} = 0$ for each *positive* l'sequence of E. This follows since $\|x_n\|_{A^\circ} \leq \|x_n^+\|_{A^\circ} + \|x_n^-\|_{A^\circ}$ and $\{x_n^+\}$ (respectively $\{x_n^-\}$) is a positive l'-sequence whenever $\{x_n\}$ is an l'sequence of E.

Proposition 2.4. *Let A be a subset of E^b; the following are equivalent:*

(1) *A is equi-l'-continuous on E.*

(2) *Every bounded monotone net in E is $\|\cdot\|_{A^\circ}$-Cauchy.*

(3) *Every bounded monotone sequence in E is $\|\cdot\|_{A^\circ}$-Cauchy.*

Proof. (1) $\Rightarrow$ (2) Suppose (2) does not hold, then there exists a bounded monotone increasing net $\{x_\alpha\}$ which is not $\|\cdot\|_{A^\circ}$-Cauchy. Thus there exist $\epsilon > 0$ and $\alpha_1 \leq \alpha_2 \leq \cdots$ such that $\|x_{\alpha_{n+1}} - x_{\alpha_n}\|_{A^\circ} > \epsilon$. Now $\{x_\alpha\}$ is order bounded by some element x on E, so $\Sigma_{k=1}^n (x_{\alpha_{k+1}} - x_{\alpha_k}) = (x_{\alpha_{n+1}} - x_{\alpha_1}) \leq (x - x_{\alpha_1})$. Thus $\{(x_{\alpha_{k+1}} - x_{\alpha_k})\}$ is an l'-sequence in E and we have a contradiction.

Of course, (2) implies (3).

(3) $\Rightarrow$ (1) Consider a positive l'-sequence $\{x_n\}$ in E. Set $y_n = \Sigma_1^n x_k$. Then $\{y_n\}$ is a bounded monotone increasing sequence of E. Thus

$$\lim_n \|x_n\|_{A^\circ} = \lim_n \|y_n - y_{n-1}\|_{A^\circ} = 0$$

and the proof is complete.

Contained in the above proof is the following useful observation:

Corollary 2.5. *A subset A of E^b is equi-l'-continuous on E if and only if every countable subset of A is equi-l'-continuous on E.*

The main result (2.8) of this section is the characterization of vaguely compact subsets of $E^{\sigma c}$ in terms of the order structure on E, in particular, in terms of equi-σ-continuity on E.

Proposition 2.6. *Let E be σ-complete. Then for $A \subset E^{\sigma c}$ the following are equivalent:*

(1) *A is equi-σ-continuous on E.*

(2) *A is equi-l'-continuous on E.*

Proof. Since E is σ-complete, it follows that $x_n \to 0$ for any l'-sequence $\{x_n\}$ in E. Hence (1) implies (2).

Assume (2) holds. Note that we may take A solid. Let $x_n \to 0$ in E, then there exist $y_n \downarrow 0$ such that $|x_n| \le y_n$. Let $\epsilon > 0$. By (2.4) choose k such that $\|y_n - y_m\|_{A^\circ} < \epsilon$ for $n, m \ge k$. Fix n $(n \ge k)$. Since $\|x_n\|_{A^\circ} = \sup_{z \in A} |\langle x_n, z\rangle|$, choose $z \in A$ such that $\|x_n\|_{A^\circ} \le |\langle x_n, z\rangle| + \epsilon$. Then for $m \ge k$

$$\|x_n\|_{A^0} \le \langle y_n, |z|\rangle + \epsilon \le \|y_n - y_m\|_{A^0} + \langle y_m, |z|\rangle + \epsilon \le \langle y_m, |z|\rangle + 2\epsilon.$$

But $\lim_m \langle y_m, |z|\rangle = 0$, thus $\|x_n\|_{A^\circ} \le 3\epsilon$ for $n \ge k$. Hence $\lim_n \|x_n\|_{A^\circ} = 0$ and the proof is complete.

Proposition 2.7. *Let E be σ-complete. Then for $A \subset E^{\sigma c}$ the following are equivalent:*

(1) *A is equi-σ-continuous on E.*

(2) *$x_n \downarrow 0$ in E implies $\lim_n \|x_n\|_{A^\circ} = 0$.*

Proof. That (1) implies (2) follows from the definition of A being equi-σ-continuous on E. Assume (2) holds, and suppose A is not equi-σ-continuous on E. Then there exist $\epsilon > 0$ and a sequence $x_n \to 0$ in E such that $\|x_n\|_{A^\circ} > \epsilon$.

Since $x_n \to 0$ in E, there exists a sequence $\{y_n\}$ in E with $|x_n| \le y_n$ and $y_n \downarrow 0$. Choose z_1 in A such that $|\langle x_1, z_1\rangle| > \epsilon$. Since $y_n \downarrow 0$, $(y_n \vee x_1) \downarrow x_1$, hence since z_1 belongs to $E^{\sigma c}$, there exists n_1 such that $|\langle y_{n_1} \vee x_1, z_1\rangle| > \epsilon$. There exists z_2 in A such that $|\langle x_{n_1}, z_2\rangle| > \epsilon$. Since $(y_n \vee x_{n_1}) \downarrow x_{n_1}$, there exists $n_2 > n_1$ such that $|\langle y_{n_2} \vee x_{n_1}, z_2\rangle| > \epsilon$. Proceeding inductively we obtain $n_1 < n_2 < n_3 < \cdots$ and $\{z_k\} \subset A$ satisfying $|\langle y_{n_k} \vee x_{n_{k-1}}, z_k\rangle| > \epsilon$. Set $w_k = y_{n_k} \vee x_{n_{k-1}}$, then $w_k \downarrow 0$ and $\|w_k\|_{A^\circ} > \epsilon$. Thus (2) fails to hold, and the proof is complete.

For a σ-complete vector lattice E, vague compactness in $E^{\sigma c}$ is completely characterized by equi-σ-continuity on E.

Theorem 2.8. *If E is σ-complete, then for $A \subset E^{\sigma c}$ the following are equivalent:*

(1) *A is equi-σ-continuous on E.*

(2) *A is relatively $w(E^{\sigma c}, E)$-compact in $E^{\sigma c}$.*

Proof. (1) $\Rightarrow$ (2) It follows from (2.2) that A is $|w|(E^{\sigma c}, E)$-bounded, hence $w(E^{\sigma c}, E)$-bounded. Thus its vague closure B in the algebraic dual E^* of E is $w(E^*, E)$-compact. Thus one need only show that $B \subset E^{\sigma c}$. This follows easily from the fact that order convergent sequences of E must converge uniformly on A.

(2) $\Rightarrow$ (1) Suppose A is not equi-σ-continuous on E, then there exist $\epsilon > 0$ and a positive l'-sequence $\{w_k\}$ in E such that $\|w_k\|_{A^\circ} > 2\epsilon$.

Set $k_1 = 1$ and choose y_1 in A such that $|\langle w_{k_1}, y_1\rangle| > 2\epsilon$. Now $\lim_k \langle w_k, y_1\rangle = 0$, since $\{w_k\}$ is an l'-sequence. It follows that we can choose an integer k_2 and an element y_2 in A so that $|\langle w_{k_2}, y_1\rangle| \le \epsilon$ and $|\langle w_{k_2}, y_2\rangle| > 2\epsilon$. Proceeding inductively, we obtain sequences $\{w_{k_n}\}, \{y_n\} \subset A$ such that $|\langle w_{k_n}, y_n\rangle| > 2\epsilon$ and $|\langle w_{k_{n+1}}, y_m\rangle| \le \epsilon$ for $m \le k_n$. Hence $|\langle w_{k_n}, y_n - y_m\rangle| \ge \epsilon$ for $m < k_n$. For simplicity of notation, let our original sequences have this property: $|\langle w_k, y_k - y_m\rangle| > \epsilon$ for $m < k$.

Since $\{y_k\}$ is relatively $w(E^{\sigma c}, E)$-compact it has a $w(E^{\sigma c}, E)$ accumulation point y in $E^{\sigma c}$. By a diagonal method we can choose a subsequence $\{y_{k_n}\}$ such that $\lim_n \langle w_k, y_{k_n}\rangle = \langle w_k, y\rangle$ for each k.

Set $z_n = (y_{k_n} - y_{k_{n-1}})$ and $x_n = w_{k_n}$. Then we have that

$$\text{(i)} \qquad |\langle x_n, z_n\rangle| \ge \epsilon \quad \text{and} \quad \lim_k \langle x_n, z_k\rangle = 0 \quad \text{for each } n.$$

We will construct an element v in E such that $|\langle v, z_{n_k}\rangle| \ge \epsilon/3$ for an infinite subsequence n_k, where $v = \sum_{k=1}^{\infty} x_{n_k}$.

Suppose this construction is completed. Since $\{z_{n_k}\}$ is also relatively $w(E^{\sigma c}, E)$-compact, it has a $w(E^{\sigma c}, E)$ accumulation point z. By line (i) $\lim_k \langle x_n, z_{n_k}\rangle = 0$. Therefore, since z is a $w(E^{\sigma c}, E)$ accumulation point of $\{z_{n_k}\}$, if follows that $\langle x_n, z\rangle = 0$ for each n. But $v = \sum_{k=1}^{\infty} x_{n_k}$ and $z \in E^{\sigma c}$, so $\langle v, z\rangle = \sup_m \langle \sum_1^m x_{n_k}, z\rangle = 0$.

Thus we have that $|\langle v, z_{n_k}\rangle| \ge \epsilon/3$ and $\langle v, z\rangle = 0$, which contradicts z being a $w(E^{\sigma c}, E)$ accumulation point of $\{z_{n_k}\}$.

We now construct the element $v = \sum_1^{\infty} x_{n_k}$ by induction. Setting $n_0 = 1$, we will define inductively an increasing sequence of integers n_j such that

$$\text{(ii)} \qquad \sum_{i=1}^{j-1} |\langle x_{n_i}, z_{n_j}\rangle| < \epsilon/3 \quad \text{and} \quad \sum_{n=n_j}^{\infty} |\langle x_n, z_{n_{j-1}}\rangle| < \epsilon/3.$$

Assume $n_1, \cdots, n_j$ are defined. Since $\{z_n\}$ converges to 0 on the x_n's there exists $m_1 > (n_j + 1)$ with $\sum_{i=1}^{j} |\langle x_{n_i}, z_n\rangle| < \epsilon/3$ for $n \ge m_1$. Since $\{x_n\}$ is a positive l'-sequence, there exists an x in E with $\sum_{k=1}^{n} x_k \le x$ for all n. Therefore

$$\sum_{n=1}^{\infty} |\langle x_n, z_{n_j}\rangle| \le \sum_{n=1}^{\infty} \langle x_n, |z_{n_j}|\rangle \le \langle x, |z_{n_j}|\rangle.$$

Thus there exists $m_2 > m_1$ with $\sum_{n=m_2}^{\infty} |\langle x_n, z_{n_j}\rangle| < \epsilon/3$. Set $n_{j+1} = m_2$ and we have that $\sum_{i=1}^{j} |\langle x_{n_i}, z_{n_{j+1}}\rangle| < \epsilon/3$ and $\sum_{n=n_{j+1}}^{\infty} |\langle x_n, z_{n_j}\rangle| < \epsilon/3$. This completes the induction.

By line (ii) we have

$$\text{(iii)} \qquad \sum_{i=j+1}^{\infty} |\langle x_{n_i}, z_{n_j}\rangle| \le \sum_{n=n_{j+1}}^{\infty} |\langle x_n, z_{n_j}\rangle| < \epsilon/3.$$

Set $v = \sum_{i=1}^{\infty} x_{n_i}$, v exists in E since $\{x_n\}$ is an l'-sequence and E is σ-complete.

Note that $v = \sup_m (\sum_{i=1}^{m} x_{n_i})$ and $\{z_{n_j}\} \subset E^{\sigma c}$, thus

$$|\langle v, z_{n_j}\rangle| = \sup_m \left|\sum_{i=1}^{m} \langle x_{n_i}, z_{n_j}\rangle\right|,$$

$$|\langle v, z_{n_j}\rangle| \geq \sup_m \left|\sum_{i=1}^{j-1} \langle x_{n_i}, z_{n_j}\rangle + \langle x_{n_j}, z_{n_j}\rangle + \sum_{i=j+1}^{m} \langle x_{n_i}, z_{n_j}\rangle\right|,$$

$$|\langle v, z_{n_j}\rangle| \geq |\langle x_{n_j}, z_{n_j}\rangle| - \sum_{i=1}^{j-1} |\langle x_{n_i}, z_{n_j}\rangle| - \sum_{i=j+1}^{m} |\langle x_{n_i}, z_{n_j}\rangle|.$$

By lines (i), (ii), and (iii), we have $|\langle v, z_{n_j}\rangle| \geq \epsilon/3$, and this completes the proof.

Combining (2.6) and (2.8) with (2.3) we have

Corollary 2.9. *Let E be σ-complete. If A is relatively $w(E^{\sigma c}, E)$-compact in $E^{\sigma c}$ then so is its convex solid envelope.*

Consider an order bounded set $\{x_n\}$ of mutually disjoint elements of E. Then $\bigvee_1^n |x_k| = \Sigma_1^n |x_k|$, so $\{x_n\}$ is an l'sequence of E. This very special class of l'-sequences will also characterize vaguely compact sets of $E^{\sigma c}$.

Proposition 2.11. *If E is σ-complete and $A \subset E^{\sigma c}$, then the following are equivalent:*

(1) *A is relatively $w(E^{\sigma c}, E)$-compact.*

(2) (a) *A is $|w|(E^{\sigma c},E)$ bounded.* (b) *If $\{x_n\}$ is a bounded set of mutually disjoint elements of E, then $\lim_n \|x_n\|_{A^\circ} = 0$.*

Proof. If the x_n's are bounded and mutually disjoint, then $\{x_n\}$ is an l'-sequence; hence (1) implies (2).

We complete the proof by showing that (2) above implies (2) of (2.7). Thus A will be equi-σ-continuous on E, and hence by (2.8) relatively $w(E^{\sigma c}, E)$-compact.

It is easy to show that (2) above must also hold for the solid envelope of A. Hence we may suppose A is solid. Now suppose that (2.7) does not hold. Then there exist $\epsilon > 0$, $x_n \downarrow 0$ in E, and $\{y_n\} \subset A$ such that $|\langle x_n, y_n\rangle| > 3\epsilon$ and $|\langle x_{n+1}, y_n\rangle| < \epsilon^2$. Moreover since $|y_n| \in A$, we may take $y_n \geq 0$. Also, A is $|w|(E^{\sigma c}, E)$ bounded so there exist real $\lambda > 0$ such that $\langle x_1, y_n\rangle < \lambda$ for all n. There is no loss of generality in supposing that $\lambda = 1$.

The following elementary relations are easily verified:

Let $x \geq 0$ and z in the closed ideal generated by x in E; then

(i) If $f = x_{z^+}$, then $z_f = z^+$.

(ii) If $f = x_{(z-\lambda x)^+}$, then $\lambda f \leq z_f$.

Let $f_n + g_n = x_1$, where f_n is the projection of x_1 on the closed ideal generated by $(x_n - \epsilon x_1)^+$. It is easily verified that $f_n \downarrow 0$. By (i) and (ii), it follows that $\epsilon f_n \leq x_n$ and $x_n \wedge g_n \leq \epsilon x_1$, thus $x_n = x_n \wedge x_1 = x_n \wedge f_n + x_n \wedge g_n \leq f_n + \epsilon x_1$. Hence

$$\langle x_n, y_n\rangle \le \langle f_n, y_n\rangle + \langle \epsilon x_1, y_n\rangle \le \langle f_n - f_{n+1}, y_n\rangle + \langle f_{n+1}, y_n\rangle + \epsilon$$
$$\le \|f_n - f_{n+1}\|_{A^0} + 2\epsilon.$$

Note that $\{(f_n - f_{n+1})\}$ are mutually disjoint and bounded, hence $\lim_n \|f_n - f_{n+1}\|_{A^\circ} = 0$. Thus we obtain $\langle x_n, y_n\rangle \le 3\epsilon$ for n large enough. This contradicts the choice of y_n's and completes the proof.

Since the elements of E^c are continuous with respect to order convergence of nets in E, we can state (2.6) in terms of nets.

Proposition 2.12. *Let E be σ-complete. Then for $A \subset E^c$ the following are equivalent*:

(1) *A is equicontinuous on E.*

(2) *A is equi-σ-continuous on E.*

(3) *A is equi-l'-continuous on E.*

Proof. From Definition (2.1) it is clear that (1) implies (2). Note that $A \subset E^c \subset E^{\sigma c}$; thus, by (2.6), (2) is equivalent to (3). That (3) implies (1) follows by an argument similar to the proof of (2.6).

Proposition 2.13. *Let E be σ-complete. Then for $A \subset E^c$, the following are equivalent*:

(1) *A is equicontinuous on E.*

(2) *$x_\alpha \downarrow 0$ in E implies $\lim_\alpha \|x_\alpha\|_{A^\circ} = 0$.*

A characterization of vague compactness in E^c was first given for a special case by Nakano [9, § 28]. The general case for σ-complete spaces was proved by Kaplan [8, (3.4)]. We obtain this result as a corollary of (2.8) by considering E^c as a sublattice of $E^{\sigma c}$.

Corollary 2.14. *If E is σ-complete, then for $A \subset E^c$ the following are equivalent*:

(1) *A is equicontinuous on E.*

(2) *A is relatively $w(E^c, E)$-compact in E^c.*

We now give a characterization of $w(E^c, E)$-compactness which is most simply stated for a solid set. Later we will be able to extend this result to $E^{\sigma c}$ and also obtain a partial extension to E^b.

Proposition 2.15. *If E is σ-complete, then for a solid set A in E^c the following are equivalent*:

(1) *A is relatively $w(E^c, E)$-compact.*

(2) (a) *A is $|w|(E^c, E)$-bounded, and* (b) *every countable set $\{y_n\}$ of mutually disjoint elements of A converges to 0 in $|w|(E^c, E)$.*

Proof. (1) $\Rightarrow$ (2) (a) above follows from (2.2). Suppose (b) does not hold, then there exist $\epsilon > 0$ and $x \in E^+$ and a countable set $\{y_n\}$ of mutually disjoint elements of A such that $\langle x, |y_n|\rangle > \epsilon$.

Since A is solid, take $\{y_n\}$ positive. Let I_n be the closed ideal in E^c generated by y_n and $J_n = (I_n^\perp)'$ the dual ideal in E. By Luxemburg and Zaanen [8, (3.3)] J_n is a band in E. Let z_n be the component of x in J_n.

Since y_n's are mutually disjoint, the z_n's are also mutually disjoint and order bounded by x. Thus $\{z_n\}$ is an l'-sequence in E. But for every n we have $\|z_n\|_{A^\circ} \geq \langle z_n, y_n\rangle = \langle x, y_n\rangle \geq \epsilon$, and hence a contradiction.

(2) $\Rightarrow$ (1) We will show (2) of (2.11) holds. Suppose not. Then there exist $\epsilon > 0$, $\{x_n\}$ bounded mutually disjoint positive elements of E, and $\{y_n\} \subset A$, $y_n \geq 0$ such that $\langle x_n, y_n\rangle \geq \epsilon$ for all n.

Let J_n be the closed ideal in E generated by x_n. Then $I_n = (J_n^\perp)'$ in E^c is a band, so let z_n be the component of y_n in I_n.

Then $z_n \geq 0$ and $z_n \in A$ since A is solid. There exist $x \geq x_n$ for all n; then $\langle x, z_n\rangle \geq \langle x_n, z_n\rangle = \langle x_n, y_n\rangle \geq \epsilon$. But z_n's are mutually disjoint, since the x_n's are, hence by (2) above $\lim_n \langle x, z_n\rangle = 0$, and again we have a contradiction.

3. Compactness in E^b. We now consider the question of characterizing compactness in E^b in terms of equi-l'-continuity. But first we need to prove some results which give a deeper relationship between equi-l'-continuity and the order structure on both E and E^b.

Each element s in E^b generates a closed ideal S which is a band in E^b. So $E^b = S \oplus S'$. Hence there is a canonical projection of E^b onto S. We will denote the image of a subset A of E^b under this projection by A_s.

The idea for the next proposition essentially comes from a construction, in a measure space, used by Ando [1]. When translated to a vector lattice, it has the surprising property of being equivalent to equi-l'-continuity. The importance of Proposition (3.1) is that it allows us to take any order bounded sequence in E and, in some sense, $\|\cdot\|_{A^\circ}$-approximate it by a bounded monotone sequence.

Proposition 3.1. *Given a solid set A in E^b, the following are equivalent:*

(1) *A is equi-l'-continuous on E.*

(2) *For each order bounded sequence $\{x_n\}$ in E and $\epsilon > 0$, there exist two sequences $\{y_n\}$ and $\{z_n\}$ such that*

(a) $y_n = x_n \vee x_{n+1} \vee \cdots \vee x_{j(n)}$ *and* $z_n = \bigwedge_1^n y_k$ *where* $j(n+1) \geq j(n) \geq n$,

(b) $\|y_n - z_n\|_{A^\circ} < \epsilon$.

Proof. (1) $\Rightarrow$ (2) The sequence $x_{n,k} = \bigvee_{i=n}^k x_i$ $(k \geq n)$ is a bounded monotone sequence for each fixed n. By (3) of (2.4) there exists a sequence of positive integers $j(n)$ with $j(n+1) \geq j(n) \geq n$ and $\|x_{n,k} - x_{n,j(n)}\|_{A^\circ} < \epsilon/2^n$ for all $k \geq j(n)$. Set $y_n = x_{n,j(n)}$ and $z_n = \bigwedge_1^n y_k$. Then

$$0 \leq y_n - z_n = \left(y_n - \bigwedge_1^n y_k\right) \leq \sum_{k=1}^{n-1} (y_{k+1} - y_{k+1} \wedge y_k).$$

It follows since A is solid that

$$\|y_n - z_n\|_{A^\circ} \leq \sum_{k=1}^{n-1} \|y_{k+1} - y_{k+1} \wedge y_k\|_{A^\circ}.$$

Now for any two elements of a vector lattice the following hold: $y_{k+1} - y_{k+1} \wedge y_k = y_{k+1} \vee y_k - y_k$, so $y_{k+1} - y_{k+1} \wedge y_k = x_{k,j(k+1)} - x_{k,j(k)}$. Thus

$$\|y_n - z_n\|_{A^\circ} \leq \sum_{k=1}^{n-1} \|x_{k,\,j(k+1)} - x_{k,\,j(k)}\|_{A^\circ} \leq \epsilon.$$

(2) $\Longrightarrow$ (1) Let $\{x_n\}$ be a bounded monotone increasing sequence in E and $\epsilon > 0$. Apply (1) above to $\{x_n\}$ and ϵ, getting $y_n = x_n \vee x_{n+1} \vee \cdots \vee x_{j(n)} = x_{j(n)}$ and $z_n = \bigwedge_{k=1}^n y_k = x_{j(1)}$ such that $\|y_n - z_n\|_{A^\circ} < \epsilon/2$.

Since A is solid, we have for $n, m \geq j(1)$

$$\|x_n - x_m\|_{A^\circ} \leq \|x_n - x_{j(1)}\|_{A^\circ} + \|x_m - x_{j(1)}\|_{A^\circ},$$

$$\|x_n - x_m\|_{A^\circ} \leq \|x_{j(n)} - x_{j(1)}\|_{A^\circ} + \|x_{j(m)} - x_{n(1)}\|_{A^\circ} \leq 2\epsilon.$$

Thus by (2.4) A is equi-$\overline{l}'$-continuous on E; and the proof is complete.

Consider $s \in E^b$, $s \geq 0$. For simplicity we will denote the seminorm $\|\cdot\|_{[-s,s]^\circ}$ on E by $\|\cdot\|_s$. It is easy to show that $\|x\|_s = \langle |x|, s\rangle$ for all $x \in E$. Also, consider any z in the closed ideal S generated by s in E^b. It is then easy to show that if $\{x_n\}$ is order bounded and if $\lim_n \|x_n\|_s = 0$, then $\lim_n \|x_n\|_z = 0$. Thus any element z of S is $\|\cdot\|_s$-continuous on each interval of E. We now give one of the main results of this section.

Theorem 3.2. *Let A be a subset of E^b, the following are equivalent*:

(1) *A is equi-l'-continuous on E.*

(2) (a) *A is $|w|(E^b, E)$-bounded, and* (b) *for each $s \in E^b$, A_s is equi-l'-continuous on E.*

(3) (a) *A is $|w|(E^b, E)$-bounded, and* (b) *for each $s \in E^b$, A_s is $\|\cdot\|_s$-equicontinuous on each interval of E.*

(4) *For each $x \in E$ and $\epsilon > 0$, there exist $\delta > 0$ and a finite set $\{z_i\}_1^n \subset A$ such that: if $|y| \leq |x|$ and $\|y\|_{z_i} < \delta$, $i = 1, \cdots, n$, then $\|y\|_{A^\circ} < \epsilon$.*

(5) *For each $x \in E$, there exists z in E^b such that: A is $\|\cdot\|_z$-equicontinuous on the interval $[-x, x]$.*

(6) *If $\{x_n\}$ is order bounded and $|w|(E, E^b)$-convergent to 0, then $\lim_n \|x_n\|_{A^\circ} = 0$.*

Proof. (1) $\Rightarrow$ (2) We may assume that A is solid; then $A_s \subset A$. Thus A_s must also be equi-l'-continuous on E.

(2) $\Rightarrow$ (3) By (2.3) we may assume A_s is solid. Suppose (3) does not hold, then there exist $\epsilon > 0$, $s \geq 0$ in E^b and an order bounded sequence $\{x_n\} \subset E$ such that: $\|x_n\|_s < 1/2^n$ and $\|x_n\|_{A_s^\circ} > 2\epsilon$.

Since A_s is solid, we may take $x_n \geq 0$. A_s is equi-l'-continuous on E, hence by (3.1) there exist $y_n = x_n \vee x_{n+1} \vee \cdots \vee x_{j(n)}$ and $z_n = \bigwedge_1^n y_k$ such that $\|y_n - z_n\|_{A_s^\circ} < \epsilon$. Note that $y_n \geq 0$, $z_n \geq 0$ and $\{z_n\}$ is a bounded monotone decreasing sequence.

$$\|z_n\|_s = \langle z_n, s\rangle \leq \langle y_n, s\rangle \leq \sum_{k=n}^{j(n)} \langle x_k, s\rangle \leq 1/2^{n-1}.$$

Thus $\lim_n \|z_n\|_s = 0$. But each element of A_s is $\|\cdot\|_s$-continuous on the order bounded set $\{z_n\}$, thus $\lim_n \langle z_n, w\rangle = 0$ for each $w \in A_s$. Note that $\|z_n\|_{A_s^\circ} \geq \|y_n\|_{A_s^\circ} - \|y_n - z_n\|_{A_s^\circ} \geq \epsilon$.

By (2.4) there exist k such that $\|z_n - z_m\|_{A_s^\circ} < \epsilon/3$. for $n, m \geq k$. Fix $n \geq k$ and choose w in A_s such that $\|z_n\|_{A_s^\circ} \leq \langle z_n, w\rangle + \epsilon/3$. Then

$$\|z_n\|_{A_s^\circ} \leq \|z_n - z_m\|_{A_s^\circ} + \langle z_m, w\rangle + \epsilon/3 \leq 2\epsilon/3 + \langle z_m, w\rangle.$$

But $\lim_m \langle z_m, w\rangle = 0$, hence $\|z_n\|_{A_s^\circ} < \epsilon$ and we have a contradiction.

(3) $\Rightarrow$ (4) Suppose (4) does not hold, then there exist $x \in E$, $\epsilon > 0$, and sequences $|x_n| \leq x$, $\{z_n\} \subset A$ such that

$$\langle |x_n|, |z_k|\rangle < 1/2^n \quad \text{for } 1 \leq k \leq n \quad \text{and} \quad |\langle x_n, z_{n+1}\rangle| > \epsilon.$$

A is $|w|(E^b, E)$-bounded, so $z = \sum_{k=1}^\infty |z_k|/2^k$ exists in E^b. Then $\lim_n \|x_n\|_z = 0$, and hence by (3) $\lim_n \|x_n\|_{A_z^\circ} = 0$. Now $\{z_n\}$ is contained in the ideal generated by z, thus $\{z_n\} \subset A_z$. Hence $\|x_n\|_{A_z^\circ} \geq |\langle x_n, z_{n+1}\rangle| \geq \epsilon$, which again gives a contradiction.

(4) $\Rightarrow$ (5) Let $x \in E$. Let $\epsilon_n = 1/n$, so by (4) there exist $\delta_n > 0$ and a finite set $B_n \subset A$ such that

If $|y| \leq |x|$ and $\|y\|_z < \delta_n$ for all z in B_n, then $\|y\|_{A^\circ} < 1/n$.

Let $B = \bigcup_1^\infty B_n$, so B is a countable subset of A, denote B by $\{z_n\}$ where the z_n's are elements of A. A is $|w|(E^b, E)$-bounded, hence $z = \sum_1^\infty |z_n|/2^n$ exist in E^b. It then follows that A is $\|\cdot\|_z$-equicontinuous on $[-x, x]$.

(5) $\Rightarrow$ (6) Let $\{x_n\}$ be order bounded by x and $|w|(E, E^b)$-convergent to 0.

By (5) choose z such that A is $\|\cdot\|_z$-equicontinuous on $[-x, x]$. But $\lim_n \|x_n\|_z = 0$, so $\lim_n \|x_n\|_{A^\circ} = 0$, and hence (6) holds.

(6) $\Rightarrow$ (1) Let $\{x_n\}$ be an l'-sequence in E, then note that $\{x_n\}$ is $|w|(E, E^b)$-convergent to 0. Thus by (6) $\lim \|x_n\|_{A^\circ} = 0$, so A is equi-l'-continuous on E, and the proof is complete.

I_E and $Ba^{1/2}$. Consider a vector lattice E and its bounded dual E^b. Then E^b is an order complete vector lattice and has an order continuous dual which we denote by $(E^b)^c$. Since we always take E^b separating on E, we have a canonical imbedding of E in $(E^b)^c$. This imbedding is, in fact, a vector lattice isomorphism of E with a linear sublattice of $(E^b)^c$ [6, (2.6)]. We will thus consider E as contained in $(E^b)^c$.

Consider two elements x, y in E; we point out that $x \vee y$-in-$E = x \vee y$-in-$(E^b)^c$. However, the infinite sup or inf of elements in E may not agree with the sup or inf in $(E^b)^c$.

We will denote by I_E the ideal generated by E in $(E^b)^c$. Thus $E \subset I_E \subset (E^b)^c$ where $I_E = \{y \in (E^b)^c\colon$ there exists x in E with $|y| \leq |x|\}$. I_E considered as a vector lattice is Dedekind complete since $(E^b)^c$ is. Also, note that if $y = \bigvee y_\alpha$-in-I_E, then $y = \bigvee y_\alpha$-in-$(E^b)^c$.

We now give (without proof) some known properties of I_E. Note that $E \subset I_E$, thus E has an order closure $\overline{E}$ in the vector lattice I_E. As might be expected, $\overline{E}$ is exactly I_E.

Proposition 3.3. $\overline{E} = I_E$.

Since I_E is a vector lattice, it has an order continuous dual $(I_E)^c$. We now explictly state what this dual is.

Proposition 3.4. $(I_E)^c = E^b$.

Combining (3.3) and (3.4), we get the following:

Proposition 3.5. *E is $|w|(I_E, E^b)$-dense in I_E.*

Let $Ba^{1/2}$ be the subspace of I_E generated by the elements of the form: $x = \bigvee x_n$-in-I_E where $\{x_n\} \subset E$. Then $E \subset Ba^{1/2} \subset I_E$. Each element of $Ba^{1/2}$ can be written as $(f - g)$ where f and g are each the sup in I_E of a countable subset of E. $Ba^{1/2}$ is a subspace of I_E, and it is easy to show that, in fact, $Ba^{1/2}$ is a linear sublattice of I_E. Also, $Ba^{1/2}$ is not σ-complete, but it has the property that if $\{x_n\}$ is an order bounded sequence of E, then $\bigvee x_n$ is an element of $Ba^{1/2}$. It is exactly this property that makes $Ba^{1/2}$ such an important sublattice of I_E. Also, note that if $\{x_n\}$ is an l'-sequence of E, then $\{\Sigma_1^n |x_n|\}$ is an order bounded sequence in I_E. Therefore, the order sum $x = \Sigma_1^\infty |x_k| = \sup_n (\Sigma_1^n |x_k|)$ is an element of $Ba^{1/2}$.

Remark. It can be shown that $(Ba^{1/2})^{\sigma c} = E^b$ by modifying the proofs of (9.3) and (9.6) in [8].

Let $E = C(X)$ be the space of continuous functions on a compact set X. Then the $Ba^{1/2}$ associated with $C(X)$ is a subspace of the first Baire class Ba^1, hence the use of the notation $Ba^{1/2}$.

Since E is contained in I_E, each l'-sequence in E is also an l'-sequence in I_E, but I_E has many more l'-sequences than those contained in E. Surprisingly, if $A \subset E^b$ is equi-l'-continuous on E, then A is equi-l'-continuous on $\overline{E} = I_E$.

Proposition 3.6. $A \subset E^b$; *then the following are equivalent:*

(1) A *is equi-l'-continuous on* E.

(2) A *is equi-l'-continuous on* I_E.

Proof. (1) $\Rightarrow$ (2) We will show that (5) of (3.2) holds for the spaces I_E and $(I_E)^c = E^b$. Consider an interval $[-x_0, x_0]$-in-I_E. Choose x in E such that $|x_0| \leq x$. By (3.2) there exists an element z in E^b such that A is $\|\cdot\|_z$-equicontinuous on the interval $[-x, x]$-in-E. It follows from (3.5) that the interval $[-x, x]$-in-E is $|w|(I_E, E^b)$-dense in the interval $[-x, x]$-in-I_E. It then can be shown (from the denseness) that A is $\|\cdot\|_z$-equicontinuous on $[-x, x]$-in-I_E.

(2) $\Rightarrow$ (1) Since $E \subset I_E$, (1) must hold and the proof is complete.

Combining (3.6) with (2.12) applied to the spaces I_E and $(I_E)^c = E^b$, we have

Corollary 3.7. *Let* $A \subset E^b$; *then the following are equivalent:*

(1) A *is equi-l'-continuous on* E.

(2) A *is equicontinuous on* I_E.

We will now complete the task of characterizing compactness in E^b in terms of equi-l'-continuity. The following is the main result on this.

Theorem 3.8. *Let* $A \subset E^b$; *then the following are equivalent:*

(1) A *is equi-l'-continuous on* E.

(2) A *is relatively* $w(E^b, I_E)$*-compact.*

(3) A *is relatively* $w(E^b, Ba^{1/2})$*-compact.*

Proof. (1) $\Rightarrow$ (2) By (3.7) A is equicontinuous on I_E. Note that I_E is a Dedekind complete vector lattice and $(I_E)^c = E^b$. By applying (2.14) to the spaces I_E and $(I_E)^c$, it follows that A is relatively $w(E^b, I_E)$-compact.

(2) $\Rightarrow$ (3) The topology $w(E^b, I_E)$ is finer than $w(E^b, Ba^{1/2})$, thus (3) must hold.

(3) $\Rightarrow$ (1) By an argument similar to (2.8) we find an l'-sqeuence $\{x_n\}$ in E and $\{y_n\} \subset A$ such that $|\langle v, y_n\rangle| \geq \epsilon$ and $\langle v, y_0\rangle = 0$ where $v = \sup_m (\Sigma_1^m x_n)$-in-$I_E$ and y_0 is a $w(E^b, Ba^{1/2})$ accumulation point. Since $v \in Ba^{1/2}$, we have a contradiction of y_0 being a $w(E^b, Ba^{1/2})$ accumulation point of $\{y_n\}$ and the proof is complete.

We now give the promised extensions of Proposition (2.15).

Corollary 3.9. *Let A be a solid set in E^b, then the following are equivalent:*

(1) A *is equi-l'-continuous on* E.

(2) (a) *A is $|w|(E^b, E)$-bounded, and* (b) *every countable set $\{y_n\}$ of mutually disjoint elements of A converges to* 0 *in $|w|(E^b, E)$.*

Proof. Note that $E^b = (I_E)^c$, then by (2.17), (2) above is equivalent to A being relatively $w(E^b, I_E)$-compact, and by (3.7) this is equivalent to A being equi-l'-continuous on E; and the proof is complete.

Combining (3.9) with (2.8) gives

Corollary 3.10. *Let E be σ-complete and A a solid set in $E^{\sigma c}$, then the following are equivalent:*

(1) *A is relatively $w(E^{\sigma c}, E)$-compact.*

(2) (a) *A is $|w|(E^{\sigma c}, E)$-bounded, and* (b) *every countable set $\{y_n\}$ of mutually disjoint elements of A converge to* 0 *in $|w|(E^{\sigma c}, E)$.*

Remark. For $x \in E$, let E_x denote the ideal generated by x in E. Then E_x is the set of all elements y in E such that $|y| \leq a|x|$ for some $a > 0$.

Let I_x denote the ideal generated by x in I_E. Then I_x is the set of all $y \in I_E$ such that $|y| \leq a|x|$ for some $a > 0$. Thus $E_x \subset I_x$.

Now E_x is a norm space where the norm is given by $\|y\| = \inf\{a \geq 0: |y| \leq a|x|\}$ for each y in E_x. Let E_x' and E_x'' denote the first and second dual of the norm space $(E_x, \|\cdot\|)$. Note that E_x' is a Banach space, and, in fact, the norm is given by $\|z\| = \langle |x|, |z| \rangle$ for each z in E_x'. Also, the norm on E_x'' is given by $\|y\| = \inf\{a \geq 0: |y| \leq a|x|\}$ for each y in E_x''. In fact, it can be shown [6, (4.1)] that $E_x' = (E_x)^b$ and $E_x'' = (E_x')^c$. It follows that the ideal generated by E_x in E_x'' is exactly E_x''. Thus for this case (3.8) becomes a statement about weak compactness in E_x'. For clarity, we state it here..

Proposition 3.11. *Let $A \subset E_x'$, then the following are equivalent:*

(1) *A is equi-l'-continuous on E_x.*

(2) *A is relatively weakly compact.*

We now show that equi-l'-continuity is closely related to sequential compactness.

Theorem 3.12. *Let $A \subset E^b$, then the following are equivalent:*

(1) *A is equi-l'-continuous on E.*

(2) *For each x in E and sequence $\{y_n\} \subset A$, there exist y in E^b and a subsequence of $\{y_n\}$ which converges pointwise to y on the interval $[-x, x]$-in-I_E.*

Proof. (1) $\Rightarrow$ (2) Let $x \in E$ and $\{y_n\} \subset A$. Consider E_x and let $T: E_x \to E$ be the identity map. Then it follows that $T^t: E^b \to E_x'$ and $T^{tt}: E_x'' \to I_x$.

Since A is equi-l'-continuous on E, it follows easily that $T^t(A)$ is equi-l'-continuous on E_x. Thus by (3.11) $T^t(A)$ is relatively $w(E_x', E_x'')$-compact. Hence by Eberlein's theorem [2, p. 430], there exists a subsequence $\{T^t(y_{n_k})\}$

converging weakly to an element z of E_x'.

Now $\{y_{n_k}\}$ is equi-l'-continuous on E, and thus by (3.8), is relatively $w(E^b, I_E)$-compact; hence has a $w(E^b, I_E)$-accumulation point y in E^b.

Since $T^t\colon E^b \to E_x'$ is continuous with respect to $w(E^b, I_E)$ and $w(E_x', E_x'')$, it follows that $T^t(y)$ is a $w(E_x', E_x'')$ accumulation point of $\{T^t(y_{n_k})\}$, and hence $\{T^t(y_{n_k})\}$ converges weakly to $T^t(y)$.

Let $s \in [-x, x]$-in-I_E. It can be shown that T^{tt} maps E_x'' onto I_x. Thus there exists r in E_x'' such that $T^{tt}(r) = s$. Thus

$$\langle s, y\rangle = \langle r, T^t(y)\rangle = \lim_k \langle r, T^t(y_{n_k})\rangle = \lim_k \langle s, y_{n_k}\rangle$$

for each s in $[-x, x]$-in-I_E.

(2) $\Rightarrow$ (1) Suppose A is not equi-l'-continuous on E, then there exist $\epsilon > 0$, an l'-sequence $\{x_n\}$ in E, and $\{y_n\} \subset A$ such that $|\langle x_n, y_n\rangle| > \epsilon$ for all n. Since $\{x_n\}$ is an l'-sequence, there exists an x in E such that $\Sigma_1^n |x_k| \le x$ for all n.

By (2) above choose a subsequence $\{y_{n_k}\}$ converging pointwise on $[-x, x]$-in-I_E to some y in E^b. Let $T\colon I_x \to I_E$ be the identity map, then it follows that $T^t\colon (I_E)^c \to (I_x)^c$. But $(I_E)^c = E^b$, so $T^t\colon E^b \to (I_x)^c$. It is clear that $\{T^t(y_{n_k})\}$ converges to $T^t(y)$ pointwise on I_x. It then follows from (2.14) that it is equicontinuous on I_x. So for k large $|\langle x_{n_k}, T^t(y_{n_k})\rangle| < \epsilon$. But $|\langle x_{n_k}, T^t(y_{n_k})\rangle| = |\langle T(x_{n_k}), y_{n_k}\rangle| = |\langle x_{n_k}, y_{n_k}\rangle| \ge \epsilon$, hence a contradiction; and this completes the proof.

Consider x in E, and E_x the ideal generated by x in E. Then $E_x^\perp$ is a closed ideal in E^b, hence a band, so $E^b = E_x^\perp \oplus (E_x^\perp)'$. Each y in E^b has a component in $(E_x^\perp)'$, we will denote this component by $(y)_x$ (an abuse of notation). Thus each element x in E determines a projection on E^b. Then equi-l'-continuity on E can be stated in terms of these projections and relatively $w(E^b, I_E)$-sequential compactness.

Proposition 3.13. *Let $A \subset E^b$, then the following are equivalent:*

(1) *A is equi-l'-continuous on E.*

(2) *A_x is relatively $w(E^b, I_E)$-sequentially compact for each x in E.*

Proof. (1) $\Rightarrow$ (2) Consider x in E and $\{y_n\} \subset A$. By (3.12) there exists a subsequence $\{y_{n_k}\}$ converging pointwise on $[-x, x]$-in-I_E to some y in E^b.

Now consider the ideal I_x in I_E, so the order closure $\overline{I}_x$ is a band in I_E, thus $I_E = \overline{I}_x \oplus (\overline{I}_x)'$. Since $I_x = \bigcup_{n=1}^\infty n\,[-x, x]$, it follows that $\{y_{n_k}\}$ converges pointwise on I_x to y. I claim that $\{y_{n_k}\}$ converges pointwise on $\overline{I}_x$. Let $z \in \overline{I}_x$, then there exists a net $\{z_\alpha\} \subset I_x$ such that $z_\alpha \to z$ in I_E. Then $z_\alpha \to z$ uniformly on $\{y_{n_k}\}$ since by (3.7) $\{y_{n_k}\}$ is equicontinuous on I_E. From the uniform convergence, it follows that $\lim_k \langle z, y_{n_k}\rangle = \langle z, y\rangle$. Therefore, $\{y_{n_k}\}$

converges pointwise on $\bar{I}_x$ to y. Since $E^b = E_x^{\perp} \oplus (E_x^{\perp})'$, $I_E = \bar{I}_x \oplus \bar{I}'_x$, and $E_x^{\perp} = I_x^{\perp} = (\bar{I}_x)^{\perp}$; it follows that $\{(y_{n_k})_x\}$ converges to $(y)_x$ in $w(E^b, I_E)$.

(2) $\Rightarrow$ (1) Suppose (1) does not hold, then there exist $\epsilon > 0$, l'-sequence $\{x_n\}$ in E, and $\{y_n\} \subset A$ such that $|\langle x_n, y_n\rangle| > \epsilon$ for all n. Choose an element x in E such that $\Sigma_1^n |x_k| \leq x$ for all n. By (2) above there exists a subsequence $\{(y_{n_k})_x\}$ converging in $w(E^b, I_E)$ to some element y in E^b. Note that $(x_n)_x = x_n$ since the x_n's are in the ideal I_x. Thus $|\langle x_n, (y_n)_x\rangle| = |\langle (x_n)_x, y_n\rangle| = |\langle x_n, y_n\rangle| > \epsilon$. Since $\{(y_{n_k})_x\}$ converges in $w(E^b, I_E)$, it follows from (3.8) that it is equi-l'-continuous on E, which contradicts $|\langle x_{n_k}, (y_{n_k})_x\rangle| > \epsilon$.

Note that by (3.8) every $w(E^b, Ba^{1/2})$ convergent sequence in E^b must be equi-l'-continuous on E and also converge in $w(E^b, I_E)$. In (3.12) and (3.13) the equi-l'-continuity of a convergent sequence was the critical fact in their proofs. Thus they could be restated in terms of $w(E^b, Ba^{1/2})$ convergent sequences.

4. Convergent sequences in E^b. As usual, l^∞, l', and c_0 denote the real space of bounded sequences, absolutely summable sequences, and sequences converging to 0 respectively, each with its usual norm and order. Then $l^\infty =$ (norm dual of l') $= (l')^c$ and $l' = (l^\infty)^c$, thus each space is the other's order continuous dual. Also $(l^\infty)^b =$ (norm dual of l^∞). Since $l' = (l^\infty)^c$, l' is a band in $(l^\infty)^b$; hence each element y in $(l^\infty)^b$ has a component $(y)_{l'}$ in l', in fact, $(l^\infty)^b = l' \oplus c_0^{\perp}$. We will make use of the following theorem due to Phillips [2, p. 296].

Proposition 4.1. *If a sequence $\{y_n\}$ in $(l^\infty)^b$ is $w[(l^\infty)^b, l^\infty]$ convergent to 0, then $\{(y_n)_{l'}\}$ is norm-convergent to 0.*

We will apply (4.1) to σ-complete vector lattices by the following technique used by Kaplan [8, (3.2)].

Proposition 4.2. *Let E be σ-complete and $\{x_n\}$ an l'-sequence in E, then there exists a positive linear mapping $F: l^\infty \to E$ satisfying $F(e_n) = |x_n|$ for all n, where e_n is the element of l^∞ with the nth coordinate 1 and the remaining coordinates 0.*

This section will be devoted to extending the results of §3 to $w(E^b, E)$-convergent sequences of E^b. Note that §3 restricted itself to the topologies $w(E^b, Ba^{1/2})$ and $w(E^b, I_E)$ on E^b. The main tool will be the deep result (4.3) that $w(E^b, E)$-convergent sequences are equi-l'-continuous on E, when E is σ-complete.

Theorem 4.3. *Let E be σ-complete, then every $w(E^b, E)$-Cauchy sequence in E^b is equi-l'-continuous on E.*

Proof. Let $\{y_n\}$ be $w(E^b, E)$-Cauchy. Suppose $A = \{y_n\}$ is not equi-l'-continuous on E, then there exist $\epsilon > 0$ and a positive l'-sequence $\{x_k\}$ in E such that $\|x_k\|_{A^\circ} > \epsilon$. Choose a subsequence $\{y_{n_k}\}$ such that $|\langle x_k, y_{n_k}\rangle| > \epsilon$. For simplicity of notation, let the original sequences have this property: $|\langle x_n, y_n\rangle| > \epsilon$.

Applying (4.2) there exists a positive linear mapping $F: l^\infty \to E$ such that $F(e_n) = x_n$. Then $F^t: E^b \to (l^\infty)^b$ is continuous with respect to the topologies $w(E^b, E)$ and $w[(l^\infty)^b, l^\infty]$. It follows that $\{F^t(y_n)\}$ is $w[(l^\infty)^b, l^\infty]$-Cauchy, and hence converges to an element z of $(l^\infty)^b$. Therefore, by (4.1) $\lim_n \|(F^t(y_n) - z)_{l'}\| = 0$. Thus for n sufficiently large $\|(F^t(y_n) - z)_{l'}\| < \epsilon/2$, hence $|\langle e_n, F^t(y_n) - (z)_{l'}\rangle| \le \epsilon/2$, so $|\langle e_n, F^t(y_n)\rangle| \le \epsilon/2 + |\langle e_n, (z)_{l'}\rangle|$ for n large enough. Note that $\{e_n\}$ converges to 0 in $w(l^\infty, l')$, thus $\lim_n |\langle e_n, (z)_{l'}\rangle| = 0$. Therefore, $|\langle e_n, F^t(y_n)\rangle| < \epsilon$ for n sufficiently large. But $|\langle e_n, F^t(y_n)\rangle| = |\langle F(e_n), y_n\rangle| = |\langle x_n, y_n\rangle| \ge \epsilon$, hence a contradiction; and the proof is complete.

Corollary 4.4. *If E is σ-complete, then E^b is $w(E^b, E)$-sequentially complete.*

Combining (3.8) with (4.3) gives the following result due to Schaefer [11]:

Corollary 4.5. *Let E be σ-complete. If a sequence $\{y_n\}$ in E^b converges in the topology $w(E^b, E)$; then it converges in the topology $w(E^b, I_E)$.*

For E σ-complete, (3.12) can be strengthened from a statement about intervals of I_E to one considering only the intervals of E.

Proposition 4.6. *If E is σ-complete and $A \subset E^b$, then the following are equivalent:*

(1) *A is equi-l'-continuous on E.*

(2) *For each x in E and sequence $\{y_n\} \subset A$, there exist y in E^b and a subsequence of $\{y_n\}$ which converges pointwise to y on the interval $[-x, x]$-in-E.*

Proof. By (3.12), (1) implies (2). Now assume (2) holds. Suppose A is not equi-l'-continuous on E. Then there exist $\epsilon > 0$, l'-sequence $\{x_n\}$ in E, and $\{y_n\} \subset A$ such that $|\langle x_n, y_n\rangle| > \epsilon$.

Choose an element x in E such that $\Sigma_1^n |x_k| \le x$ for all n. By (2) above there exists a subsequence $\{y_{n_k}\}$ converging to an element y in E^b on the interval $[-x, x]$-in-E.

Consider the identity map $T: E_x \to E$ and $T^t: E^b \to E_x'$, then $T^t(y_{n_k})$ converges to $T^t(y)$ in $w(E_x', E_x)$. Note that $E_x' = (E_x)^b$. Applying (4.3) to the spaces E_x and E_x^b, it follows that $T^t(y_{n_k})$ is equi-l'-continuous on E_x. But $|\langle x_{n_k}, T^t(y_{n_k})\rangle| = |\langle x_{n_k}, y_{n_k}\rangle| \ge \epsilon$, and $\{x_{n_k}\}$ is an l'-sequence in E_x, hence a contradiction; and this completes the proof.

For E σ-complete, we get the following strengthening of (3.13) by applying (4.5). This points out the close relationship between vague sequential compactness and equi-l'-continuity.

Proposition 4.7. *If E is σ-complete and $A \subset E^b$, then the following are equivalent*:

(1) *A is equi-l'-continuous on E.*

(2) *A_x is relatively $w(E^b, E)$-sequentially compact for each x in E.*

As stated earlier, each element w in E^b generates a closed ideal in E^b, and hence determines a projection on E^b, denoted by $(y)_w$ for y in E^b.

This projection is determined purely by the order structure on E^b; however, there is a relationship between this and vaguely convergent sequences.

Proposition 4.8. *Let E be σ-complete. If $\{y_n\}$ converges to y in $w(E^b, E)$ and $0 \leq w_n \uparrow w_0$ in E^b then $\{(y_n)_{w_n}\}$ converges to $(y)_{w_0}$ in $w(E^b, E)$.*

Proof. Consider I_E and $(I_E)^c = E^b$. Let I_n be the closed ideal generated by w_n in E^b and $J_n = (I_n^\perp)'$ the dual ideal in I_E. Then J_n is a band in I_E. For x in E, let $x_n = (x)_{J_n}$. Note that $\langle x_n, z\rangle = \langle x, (z)_{w_n}\rangle$ for each z in E^b.

Since $w_n \uparrow w_0$ in E^b, it follows that $x_n \uparrow x_0 = (x)_{J_0}$ in I_E. From (4.3) it follows that $\{y_n\}$ is equicontinuous on I_E, thus $x_n \uparrow x_0$ uniformly on the y_n's. Therefore, by the uniform convergence, it follows that

$$\langle x, (y)_{w_0}\rangle = \langle x_0, y\rangle = \lim_n \langle x_n, y_n\rangle = \lim_n \langle x, (y_n)_{w_n}\rangle$$

for each x in E. Hence $\{(y_n)_{w_n}\}$ converges to $(y)_{w_0}$ in $w(E^b, E)$.

Corollary 4.9. *Let E be σ-complete. If $\{y_n\}$ converges to y in $w(E^b, E)$ then, for any closed ideal I in E^b, the projection $\{(y_n)_I\}$ converges to $(y)_I$ in $w(E^b, E)$.*

Proof. Since $\{y_n\}$ is equi-l'continuous on E, it follows that $\{|y_n|_I\}$ is $|w|(E^b, E)$-bounded. Thus $z = \Sigma_1^\infty |y_n|_I/2^n$ is an element of E^b and $(y_n)_z = (y_n)_I$. By (4.8) $\{(y_n)_z\}$ converges to $(y)_z$ in $w(E^b, E)$; and this completes the proof.

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